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Spectral equality for bounded regularized semigroup

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In memory of professor Boussouis Brahim

Abstract. In this note we prove that the spectral inclusion for regularized semigroups holds for the approximate point spectrum and the semiregular spectrum. Also, we give conditions for which the regularized semigroups satisfy spectral equality for approximate point and semiregular spectrum. Furthermore, we give necessary and sufficient conditions for the generator of a regularized semigroup to be semiregular.

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1. Introduction

Let X be a Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. For $T \in \mathcal{B}(X)$, let $N(T)$, $R(T)$ and $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$ denote respectively the kernel, the range and the hyper-range of T. We denote by $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_r(T)$ and $\sigma_{su}(T)$ respectively the resolvent set, the spectrum, the point spectrum, the approximate point spectrum, the residual spectrum and the surjective spectrum of T. Recall that T is semiregular if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$. The semiregular spectrum is defined by:

$$\sigma_\gamma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ \text{ is not semiregular}\}.$$

For more details see [1].

Let X^* denote the dual space of X and A^* the adjoint operator of A with domain $D(A)$. The quasi-nilpotent part of A is defined by

$$H_0(A) := \{x \in \cap_{n\geq 0} D(A^n) : \lim_{n\to\infty} \|A^n x\|^{\frac{1}{n}} = 0\}.$$

Recall that the family $(T(t))_{t\geq 0}$ of closed operators on X satisfying $T(0) = I$ and $T(s)T(t) \subset T(s+t)$ is called the semigroup of unbounded operators in the sense of [6] if:

$$D := \{x \in \cap_{s,t\geq 0} D(T(s)T(t)) : \quad t \mapsto T(t)x \in \mathcal{C}([0,\infty], X)\} \neq \{0\},$$

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see for instance [12].

In the continuous case, a one-parameter family $T = (T(t))_{t\geq 0}$ of continuous linear operators on X, is a strongly continuous semigroup (or C_0-semigroup) of operators, if $T(0) = I, T(t)T(s) = T(t+s)$, for all $t, s \geq 0$, and $\lim_{t\to 0} T(t)x = x$ for all $x \in X$. The operator $A : D(A) \subseteq X \to X$ defined by $Ax = \lim_{t\to 0} \frac{T(t)x-x}{t}$ is called the generator of the C_0-semigroup T, where $D(A) = \{x \in X : \lim_{t\to 0} \frac{T(t)x-x}{t}$ exists $\}$. For further information about C_0-semigroups we refer the reader to the books [4].

Regularized semigroups have been extensively studied in functional analysis and partial differential equations, and they have important applications in various fields, including physics, engineering, and biology.

In the 1960s, L. Schwartz [13] introduced the concept of regularized semigroups as a way of extending the theory of semigroups of linear operators to the case of unbounded operators. He showed that a regularized semigroup is a family of linear operators that satisfies a number of properties, including the semigroup property, the strong continuity property, and the regularity property. Schwartz's work laid the foundation for the study of regularized semigroups, and his ideas have been extended and generalized in many different directions since then.

In [9], the main assumption on the semigroup is that $(T(t))_{t\geq 0}$ can be regularized, i.e. there is an injective operator $C \in \mathcal{B}(X)$ such that

$$D \subset R(C) \text{ and } C^{-1}T(t)C = T(t) \text{ for all } \quad t \geq 0.$$

Then $S(t) = CT(t)$ defines a C-regularized semigroup satisfying:

1. $S(0) = C$ is injective;
2. $S(t)S(s) = CS(t+s)$ for all $t, s \geq 0$;
3. $t \mapsto S(t)x \in \mathcal{C}([0,\infty], X)$ for all $x \in X$.

Let $(S(t))_{t\geq 0}$ be a C-regularized semigroup on X. We define a linear operator G and U by the strong limits:

$$D(G) = \{x \in R(C) : \exists \lim_{t\to 0^+} \frac{(C^{-1}S(t)x - x)}{t}\},$$

$$Gx := \lim_{t\to 0^+} \frac{1}{t}(C^{-1}S(t)x - x)$$

and

$$D(U) = \{x \in X : \lim_{t\to 0^+} \frac{(S(t)x - Cx)}{t} \in R(C)\},$$

$$Ux := C^{-1} \lim_{t\to 0^+} \frac{(S(t)x - Cx)}{t} \quad \text{for} \quad x \in D(U),$$

respectively. The operator G is called generator of the C-regularized semigroup $(S(t))_{t\geq 0}$ in the sense of Miyadera [11]. The operator U is called generator of the C-regularized semigroup $(S(t))_{t\geq 0}$ in the sense of Da Prato [2].

Recall also that

$$x \in D(G) \Leftrightarrow \forall t > 0 : S(t)x - Cx = \int_0^t S(r)Gxdr,$$

$$\text{For all } x \in D(G) : \frac{d}{ds}S(s)x = S(s)Gx = GS(s)x.$$

Also, if $x \in D(G)$, then for all $t \geq 0$ we have $S(t)x \in D(G)$, [3].

C-regularized semigroups have also been studied in the context of operator theory and spectral theory. For example, Ki Sik Ha [5] and Kunstman P.C,[9] developed a theory of spectral mapping theorems for C-regularized semigroups, which provides a powerful tool for studying the spectra of operators that generate such semigroups.

Ki Sik Ha in [5] proved that spectral mapping theorems such as $e^{t\sigma(G)} \subset \sigma(T(t))$, $e^{t\sigma_p(G)} \subset \sigma_p(T(t)) \subset e^{t\sigma_p(G)} \cup \{0\}$ and $e^{t\sigma_r(G)} \subset \sigma_r(T(t)) \subset e^{t\sigma_r(G)} \cup \{0\}$ hold for every $t \geq 0$, where $T(t) = C^{-1}S(t)$.
In this work we will continue in the same direction, and prove that the spectral inclusion is verified for approximate point spectrum and semiregular spectrum. After giving an example that shows the spectral inclusion is strict for the approximate point spectrum and the semiregular spectrum, we will develop a spectral theory for C-regularized semigroups and their generators by giving conditions for which the C-regularized semigroups satisfy spectral equality for approximate point and semiregular spectrum. Finally, we give, under suitable assumptions, necessary and sufficient conditions for the generator of a C-regularized semigroup to be semiregular.

2. Main results

The following lemma will be needed in the sequel.

Lemma 2.1. *For a C-regularized semigroup $(S(t))_{t\geq 0}$ with generator G, let $B(\lambda, t) := \int_0^t e^{\lambda(t-s)}S(s)ds$. Then we have $B(\lambda, t)x \in D(G)$ for all $x \in X$.*

Furthermore, for all $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$ and $t \geq 0$, we have:

1. $(e^{\lambda t}C - S(t))^n x = (\lambda - G)^n B(\lambda, t)^n x$, *for all* $x \in X$.
2. $(e^{\lambda t}C - S(t))^n x = B(\lambda, t)^n (\lambda - G)^n x$, *for all* $x \in D(G)$.
3. $R^\infty(e^{\lambda t} - C^{-1}S(t)) \subseteq R^\infty(\lambda - G)$.
4. $N(\lambda - G) \subseteq N(e^{\lambda t} - C^{-1}S(t))$.
5. $H_0(\lambda - G) \subseteq H_0(e^{\lambda t} - C^{-1}S(t))$.

Proof. It is easy to show that $B(\lambda,t)=\int_0^t e^{\lambda(t-s)}S(s)ds$, is a bounded linear operator on X, for all $\lambda\in\mathbb{C}$ and $t\geq 0$. Now, for every $x\in X$ we have:

$$\begin{aligned}\frac{S(h)-C}{h}B(\lambda,t)x &= \frac{1}{h}\int_0^t e^{\lambda(t-s)}S(s+h)xds-\frac{C}{h}\int_0^t e^{\lambda(t-s)}S(s)xds\\ &= \frac{e^{\lambda h}}{h}\int_h^{h+t} e^{\lambda(t-s)}CS(s)xds-\frac{C}{h}\int_0^t e^{\lambda(t-s)}S(s)xds\end{aligned}$$

$= \frac{e^{\lambda h}}{h}\int_h^t e^{\lambda(t-s)}CS(s)xds+\frac{e^{\lambda h}}{h}\int_t^{h+t} e^{\lambda(t-s)}CS(s)xds-\frac{C}{h}\int_0^h e^{\lambda(t-s)}S(s)xds.$
As $h\downarrow 0$, the right-hand side converges to:

$$C\lambda B(\lambda,t)x+CS(t)x-C^2e^{\lambda t}x=C(\lambda B(\lambda,t)x+S(t)x-Ce^{\lambda t}x)$$

and consequently $B(\lambda,t)x\in D(G)$ and

$$CGB(\lambda,t)x=C(\lambda B(\lambda,t)x+S(t)x-Ce^{\lambda t}x).$$

Since C is injective this implies:

$$GB(\lambda,t)x=\lambda B(\lambda,t)x+S(t)x-Ce^{\lambda t}x.$$

From the definition of $B(\lambda,t)$ it is clear that for $x\in D(G)$, $B(\lambda,t)Gx=GB(\lambda,t)x$. Proceeding by induction on n, we get the result. The assertions $(3),(4)$ and (5) easily result from (1). □

Now, we give another proof for a spectral inclusion that has been proven in [5].

Theorem 2.2. *Let G be the generator of the C-regularized semigroup $S(t)_{t\geq 0}$, then*

$$e^{t\sigma(G)}\subset\sigma(T(t)),\forall t\geq 0,$$

where $T(t)=C^{-1}S(t)$.

Proof. Let $\lambda\in\mathbb{C}$ such that $e^{\lambda t}-T(t)$ is invertible, then $e^{\lambda t}-T(t)$ is injective and surjective. As $e^{\lambda t}-T(t)$ is surjective then $R(e^{\lambda t}-T(t))=X$. From (3) in Lemma 2.1 it follows that $R(\lambda-G)=X$, hence $(\lambda-G)$ is surjective. On the other hand, as $e^{\lambda t}-T(t)$ is injective then $N(e^{\lambda t}-T(t))=\{0\}$ and from (4) of Lemma 2.1 this implies that $N(\lambda-G)=\{0\}$, it follows that $\lambda-G$ is bijective, hence $\lambda-G$ is invertible. □

In the following, we give a spectral inclusion for the approximate point spectrum.

Theorem 2.3. *Let G be the generator of the C-regularized semigroup $(S(t))_{t\geq 0}$, then*

$$e^{t\sigma_a(G)}\subset\sigma_a(T(t)),\forall t\geq 0,$$

where $T(t)=C^{-1}S(t)$.

Proof. Take $\lambda \in \sigma_a(G)$ and a corresponding approximate eigenvector $(x_n)_{n\in\mathbb{N}} \in D(G) \subseteq R(C)$. According to Lemma 2.1,

$$e^{\lambda t}x_n - T(t)x_n = e^{\lambda t}x_n - C^{-1}S(t)x_n = C^{-1}B(\lambda,t)(\lambda - G)x_n$$

These vectors satisfy for some constant $M > 0$ the estimate

$$\|C^{-1}B(\lambda,t)(\lambda - G)x_n\| \leq M\|(\lambda - G)x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence,

$$e^{\lambda t}x_n - T(t)x_n \to 0.$$

Then $e^{\lambda t}$ is an approximate eigenvalue of $T(t)$, and $(x_n)_{n\in\mathbb{N}}$ serves as the same approximate eigenvector for all $t \geq 0$. □

Now, we will give some important concepts for C-regularized semigroups.

Quotient C-regularized Semigroups. For a closed $(S(t))_{t\geq 0}-$invariant subspace Y of X, we consider the quotient space $X/ := X/Y$ with canonical quotient map $q : X \to X/$. The quotient operators $S(t)/$ given by $S(t)/q(x) := q(S(t)x)$ for $x \in X$ and $t \geq 0$ are well-defined and form a C-regularized semigroup, called the quotient C-regularized semigroup $(S(t)/)_{t\geq 0}$ on the Banach space $X/$. The generator $G/$ of the quotient C-regularized semigroups is given by

$$G/q(x) := q(Gx)$$

with domain $D(G/) := q(D(G))$.

Let $S(t)_{t\geq 0}$ be a C-regularized semigroup on a Banach space X such that $S(t) = CT(t)$, with $T(t)_{t\geq 0}$ is a semigroup. The adjoint C-regularized semigroup $S^*(t)$ is the C^*-regularized semigroup on the dual space X^* which is obtained from $S(t)$ by taking pointwise in t the adjoint operators $S^*(t) := (S(t))^*$.

Proposition 2.4. *$D(G^*)$ is a $S^*(t)$-invariant subspace of X^* for all $t \geq 0$. Furthermore, for all $x^* \in D(G^*)$ we have:*

1. *$G^*S^*(t)x^* = S^*(t)G^*x^*$, for all $t \geq 0$.*
2. *weak* $\int_0^t S^*(s)x^*ds \in D(G^*)$ for all $t > 0$ and $x^* \in X^*$, and*

$$G^*(weak^* \int_0^t S^*(s)x^*ds) = S^*(t)x^* - C^*x^*.$$

If $x^* \in D(G^*)$, then

$$G * (weak^* \int_0^t S^*(s)x^*ds) = (weak^* \int_0^t S^*(s)G^*x^*ds).$$

Proof. 1. Let $x^* \in D(G^*)$ and $x \in D(G)$ be arbitrary. Then for any fixed $t > 0$, we have

$$\begin{aligned}\langle S^*(t)x^*, Gx\rangle &= \langle x^*, S(t)Gx\rangle = \langle x^*, GS(t)x\rangle \\ &= \langle G^*x^*, S(t)x\rangle = \langle S^*(t)G^*x^*, x\rangle\end{aligned}$$

Therefore $S^*(t)x^* \in D(G^*)$ and $G^*S^*(t)x^* = S^*(t)G^*x^*$.

2. Let $x \in D(G)$ be arbitrary. The identities

$$\begin{aligned} &< weak^* \int_0^t S^*(s)x^* ds, Gx > \\ &= \int_0^t < S^*(s)x^*, Gx > ds = \int_0^t \langle x^*, S(s)Gx \rangle ds \\ &= \langle x^*, \int_0^t S(s)Gx ds \rangle = \langle x^*, G\int_0^t S(s)x ds \rangle \\ &= \langle x^*, S(t)x - Cx \rangle = \langle S^*(t)x^* - C^*x^*, x \rangle \end{aligned}$$

show that $weak^* \int_0^t S^*(s)x^* ds \in D(A^*)$ and

$$G^*(weak^* \int_0^t S^*(s)x^* ds) = S^*(t)x^* - C^*x^*.$$

The second formula follows from a similar calculation: for $x \in D(G)$ we have

$$\begin{aligned} \langle G^*(weak^* \int_0^t S^*(s)x^* ds), x \rangle &= \int_0^t \langle x^*, S(s)Gx \rangle ds \\ &= \langle weak^* \int_0^t S^*(s)G^*x^* ds, x \rangle. \end{aligned}$$

□

In [7] V. Kordula and V. Müller proved the following lemma:

Lemma 2.5. *Let $T \in \mathcal{B}(X)$.*

1. *T is semiregular if and only if there exists a closed subspace $M \subseteq X$ such that $TM = M$ and the operator $\widetilde{T} : X/M \to X/M$ induced by T is bounded below.*
2. *If T is semiregular, then the operator $\widehat{T} : X/R^\infty(T) \to X/R^\infty(T)$ induced by T is bounded below.*

Now we can prove the following theorem.

Theorem 2.6. *Let $(S(t))_{t\geq 0}$ be a C-regularized semigroup with generator G. Then:*

$$e^{t\sigma_\gamma(G)} \subset \sigma_\gamma(T(t)), \quad \forall t \geq 0$$

where $T(t) = C^{-1}S(t)$.

Proof. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $t_0 \geq 0$ such that $e^{\lambda t_0} - C^{-1}S(t_0) = e^{\lambda t_0} - T(t_0)$ is semiregular. Then according to Mbekhta [10] the subspace $R^\infty(e^{\lambda t} - T(t_0))$ is $T(t)_{t\geq 0}-$invariant and is closed.
Take the quotient semi-group $\widetilde{T(t)}_{t\geq 0}$ is defined on $X/R^\infty(e^{\lambda t} - T(t_0))$ by:

$$\widetilde{T(t)}\widetilde{x} = \widetilde{T(t)x}, \quad \widetilde{x} \in X/R^\infty(e^{\lambda t} - T(t_0)).$$

Its generator $\widetilde{G}$ is defined on $D(\widetilde{G}) = \{\widetilde{x} : x \in D(G)\}$ by $\widetilde{G}\widetilde{x} = \widetilde{Gx}$ for all $x \in D(\widetilde{G})$. As in Lemma 2.2 we have that the operator $e^{\lambda t_0} - \widetilde{T(t_0)}$ is bounded below, then $e^{\lambda t_0} \notin \sigma_a(\widetilde{T(t_0)})$, therefore as we see in Theorem 2.2 $\lambda \notin \sigma_a(\widetilde{G})$ so the operator $\lambda - \widetilde{G}$ is bounded below. The injectivity of the operator $\lambda - \widetilde{G}$ gives that:

$$N(\lambda - G) \subseteq R^{\infty}(e^{\lambda t_0} - T(t_0))$$

and by (3) of Lemma 2.1, we have :

$$N(\lambda - G) \subseteq R^{\infty}(\lambda - G).$$

Now we show that $R(\lambda - G)$ is closed. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of $R(\lambda - G)$ which converges to f, then there exists a sequence $(g_n)_{n \in \mathbb{N}}$ in $D(G)$ such that

$$(\lambda - G)g_n = f_n \to f, \quad n \to \infty.$$

Since $R(\lambda - \widetilde{G})$ is closed, then there exists $\widetilde{h} \in D(\widetilde{G})$ such that $\widetilde{f} = (\lambda - \widetilde{G})\widetilde{h}$. As a result, $f - (\lambda - G)h \in R^{\infty}(e^{\lambda t_0} - T(t_0)) \subseteq R^{\infty}(\lambda - G) \subseteq R(\lambda - G)$. Then $f \in R(\lambda - G)$. Therefore, $\lambda - G$ is semiregular. □

Song X. in [14] proved the inclusion

$$\{e^{\lambda} : \lambda \in \sigma(A)\} \subset \sigma(C^{-1}T(t)).$$

In the following example, we show that the inclusion is strict and that the spectral inclusion is strict for approximate point spectrum and semiregular spectrum. We start by the following Lemma [8].

Lemma 2.7. *The generator of a semigroup $(T(t))_{t \geq 0}$ generates a regularized semigroup if and only if there is an injective bounded operator $C \in \mathcal{B}(X)$ which commutes with all $(T(t))_{t>0}$, and satisfies*

$$R(C) \subset \{x \in E : \lim_{t \to 0} T(t)x = x\}.$$

Example 2.8. Let X be the Banach space of continuous functions on $[0,1]$ which are equal to zero at $x = 1$ with the supremum norm. Let $(T(t))_{t \geq 0}$ be a C_0-semigroup with generator A and $\lambda > \omega_0$.
Define:

$$(T(t)f)(x) := \begin{cases} f(x+t) & if x + t \leq 1, \\ 0 & if x + t > 1. \end{cases}$$

Set

$$Cf(x) = (\lambda - A)^{-1} f(x) = \int_0^{\infty} e^{-\lambda t} T(t) f(x) dt \quad for x \in [0,1].$$

We have C is a bounded linear in X, injective $CT(t) = T(t)C$ and $R(C) \subset \{x \in X : \lim_{t \to 0} T(t)x = x\}$. In fact, let $y \in R(C) \Rightarrow \exists f \in X such\ that\ y = Cf(x)$ then

$$\lim_{t \to 0} T(t)y = \lim_{t \to 0} T(t)Cf(x) = \lim_{t \to 0} T(t) \int_0^{\infty} e^{-\lambda s} T(s) f(x) ds$$

$$= \lim_{t\to 0} \int_0^{\infty} e^{-\lambda s} T(t+s) f(x) ds = \lim_{t\to 0} \int_t^{\infty} e^{-\lambda(s-t)} T(s) f(x) ds$$

$$= \lim_{t\to 0} e^{\lambda t} \int_t^{\infty} e^{-\lambda s} T(t+s) f(x) ds = Cf(x) = y.$$

Then A generates a $C-$regularized semigroup.

$S(t) = CT(t)$ is a C-regularized semigroup on X and $CT(t) = T(t)C$. Its infinitesimal generator G is given on

$$D(G) = \{f : f \in C^1([0,1]) \cap X, f^{'} \in R(C) \cap X\}$$

by

$$Gf = Cf^{'} \text{ for } f \in D(G) \subset R(C) \cap X.$$

One checks easily that for every $\lambda \in \mathbb{C}$ and $g \in R(C) \cap X$ the equation $\lambda f - Cf^{'} = g$ has an unique solution $f \in R(C) \cap X$.

Therefore $\sigma(G) = \emptyset$, hence $\sigma_{ap}(G) = \emptyset$.

On the other hand, since for every $t \geq 0$, $S(t)$ is a bounded linear operator, then $\sigma(C^{-1}S(t)) \neq \emptyset$ and as $\partial\sigma(C^{-1}S(t)) \subseteq \sigma_a(C^{-1}S(t))$ then $\sigma_a(C^{-1}S(t)) \neq \emptyset$, therefore the inclusion $e^{t\sigma_a(G)} \subset \sigma_a(C^{-1}S(t))$ is strict. The same for semiregular spectrum i.e. $e^{t\sigma_\gamma(G)} \subset \sigma_\gamma(C^{-1}S(t))$ is strict.

In the next theorem, we give conditions for which the regularized semigroups satisfy the spectral equality for approximate point and semiregular spectrum. We start with the approximate point spectrum.

Theorem 2.9. *Let $(S(t))_{t\geq 0}$ be a C-regularized semigroup and let G be its generator. If $\lambda \in \sigma_a(A)$, then $e^{\lambda t} \in \sigma_a(C^{-1}S(t))$, and if $e^{\lambda t} \in \sigma_a(C^{-1}S(t))$, such that $\lim_{t\downarrow 0} \sup_{n\in\mathbb{N}} ||(S(t) - e^{\lambda t})x_n|| = 0$, where $(x_n)_n \subseteq X$, $||x_n|| = 1$, then there exists $k \in \mathbb{Z}$ such that $\lambda_k = \lambda + \frac{2\pi i k}{t} \in \sigma_a(G)$.*

Proof. If $\lambda \in \sigma_a(G)$, then $e^{\lambda t} \in \sigma_a(C^{-1}S(t))$ by Theorem 2.3.

To prove the second inclusion, let $e^{\lambda t} \in \sigma_a(C^{-1}S(t))$, then there is an approximate eigenvector $(x_n)_{n\in\mathbb{N}} \subset X$ satisfying $||x_n|| = 1, n \in \mathbb{N}$ and $||(e^{\lambda t} - C^{-1}S(t))x_n|| \to 0, \quad n \to \infty$ such that $\lim_{t\downarrow 0} \sup_{n\in\mathbb{N}} ||(e^{\lambda t} - S(t))x_n|| = 0$. The uniform continuity of $((e^{-\lambda t}S(t)))_{t\geq 0}$ on the vectors $x_n, n \in \mathbb{N}$ implies that the maps $[0,t] \ni s \to e^{-\lambda s}S(s)x_n, n \in N$ are equicontinuous.

We take $x_n^{'} \in X^*, ||x_n^{'}|| \leq 1$, satisfying $\langle x_n, x_n^{'} \rangle \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Then the functions

$$[0,t] \ni s \mapsto \xi_n(s) := \langle e^{-\lambda s} S(s) x_n, x_n^{'} \rangle$$

are uniformly bounded and equicontinuous. Hence, by the Arzelà-Ascoli theorem, there exists a convergent subsequence $(\xi_n)_{n\in\mathbb{N}}$ which converge to the function ξ different from zero, since it does not vanish identically one of its

Fourier coefficients must be different from zero. Therefore there is a $k \in \mathbb{Z}$ such that

$$\frac{1}{t}\int_0^t e^{-(2\pi ik/t)s}\xi(s)ds \neq 0.$$

If we set

$$z_k := \frac{1}{t}\int_0^t e^{-(2\pi ik/t)s}(e^{-\lambda s}S(s))x_n ds \neq 0,$$

then

$$\begin{aligned}
&(\lambda + 2\pi kt^{-1} - G)z_k \\
&= \frac{1}{t}(\lambda + 2\pi kt^{-1} - G)\int_0^t e^{-(2\pi ik/t)s}(e^{-\lambda s}S(s))x_n ds \\
&= \frac{1}{t}(\lambda + 2\pi kt^{-1} - G)\int_0^t e^{-(2\pi ikt^{-1}+\lambda)s}S(s)x_n ds \\
&= \frac{1}{t}e^{-(2\pi ikt^{-1}+\lambda)t}(\lambda + 2\pi kt^{-1} - G)\int_0^t e^{(2\pi ikt^{-1}+\lambda)(t-s)}S(s)x_n ds \\
&= \frac{1}{t}e^{-(2\pi ikt^{-1}+\lambda)t}(\lambda + 2\pi kt^{-1} - G)B(2\pi ikt^{-1}+\lambda, t)x_n \\
&= \frac{1}{t}e^{-(2\pi ikt^{-1}+\lambda)t}(Ce^{(\lambda+2\pi kt^{-1})t} - S(t))x_n \\
&= \frac{1}{t}(C - e^{-\lambda t}S(t))x_n \qquad \to 0, \quad n \to \infty.
\end{aligned}$$

We have $z_k \in D(G)$, and $(\lambda_k - G)z_k \to 0$.
As

$$\begin{aligned}
\lim_{k\to\infty} ||z_k|| &\geq \lim_{n\to\infty} |\frac{1}{t}\int_0^t e^{-(2\pi ik/t)s}\langle e^{-\lambda s}S(s)x_n, x_n^{'}\rangle ds| \\
&\geq |\frac{1}{t}\int_0^t e^{-(2\pi ik/t)s}\xi(s)ds| > 0.
\end{aligned}$$

This shows that $z_k/||z_k||$ is an approximate eigenvector of G corresponding to $\lambda_k = \lambda + 2\pi ik/t$ as an approximate eigenvalue of G □

We turn now to the semiregular spectrum.

Theorem 2.10. *Let $(S(t))_{t\geq 0}$ be a C-regularized semigroup and let G be its generator. If $\lambda \in \sigma_\gamma(G)$, then $e^{\lambda t} \in \sigma_\gamma(C^{-1}S(t))$, and if $e^{\lambda t} \in \sigma_\gamma(C^{-1}S(t))$ and $B(\lambda, t)$ is right invertible such that $\lim_{t\downarrow 0}\sup_{n\in\mathbb{N}} ||(S(t) - e^{\lambda t})x_n|| = 0$, where $(x_n)_n \subseteq X$, $||x_n|| = 1, n \in \mathbb{N}$, then there exists $k \in \mathbb{Z}$ such that $\lambda_k = \lambda + \frac{2\pi ik}{t} \in \sigma_\gamma(G)$.*

Proof. If $\lambda \in \sigma_\gamma(G)$, then by Theorem 2.6 we have that $e^{\lambda t} \in \sigma_\gamma(C^{-1}S(t))$.

Now, let $t_0 > 0$ be fixed and suppose that $\lambda \in \{\mathbb{C} \setminus \lambda + 2\pi ikt^{-1}, k \in \mathbb{Z}\}$ is such that $(\lambda - G)$ is semiregular. We show that $(e^{\lambda t_0} - C^{-1}S(t_0)) = e^{\lambda t_0} - T(t_0)$ is semiregular. We consider the closed $(S(t))_{t\geq 0}$-invariant subspace $M =$

$R^\infty(e^{\lambda t_0} - C^{-1}S(t_0)) = R^\infty(e^{\lambda t_0} - T(t_0))$ of X and quotient $C-$ regularized semigroup, $(\widehat{T}(t))_{t\geq 0}$ defined on X/M by: $\widehat{T}(t)\widehat{x} = \widehat{T(t)x}$, for $\widehat{x} \in X/M$, with generator $\widehat{G}$ defined by:

$$D(\widehat{G}) = \{\widehat{x}, x \in D(G)\} \quad and \quad G\widehat{x} = \widehat{Gx}, \quad for \quad all \quad \widehat{x} \in D(G).$$

From Lemma 2.5 it follows that the operator $(\lambda - \widehat{G})$ is bounded below for all $\lambda \in \{\mathbb{C} \setminus \lambda + 2\pi ikt^{-1}, k \in \mathbb{Z}\}$. Thus, $\lambda \notin \sigma_{ap}(\widehat{G})$. By virtue of Theorem 2.9, we get $e^{\lambda t_0} \notin \sigma_{ap}(\widehat{C^{-1}S}(t_0)) = \sigma_{ap}(\widehat{T}(t_0))$ in consequence, the operator $(e^{\lambda t_0} - \widehat{T}(t_0))$ is bounded below. If $(e^{\lambda t_0} - T(t_0))x = 0$ then $(e^{\lambda t_0} - \widehat{T}(t_0))(x + M) = 0$ and the injectivity of $(e^{\lambda t_0} - \widehat{T}(t_0))$ implies $x \in M$. Thus $N(e^{\lambda t_0} - T(t_0)) \subset M$.
We found that

$$N(e^{\lambda t_0} - C^{-1}S(t_0)) = N(e^{\lambda t_0} - T(t_0)) \subset R^\infty(e^{\lambda t_0} - T(t_0)) = R^\infty(e^{\lambda t_0} - C^{-1}S(t_0))$$

Now, let us show $R(e^{\lambda t_0} - C^{-1}S(t_0)) = R(e^{\lambda t_0} - T(t_0))$ is closed. To do this, let a sequence $(y_n)_n$ of elements of $R(e^{\lambda t_0} - T(t_0))$ and $y_n \to \ y, n \to \infty$. Then there exists a sequence $(v_n)_n \in X$ such that

$$(e^{\lambda t_0} - T(t_0))v_n = y_n = (\lambda - G)B(\lambda, t_0)v_n = (\lambda - G)u_n \to y.$$

As $B(\lambda, t_0)$ is right invertible and $(\lambda - G)$ is closed then

$$(\lambda - G)u_n \to (\lambda - G)u = (\lambda - G)B(\lambda, t_0)B^{-1}(\lambda, t_0)u = (\lambda - G)B(\lambda, t_0)h.$$

Then $y_n = (e^{\lambda t_0} - T(t_0))v_n \to \ y = (\lambda - G)B(\lambda, t_0)h = (e^{\lambda t_0} - T(t_0))h$ $\Rightarrow y \in R(e^{\lambda t_0} - T(t_0)) \Rightarrow R(e^{\lambda t_0} - C^{-1}S(t_0))$ is closed. Consequently the operator $(e^{\lambda t_0} - C^{-1}S(t_0))$ is semiregular. □

The next theorem gives, under suitable assumptions, necessary and sufficient conditions for the generator of a C-regularized semigroup to be semiregular.

Theorem 2.11. *Let $(S(t))_{t\geq 0}$ be a C-regularized semigroup with generator G. If $(S(t))_{t\geq 0}$ satisfies $\lim_{t\to\infty} \frac{1}{t^n}\|S(t)\| = 0$, then the following assertions are equivalent:*

1. *G is semiregular,*

2. *G is invertible .*

Proof. (1) $\Rightarrow$ (2) :

Since G is semiregular, then $N(G) \subseteq R^\infty(G)$ and $R(G)$ is closed. Let $y \in N(G)$, then there exists $x \in D(G^n))$ such that $y = G^n x$. We integrate by parts in the following formula:

$$S(t)x = Cx + \int_0^t S(s)Gxds,$$

We obtain that

$$S(t)x = Cx + tCG + \frac{t^2}{2!}CG^2 + \int_0^t \frac{(t-s)^2}{2!} S(s)G^3 x ds.$$

We repeat this operation for n times, we obtain that

$$S(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!} CG^k x + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} S(s)G^n x ds.$$

Hence,

$$\begin{aligned} S(t)x &= \sum_{k=0}^{n-1} \frac{t^k}{k!} CG^k x + Cy \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} ds \\ &= \sum_{k=0}^{n-1} \frac{t^k}{k!} CG^k x + \frac{t^n}{n!} Cy. \end{aligned}$$

As $\lim_{t\to\infty} \frac{1}{t^n} \|S(t)\| = 0$, then $y = 0$, this implies that

$$N(G) = \{0\}.$$

On the other hand, let $(S(t)^*)_{t\geq 0}$ with generator G^* the adjoint semigroup of $(S(t))_{t\geq 0}$. Since G is semiregular, then G^* is also semiregular, see [10, Proposition 1.6]. Using the following formula

$$S(t)^* x' - C^* x' = weak^* \int_0^t S(s)^* G^* x' ds, \quad \forall x' \in D(G^*), \quad \forall t \geq 0$$

which is proved in Proposition 2.4 and by the same argument as above, we get that $N(G^*) = \{0\}$. This is equivalent to the fact that

$$\overline{R(G)} = X.$$

Since $R(G)$ is closed, then $R(G) = X$. From this it follows that G is surjective and hence it is invertible. Finally, G is invertible.

□

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