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Orders of solutions of fractional differential equation with entire coefficients

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Abstract. We study the solutions of the fractional differential equation

$$^{LC}D_z^{\alpha} f'(z) + A(z)^{LC}D_z^{\beta} f(z) + B(z)f(z) = 0$$

where $^{LC}D_z^{\alpha}$ and $^{LC}D_z^{\beta}$ are the Liouville-Caputo fractional derivatives of orders $n-1 < \alpha, \beta \leq n \in \mathbb{N}^*$, and z is complex number, $A(z), B(z)$ be entire functions. We find conditions on the coefficients so that every solution that is not identically zero has infinite order.

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1. Introduction

Recently, the complex modelings of phenomena in nature and society have been the object of several investigations based on the methods originally developed in a physical context. These systems are the consequence of the ability of individuals to develop strategies. They occur in kinetic theory [3], complex dynamical systems [12], chaotic complex systems and hyperchaotic complex systems [21], and the complex Lorenz-like system which has been found in laser physics while analyzing baroclinic instability of the geophysical flows in the atmosphere (or in the ocean) [14, 22]. Sainty [15] considered the complex heat equation using a complex valued Brownian.

Kilbas and Baleanu et al.[1, 2, 10, 11], imposed several applications of fractional calculus including complex modelings, and theory and applications of fractional differential equations.
The author studied various types of fractional differential equations in complex domain such as the Cauchy equation, the diffusion equation and telegraph equations [9]-[8]. Transform is a significant technique to solve mathematical problems. Many useful transforms for solving various problems appeared in open literature such as wave transformation, the Laplace transform, the Fourier transform, the Bücklund transformation, the integral transform, the local fractional integral transforms and the fractional complex transform (see [4, 13]). And also we must also not forget the scholar Hari Mohan Srivastava who has

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touched upon in several articles on fractional arithmetic in the complex plane (see [18, 19, 20, 16, 17]).

In [20], H. M. Srivastava investigated and closely examined the so-called so-called k-gamma function and the corresponding $k-$Pochhammer symbol and $k-$Laplace transform, the pathway integral version and the conformable or non-conformable version as well as the so-called $(k, s)-$extention of the operators of the traditional Riemann-Liouville fractional calculs and such other familiar operators of calculus as the Liouville- Caputo fractional derivative operator, the Sumudu transform and the $P-$version of the classical Laplace transform, the so-called post-quantum or, briefly, $(p, q)-$version of the familiar basic or quantum (or $q-$) analysis, the parametric variation of the Bessel and related functions, and so on. We also look into the current literature which is full of repeated or translational usages of the classical Laplace transform operators L in order to successfully solve initial-value problems involving ordinary and partial differential equations.

2. Definitions and Lemmas

In this section [18], we introduce some notations and definitions for fractional operators (derivative and integral) in the complex $z-$plane $\mathbb{C}$ as follows.

Definition 1. *The fractional derivative of order α is defined, for a function $f(z)$, by*

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \int_0^z \frac{f(\xi)}{(z-\xi)^\alpha} d\xi, \quad 0 \le \alpha < 1 \tag{2.1}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z-$plane containing the origin, and the multiplicity of $(z-\xi)^{-\alpha}$ is removed by requiring $\log(z-\xi)$ *to be real when* $(z-\xi) > 0$.

Definition 2. *The fractional integral of order α is defined, for a function $f(z)$, by*

$$I_z^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-\xi)^{\alpha-1} f(\xi) d\xi, \quad 0 < \alpha, \tag{2.2}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z-$plane containing the origin, and the multiplicity of $(z-\xi)^{\alpha-1}$ is removed by requiring $\log(z-\xi)$ *to be real when* $(z-\xi) > 0$.

Applying the Caputo method, we know the Caputo derivative by the formula:

Definition 3. *The Liouville-Caputo fractional derivative of order* $n-1<\alpha<n, n\in\mathbb{N}^*$*, for a function* $f(z)$ *is defined as*
(2.3)

$$^{LC}D_z^{\alpha}f(z)=I_z^{n-\alpha}f^{(n)}(z)=\frac{1}{\Gamma(n-\alpha)}\int_0^z(z-\xi)^{n-\alpha-1}f^{(n)}(\xi)d\xi,\quad n-1<\alpha\leq n, n\in\mathbb{N}^*$$

where the function $f(z)$ *is analytic in a simply-connected region of the complex* $z-$*plane containing the origin, and the multiplicity of* $(z-\xi)^{n-\alpha-1}$ *is removed by requiring* $\log(z-\xi)$ *to be real when* $(z-\xi)>0$.

Remark 1. *In the following we put*

$$^{LC}D^{\alpha}f(z)=f^{(\alpha)}(z)$$

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [7]). Let $\rho(f)$ denote the order of an entire function f, that is,

$$\rho(f)=\overline{\lim_{r\to+\infty}}\frac{\log T(r,f)}{\log r}=\overline{\lim_{r\to+\infty}}\frac{\log\log M(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f (see [7]), and

$$M(r,f)=\max_{|z|=r}|f(z)|.$$

For example, the function $f(z)=e^{z^2}$ satisfies $\rho(f)=2$.

In the study of the differential equation

$$f''+A(z)f'+B(z)f=0 \tag{2.4}$$

where $A(z)$ and $B(z)\not\equiv 0$ are entire functions, Gundersen proved the following results.

Theorem 1. ([6], *p.418*) *Let* $A(z)$ *and* $B(z)\not\equiv 0$ *be entire functions such that for real constants* $\lambda,\eta,\theta_1,\theta_2$ *where* $\lambda>0,\eta>0$*, and* $\theta_1<\theta_2$*, we have*

$$|B(z)|\geq\exp\{(1+o(1))\lambda|z|^{\eta}\} \tag{2.5}$$

and

$$|A(z)|\leq\exp\{o(1)|z|^{\eta}\} \tag{2.6}$$

as $z\to\infty$ *in* $\theta_1\leq\arg z\leq\theta_2$. *Then every solution* $f\not\equiv 0$ *of 2.4 has infinite order.*

Lemma 1. *([5, 6]) Let f be a transcendental entire function of finite order ρ. Let $\Lambda = \{(k_1, j_1), (k_2, j_2), \cdots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0, i = 1, ..., m$, and let $\varepsilon > 0$ be a given constant. Then the following three statements hold:*

(i) There exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0) > 0$ so that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, and for all $(k, j) \in \Lambda$ we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)((\rho-1+\varepsilon))}. \tag{2.7}$$

(ii) There exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E_2 \cup [0, 1]$, and for all $(k, j) \in \Lambda$, the inequality 2.7 holds.

(iii) There exists a set $E_3 \subset (0, \infty)$ that has finite linear measure, such that for all z satisfying $|z| \notin E_3$, and for all $(k, j) \in \Lambda$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)((\rho+\varepsilon))}.$$

3. Main results

Consider the fractional differential equations

$$^{LC}D_z^{\alpha} f'(z) + A(z) {}^{LC}D_z^{\beta} f(z) + B(z) f(z) = 0 \tag{3.1}$$

where $^{LC}D_z^{\alpha}$ and $^{LC}D_z^{\beta}$ are the Liouville-Caputo fractional derivatives of orders $n-1 < \alpha, \beta \leq n \in \mathbb{N}^*$, and z is complex number, $A(z), B(z)$ be entire functions.

Theorem 2. *Let $A(z), B(z)$ be entire functions such that for real constants $\lambda, \eta, \theta_1, \theta_2$ where $\lambda > 0, \eta > 0$, and $\theta_1 < \theta_2$, we have*

$$|B(z)| \geq \frac{1}{\Gamma(n+1-\alpha)} \exp\left\{(1 + o(1)) \lambda |z|^{\beta\eta}\right\} \tag{3.2}$$

and

$$|A(z)| \leq \frac{\Gamma(n+1-\beta)}{\Gamma(n+1-\alpha)} \exp\{o(1) |z|^{\eta\alpha}\} \tag{3.3}$$

as $z \to \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Then every solution $f \not\equiv 0$, and $\max_{\zeta \in [0,z]} \left|f^{(n)}(\xi)\right| = \left|f^{(n)}(z)\right|$ of equation 3.1 has infinite order.

4. Proof of Theorem 2

Suppose that $f\ (\not\equiv 0)$ is a solution of (3.1) of finite order $\rho(f)=\sigma<\infty$, and $\max\limits_{\zeta\in[0,z]}\left|f^{(n)}(\xi)\right| = \left|f^{(n)}(z)\right|$. Then from Lemma 1 there exists a real constant ψ_0 where $\theta_1\le\psi_0\le\theta_2$, such that

$$
\begin{aligned}
(4.1)\qquad \left|\frac{(f^{(\alpha)}(z)}{f(z)}\right| &= \frac{\frac{1}{\Gamma(n-\alpha)}\left|\int\limits_0^z (z-\xi)^{n-\alpha-1} f^{(n)}(\xi)d\xi\right|}{|f(z)|}\\
&\le \frac{|z|^{n-\alpha}}{\Gamma(n-\alpha+1)}\frac{\max\limits_{\zeta\in[0,z]}\left|f^{(n)}(\xi)\right|}{|f(z)|}\\
&\le \frac{|z|^{n-\alpha}}{\Gamma(n-\alpha+1)}\frac{\left|f^{(n)}(z)\right|}{|f(z)|}\\
&= o(1)\frac{|z|^{n(\sigma+1)-\alpha}}{\Gamma(n-\alpha+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.2)\qquad \left|\frac{(f^{(\beta)}(z)}{f(z)}\right| &= \frac{\frac{1}{\Gamma(n-\beta)}\left|\int\limits_0^z (z-\xi)^{n-\beta-1} f^{(n)}(\xi)d\xi\right|}{|f(z)|}\\
&\le \frac{|z|^{n-\beta}}{\Gamma(n-\beta+1)}\frac{\max\limits_{\zeta\in[0,z]}\left|f^{(n)}(\xi)\right|}{|f(z)|}\\
&\le \frac{|z|^{n-\beta}}{\Gamma(n-\beta+1)}\frac{\left|f^{(n)}(z)\right|}{|f(z)|}\\
&= o(1)\frac{|z|^{n(\sigma+1)-\beta}}{\Gamma(n-\beta+1)}
\end{aligned}
$$

as z$\to\infty$ along $\arg\psi_0$. Then from (4.1), (4.2) and (3.1), we obtain

$$
\begin{aligned}
|B(z)| &\le \frac{\left|{}^{LC}D_z^{\alpha}f(z)\right|}{|f(z)|} + |A(z)|\,\frac{\left|{}^{LC}D_z^{\beta}f(z)\right|}{|f(z)|}\\
&\le \left(\frac{|z|^{n-\alpha}}{\Gamma(n+1-\alpha)} + \frac{|A(z)|\,|z|^{n-\beta}}{\Gamma(n-\beta+1)}\right)\left|\frac{f^{(n)}(z)}{f(z)}\right|\\
&\le \left(\frac{|z|^{n-\alpha}}{\Gamma(n+1-\alpha)} + \frac{|A(z)|\,|z|^{n-\beta}}{\Gamma(n+1-\beta)}\right)|z|^{n(\rho-1+\varepsilon)}\\
&\le \frac{o(1)\,|z|^{n(\rho-1)-\alpha}}{\Gamma(n+1-\alpha)} + \frac{o(1)\,|z|^{n(\rho-1)-\beta}}{\Gamma(n+1-\beta)}\,|A(z)|
\end{aligned}
$$

and therefore

$$\Gamma(n+1-\alpha)\,|B(z)| \leq o(1)\,|z|^{n(\rho-1)-\alpha} + \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1-\beta)}o(1)\,|z|^{n(\rho-1)-\beta}\,|A(z)|$$

as z$\to\infty$ along $\arg\psi_0$, and this contradicts (3.2) and (3.3).

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References

[1] Baleanu, D., Diethelm, K., Scalas, E., and Trujillo, J. J. *Fractional calculus*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. Models and numerical methods.

[2] Baleanu, D., and Trujillo, J. J. On exact solutions of a class of fractional Euler-Lagrange equations. *Nonlinear Dynam. 52*, 4 (2008), 331–335.

[3] Bianca, C. Onset of nonlinearity in thermostatted active particles models for complex systems. *Nonlinear Anal. Real World Appl. 13*, 6 (2012), 2593–2608.

[4] Gordoa, P. R., Pickering, A., and Zhu, Z. N. Bäcklund transformations for a matrix second Painlevé equation. *Phys. Lett. A 374*, 34 (2010), 3422–3424.

[5] Gundersen, G. G. Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates. *J. London Math. Soc. (2) 37*, 1 (1988), 88–104.

[6] Gundersen, G. G. Finite order solutions of second order linear differential equations. *Trans. Amer. Math. Soc. 305*, 1 (1988), 415–429.

[7] Hayman, W. K. *Meromorphic functions.* Oxford Mathematical Monographs. Clarendon Press, Oxford, 1964.

[8] Ibrahim, R., Elobaid, M., and Suzan, J. Obaiys a class of quantum briot-bouquet differential equations with complex coefficients. *Mathematics 8*, 5 (2020), 8.

[9] Ibrahim, R. W. Ulam stability for fractional differential equation in complex domain. *Abstr. Appl. Anal.* (2012), Art. ID 649517, 8.

[10] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. *Theory and applications of fractional differential equations*, vol. 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006.

[11] Magin, R., Feng, X., and Baleanu, D. Solving the fractional order bloch equation. *Concepts Magn. Reson*, 1 (2009), 16–23.

[12] Mahmoud, G. M., and Mahmoud, E. E. Lag synchronization of hyperchaotic complex nonlinear systems. *Nonlinear Dynam. 67*, 2 (2012), 1613–1622.

[13] Molliq, R., and Batiha, B. Approximate analytic solutions of fractional zakharov-kuznetsov equations by fractional complex transform. *Int. J. Eng. Technol 1*, 1 (2012), 1–13.

[14] Rauh, A., Hannibal, L., and Abraham, N. B. Global stability properties of the complex Lorenz model. *Phys. D 99*, 1 (1996), 45–58.

[15] Sainty, P. Construction of a complex-valued fractional Brownian motion of order *N*. *J. Math. Phys. 33*, 9 (1992), 3128–3149.

[16] Srivastava, H. An introductory overview of fractional-calculus operators based upon the fox-wright and related higher transcendental functions. *J. Adv. Eng. Comput 5* (2021), 135–166.

[17] Srivastava, H. A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. *Symmetry 13*, 2294 (2021).

[18] Srivastava, H. M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In *Univalent functions, fractional calculus, and their applications*, Ellis Horwood Ser. Math. Appl. Horwood, Chichester, 1989, pp. 329–354.

[19] Srivastava, H. M. Fractional-order derivatives and integrals: introductory overview and recent developments. *Kyungpook Math. J. 60*, 1 (2020), 73–116.

[20] Srivastava, H. M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal. 22*, 8 (2021), 1501–1520.

[21] Tsuji, M. *Potential theory in modern function theory.* Chelsea Publishing Co., New York, 1975. Reprinting of the 1959 original.

[22] Yang, L. Springer-verlag science press, berlin-beijing. *Springer-Verlag Science Press, Berlin-Beijing* (1993).

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