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Spectral analysis of special perturbations of diagonal operators on non-Archimedean Banach spaces

Aziz Blali[1], Abdelkhalek El amrani[23] and Mohamed Amine Taybi[4]

Abstract. In this paper we are concerned with the spectrum of the operator $T = D + T_\mu$ where D is a diagonal operator and $T_\mu = \sum_{i=1}^{\infty} \mu_i P_i$ is a compact and self-adjoint operator in the non-Archimedean Banach space c_0, where $\mu = (\mu_i)_{i\in\mathbb{N}} \in c_0$ and for each $i \geq 1$, $P_i = \frac{<.,y_i>}{<y_i,y_i>} y_i$ is the normal projection defined by $(y_i)_{i\in\mathbb{N}} \in c_0$. Using Fredholm theory in the non-Archimedean setting and the concept of essential spectrum for linear operator, we compute the spectrum of T.

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1. Introduction

Non-Archimedean analysis is a well-developed branch of mathematics comparable to its classical counterpart, dealing over $\mathbb{R}$ and $\mathbb{C}$, see for example the monographs [7], [3] and [11]. The previous references includes some basic information on non-archimedean Banach spaces and operator theory and a rather complete theory of compact operators, see [9]. Moreover, a characterization of compact and self-adjoint operators on free Banach spaces is given in [3].

The problem of perturbation of p-adic linear operator has been long studied through several steps. A first approach was carried out by Serre in [9], where he dealt with compact perturbation of identity on Banach space having an orthogonal base. A step further was taking by Gruson [5] for more general class of Banach spaces, always working on perturbation of the identity. A complete study of perturbation of the identity was finally done by Schikhof in [8].

[1]Ecole Normale Superieure, Department of Mathematics, Sidi Mohamed Ben Abdellah University, B. P. 5206 Bensouda-Fès, Morocco, e-mail: aziz.blali@usmba.ac.ma, ORCID iD: orcid.org/0000-0001-5788-2399

[2]Department of mathematics and computer science, Sidi Mohamed Ben Abdellah University , Faculty of Sciences Dhar El Mahraz, Atlas Fès, Morocco.
e-mail: abdelkhalek.elamrani@usmba.ac.ma, ORCID iD: orcid.org/0000-0003-4841-8038

[3]Corresponding author

[4]Department of mathematics and computer science, Sidi Mohamed Ben Abdellah University , Faculty of Sciences Dhar El Mahraz, Atlas Fès, Morocco.
e-mail: mohamedamine.ettayb@usmba.ac.ma, ORCID iD: orcid.org/0009-0004-6553-828X

Let $\mathbb{K}$ denote a non trivial field which is complete with respect to a non archimedean valuation denoted $|.|$ and its residue class fields is formally real, i.e., for any finite subset $\{a_1, ..., a_n\}$ of $\mathbb{K}$, $\sum_{i=1}^{n} a_i = 0$ implies that each $a_i = 0$, see [6]. For a given sequence $(\lambda_j)_{j\in\mathbb{N}}$ with $\lambda_j \in \mathbb{K}$ for all $j \in \mathbb{N}$, we set $\Lambda = \{\lambda_j \in \mathbb{K} : j \in \mathbb{N}\}$. For each λ in Λ, $I_\lambda = \{j \in \mathbb{N} : \lambda_j = \lambda\}$. Further, $r_\lambda = \textit{cardinality of } I_\lambda$. Moreover $\Lambda^* = \{\lambda \in \Lambda : r_\lambda < \infty\}$. The set $\overline{\Lambda}$ is the closure of Λ in $\mathbb{K}$ and $\Lambda' = \{\lambda \in \Lambda : \lambda \textit{ is an accumulation point of } \Lambda\}$. Then the essential spectrum $\sigma_e(D)$ of D is characterized by T. Diagana in [4] and is given by

$$\sigma_e(D) = (\overline{\Lambda} \backslash \Lambda^*) \cup (\Lambda^* \cap \Lambda').$$

In this paper, we introduce a spectral analysis for compact and self-adjoint perturbation of diagonal operator in non-Archimedean Banach space of countable type. Namely we study the spectral analysis for operator T of the form:

$$T = D + T_\mu,$$

where $D = \sum_{i\in\mathbb{N}} a_i < ., e_i > e_i$, $(a_i)_{i\in\mathbb{N}} \in c_0$, is a bounded diagonal operator and $T_\mu = \sum_{i=1}^{\infty} \mu_i \frac{< ., y_i >}{< y_i, y_i >} y_i$ is compact and self-adjoint operator. Under some suitable assumptions, we will show that the spectrum $\sigma(T)$ of the bounded linear operator T is given by

$$\sigma(T) = \sigma_e(D) \cup \sigma_p(T),$$

where $\sigma_e(D)$ is the essential spectrum of D and $\sigma_p(T)$ is the point spectrum of T, that is the set of eigenvalues of T given by $\sigma_p(T) = \{a_n + \mu_n : n \in \mathbb{N}\}$.

2. Preliminaries

Define the space c_0 as the collection of all $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $\lambda_i \in \mathbb{K}$ for all $i \in \mathbb{N}$ such that λ_i tends to 0 in $\mathbb{K}$ as $i \to \infty$. Namely, c_0 is given by

$$c_0 = \{\lambda = (\lambda_n)_n \subset \mathbb{K} \ : \ \lim_n \lambda_n = 0\}.$$

It is known that the space c_0 equipped with the norm defined by for each $\lambda = (\lambda_i)_{i\in\mathbb{N}} \in c_0$

$$\|\lambda\|_\infty = \sup_{i\in\mathbb{N}} |\lambda_i|$$

is a non-Archimedean Banach space see [7]. The bilinear form $\langle ., . \rangle : c_0 \times c_0 \to \mathbb{K}$ defined by $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$ with $x = (x_i)$, $y = (y_i) \in c_0$, is an inner product in the non-archimedean sense. Since the residue class field of $\mathbb{K}$ is formally real, then $\|x\|_\infty^2 = \langle x, x \rangle$. The non archimedean Banach space c_0 has a special base denoted by $(e_i)_{i\in\mathbb{N}} = (\delta_{ij})_{i\in\mathbb{N}}$ where δ_{ij} is the usual Kronecker symbol.

Recall that a topological space is called separable if it has a countable dense subset. Now let E be a non trivial normed space over $\mathbb{K}$, $E \neq \{0\}$ and suppose that E is separable, then its one-dimensional subspaces are homeomorphic to $\mathbb{K}$, so $\mathbb{K}$ must be separable as well. Thus, for normed space the concept of separability is of no use if $\mathbb{K}$ is not separable, however linearizing the notion of separability we obtain a generalization useful for every scalar field $\mathbb{K}$. A normed space E is of countable type if it contains a countable set whose linear hull is dense in E. Clearly the span of unit vectors $e_1 = (1, 0, ...), e_2 = (0, 1, 0, ...), ...$ is dense in c_0, which is a Banach space of countable type. Each normed space is linearly heomeorphic to a subspace of c_0. Each infinite-dimensional Banach space of countable type is linearly heomeorphic to c_0 see [7]. This result shows that, up to linear homeomorphisms, there exists, for given $\mathbb{K}$, only one infinite-dimensional Banach space of countable type viz c_0.

A mapping $T : c_0 \to c_0$ is said to be a bounded linear operator on c_0 when it is linear and bounded. That is, there exists $C > 0$ such that

$$\|Tu\|_\infty \leq C\|u\|_\infty$$

for all $u \in c_0$.

$\mathcal{B}(c_0)$ denotes the collection of all bounded linear operators on c_0, $\mathcal{B}(c_0)$ is a Banach space with the norm $\|T\| = \sup_{u \neq 0} \dfrac{\|Tu\|_\infty}{\|u\|_\infty}$.

For all $T \in \mathcal{B}(c_0)$, its kernel and range are, respectively, defined by $N(T) = \{u \in c_0 : Tu = 0\}$ and $R(T) = \{Tu : u \in c_0\}$.

A linear operator $T : c_0 \to c_0$ is said to be a compact operator if $T(B_{c_0})$ is compactoid, where $B_{c_0} = \{x \in c_0 : \|x\|_\infty \leq 1\}$ is the unit ball of c_0. It was proved in [11], that T is compact if and only if, for each $\epsilon > 0$, there exist a linear operator of finite dimensional range S in $\mathcal{B}(c_0)$ such that $\|T - S\| < \epsilon$.

An operator $T \in \mathcal{B}(c_0)$ is said to be a Fredholm operator if it satisfies the following conditions:
1. $\eta(T) = \dim N(T)$ is finite;
2. $R(T)$ is closed;
3. $\delta(T) = \dim(c_0/R(T))$ is finite.

The collection of all Fredholm linear operators on c_0 will be denoted by $\Phi(c_0)$. If $T \in \Phi(c_0)$, then we define its index by setting $\chi(T) = \eta(T) - \delta(T)$. An example of a Fredholm operator is an invertible bounded linear operator, in particular, the identity operator $I : c_0 \to c_0, I(x) = x$ is a Fredholm operator with index $\chi(I) = 0$ as $\delta(I) = \eta(I) = 0$.

The adjoint T^* of $T \in B(c_0)$, if it exists, is defined by $< Tu, v >=< u, T^*v >$ for all $u, v \in c_0$. In contrast with the classical case, the adjoint of an operator may or may not exist. Note that if it exists, the adjoint T^* of an operator T is unique and has the same norm as A, and hence, lies in $B(c_0)$ as well. Since c_0 is not orthomodular, there exist operator in $\mathcal{B}(c_0)$ which do not admit an adjoint; for example the linear operator $T : c_0 \to c_0$ defined by $T(x) = (\sum_{i=1}^{\infty} x_i)e_1$, $x = (x_i)_{\in\mathbb{N}} \in c_0$, does not admit an adjoint. We will denote by $A_0 = \{T \in \mathcal{B}(c_0) : \lim_{i\to\infty} < Te_i, y >= 0, \ for\ all\ y \in c_0\}$ the collection of all element of $B(c_0)$ which admit an adjoint.

Set

$$A_1 = \{T \in \mathcal{B}(c_0) : \lim_{n\to\infty} Te_n = 0\},$$

and note that $A_1 \subsetneq A_0$, because $| < Te_n, y > | < \|Te_n\|_\infty \|y\|_\infty$, for all $n \in \mathbb{N}$ and $y \in c_0$ and I_d doesn't in A_1. We know that each $T \in \mathcal{B}(c_0)$ can be represented by:

$$T = \sum_{i,j=1}^{\infty} a_{ij} e'_j \otimes e_i,$$

where $\lim_{i\to\infty} a_{ij} = 0$ for each $j \in \mathbb{N}$. Also,

$$\begin{aligned}\|T\| &= \sup\{\|T(e_i)\|_\infty : i \in \mathbb{N}\}\\ &= \sup\{| < T(e_i), e_j > | : i, j \in \mathbb{N}\}.\end{aligned}$$

And T is compact if and only if : $\lim_{j\to\infty} \sup\{|a_{ij}| : i \in \mathbb{N}\} = 0$.

Now, note that for all $n \in \mathbb{N}$,

$$\begin{aligned}\|Te_n\|_\infty &= \|(\sum_{i,j=1}^{\infty} a_{ij} e_j^{'} \otimes e_i)(e_n)\|_\infty\\ &= \|\sum_{i,j=1}^{\infty} a_{ij} e_j^{'}(e_n) e_i\|_\infty\\ &= \|\sum_{i=1}^{\infty} a_{in} e_i\|_\infty\\ &= \sup\{|a_{in}| : i \in \mathbb{N}\}.\end{aligned}$$

Thus

$$T \in A_1 \Longleftrightarrow T \in A_0 \ \ and\, T\ is\ compact.$$

We will call a normal projection any projection $P : c_0 \to c_0$ such that $< x, y >= 0$ for each pair $(x, y) \in N(P) \times R(P)$. An example of a normal projection is $P(.) = \frac{<.,y>}{<y,y>} y$, for a fixed $y \in c_0 \backslash \{0\}$.

Let us take a fixed orthonormal sequence $(y_i)_{i\in\mathbb{N}} \in c_0$ that is, $< y_i, y_j >= 0$, for all i, j; $i \neq j$ and $\|y_i\|_\infty = 1$.

The next theorem involves normal projections with compact and self-adjoint operators. The proof can be found in [1].

Theorem 2.1. *If the linear operator $T : c_0 \to c_0$ is compact and self-adjoint, then there exist an element $\lambda = (\lambda_i)_{i\in\mathbb{N}} \in c_0$ such that:*

$$T = \sum_{i=1}^{\infty} \lambda_i P_i.$$

Where for all $i \in \mathbb{N}$, $P_i = \frac{<.,y_i>}{<y_i,y_i>} y_i$ is the normal projection defined by $(y_i)_{i\in\mathbb{N}} \in c_0$. Moreover $\|T\| = \|\lambda\|_\infty$.

3. Main Results

From now we will consider a fixed orthonormal sequence $Y = (y_i)_{i\in\mathbb{N}} \in c_0$. We will denote by C_Y the collection of all compact operator T_μ, $\mu \in c_0$, where $T_\mu = \sum_{i=1}^{\infty} \mu_i P_i$.

The adjoint T_μ^* of T_μ is itself and $\lim_{n\to\infty} T_\mu(e_n) = 0$. On the other hand, since Y is orthonormal for all $i \in \mathbb{N}$, $T_\mu(y_i) = \mu_i y_i$, then μ_i is eigenvalues of T_μ. Let us denote by $\sigma_p(T_\mu)$ the set of eigenvalues of T_μ.

Now, the collection C_Y is a linear space with the operations

$$T_\lambda + T_\mu = T_{\lambda+\mu}; \quad \alpha T_\lambda = T_{\alpha\lambda}.$$

On the other hand, since c_0 is a commutative algebra with the operation $\lambda.\mu = (\lambda_i.\mu_i)$, we have

$$T_\lambda \circ T_\mu = T_{\lambda.\mu} = T_\mu \circ T_\lambda.$$

In order to simplify the notation, $T_\lambda \circ T_\mu$ will be denoted by $T_\lambda.T_\mu$.

With the operations described above, C_Y becomes a commutative algebra without unit. Even more, by the fact that $T_\lambda = T_\mu$ implies $\lambda = \mu$, the map

$$\Gamma : c_0 \to C_Y; \quad \lambda \mapsto \Gamma(\lambda) = T_\lambda$$

is an isometric isomorphism of algebras.

The resolvent of a bounded linear operator $T : c_0 \to c_0$ is defined by $\rho(T) = \{\lambda \in \mathbb{K} : \lambda I - T \ \textit{is a bijection and } (\lambda I - T)^{-1} \in \mathcal{B}(c_0)\}$. The spectrum $\sigma(T)$ of T is then defined by $\sigma(T) = \mathbb{K} \setminus \rho(T)$. A scalar $\lambda \in \mathbb{K}$ is called an eigenvalue of $T \in \mathcal{B}(c_0)$, whenever there exists a nonzero $u \in c_0$ (called eigenvector associated with λ) such that $Tu = \lambda u$.

Clearly, eigenvalues of T consist of all $\lambda \in \mathbb{K}$, for with $\lambda I - T$ is not one-to-one, that is $N(\lambda I - T) \neq \{0\}$. The collection of all eigenvalues of T is denoted by $\sigma_p(T)$ (called punctual Spectrum) and is defined by

$$\sigma_p(T) = \{\lambda \in \sigma(T) : N(\lambda I - T) \neq \{0\}\}.$$

Example 3.1. Consider the diagonal operator $D : c_0 \to c_0$ defined by

$$Du = \sum_{j=0}^{\infty} \lambda_j u_j e_j \quad \text{for all} \quad u = (u_j)_{j\in\mathbb{N}} \in c_0$$

where $\sup_{j\in\mathbb{N}} |\lambda_j| < +\infty$. Then $\sigma(D) = \overline{\{\lambda_k : k \in \mathbb{N}\}}$ the closure of $\{\lambda_k : k \in \mathbb{N}\}$, i.e:

$$\sigma(D) = \{\lambda \in \mathbb{K} : \inf_{j\in\mathbb{N}} |\lambda - \lambda_j| = 0\}.$$

Definition 3.2. Define the essential spectrum $\sigma_e(T)$ of a bounded linear operator $T : c_0 \to c_0$ as follows

$$\sigma_e(T) = \{\lambda \in \mathbb{K} : \lambda I - T \ \textit{is not Fredholm operator of index}\ 0\}.$$

Clearly, if $\lambda \in \mathbb{K}$ does not belong to neither $\sigma_p(T)$ nor $\sigma_e(T)$, then $(\lambda I - T)$ must be injective. $N(\lambda I - T) = \{0\}$ and $R(\lambda I - T)$ is closed with $0 = \dim N(\lambda I - T) = \dim(c_0/R(\lambda I - T))$. Consequently $(\lambda I - T)$ must be bijective (injective and surjective) which yields that $\lambda \in \rho(T)$. In view of the previous fact, we have

$$\sigma(T) = \sigma_p(T) \cup \sigma_e(T).$$

Theorem 3.3. 1. *Let $T_\lambda \in C_Y$ be a compact and self-adjoint operator and let $\mu \in \mathbb{K}$, $\mu \neq 0$ be an eigenvalue of T_λ. Then $\mu = \lambda_i$ for some i.*
2. *If $T \in \mathcal{B}(c_0)$, then for all $T_\mu \in C_Y$, we have*

$$\sigma_e(T + T_\mu) = \sigma_e(T).$$

3. *If $T = D + T_\mu$, where $T_\mu \in C_Y$ and D is a diagonal operator, then its spectrum $\sigma(T)$ is given by $\sigma(T) = \sigma_e(D) \cup \sigma_p(T)$.*
4. *The punctual Spectrum of $T = D + T_\mu$, is given by:*

$$\sigma_p(T) = \{\mu_n + a_n : n \in \mathbb{N}\}.$$

We use the following lemma to show the second assertion of the theorem.

Lemma 3.4. *If $T \in \Phi(c_0)$ and $T_\mu \in \mathcal{C}_Y$, then $T + T_\mu \in \Phi(c_0)$, with $\chi(T + T_\mu) = \chi(T)$.*

Proof. see [10] and [2]. □

Proof of Theorem 3.3
1. Let $x \in c_0$ an eigenvector corresponding to μ. Then

$$T_\lambda x = \sum_{i=1}^{\infty} \lambda_i \frac{< x, y_i >}{< y_i, y_i >} y_i$$

$$T_\lambda(T_\lambda x) = T_\lambda(\mu x) = \mu T_\lambda x.$$

It follows from the last equation that

$$T_\lambda\Big(\sum_{i=1}^{\infty} \lambda_i \frac{< x, y_i >}{< y_i, y_i >} y_i\Big) = \mu\Big(\sum_{i=1}^{\infty} \lambda_i \frac{< x, y_i >}{< y_i, y_i >} y_i\Big).$$

Thus

$$\sum_{i=1}^{\infty} \lambda_i^2 \frac{< x, y_i >}{< y_i, y_i >} y_i = \sum_{i=1}^{\infty} \lambda_i \mu \frac{< x, y_i >}{< y_i, y_i >} y_i.$$

Since $T_\lambda x = \mu x \neq 0$, it follows that $< x, y_i > \neq 0$ for some i. Hence

$$\bigcup_{i=1}^{\infty} \{\lambda_i\} \neq \emptyset.$$

Thus

$$\sum_{i=1}^{\infty} \lambda_i(\lambda_i - \mu) \frac{< x, y_i >}{< y_i, y_i >} y_i = 0 \;\; for \;\; < x, y_i > \neq 0.$$

The normality of the sequence $\{y_i\}$ implies that

$$for\ all\ i \in \mathbb{N},\ \lambda_i(\lambda_i - \mu)\frac{< x, y_i >}{< y_i, y_i >} = 0;\ \ < x, y_i > \neq 0.$$

Since the eigenvectors corresponding to different eigenvalues are normal and since $x \neq 0$, it follows that $\lambda_i \neq 0$ for some $i \in \mathbb{N}$, then $\lambda_i - \mu = 0$ for $i \in \mathbb{N}$.

Hence $\lambda_i = \mu$ for some i.

2. If λ does not belong to $\sigma_e(T)$, then $\lambda I - T$ belongs to $\Phi(c_0)$ with $\chi(\lambda I - A) = 0$, therefore $\lambda I - T - T_\mu$ belongs to $\Phi(c_0)$ with $\chi(\lambda I - (T + T_\mu)) = 0$ for all $T_\mu \in C_Y$. Then λ does not belong to $\sigma_e(T + T_\mu)$.

3. We have $\sigma(T) = \sigma_e(T) \cup \sigma_p(T)$. In view of the second assertion of theorem, we have $\sigma_e(T) = \sigma_e(D+T_\mu) = \sigma_e(D)$. So, it follows that $\sigma(T) = \sigma_e(D) \cup \sigma_p(T)$.

4. Let $y = (y_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in c_0 then:

$$Ty = (D + T_\mu)(y);$$

since y is orthonormal, we have: $Dy = a_n < y_n, e_n > e_n$ and $T_\mu y = \mu_n y_n$. Then:

$$Ty = Ty_n = a_n < y_n, e_n > e_n + \mu_n y_n. \qquad (*)$$

Taking the inner product of equality $(*)$ with the canonical basis of c_0 we obtain:

$$< Ty_n, e_n >= a_n < y_n, e_n > +\mu_n < y_n, e_n >= (\mu_n + a_n) < y_n, e_n > .$$

Then $< Ty_n - (\mu_n + a_n)y_n, e_n >= 0$ it follows that $Ty_n - (\mu_n + a_n)y_n = 0$, if not, there exists a nonzero $(\alpha_i)_{i\in\mathbb{N}} \subseteq \mathbb{K}$ such that $Ty_n - (\mu_n + a_n)y_n = \sum_{i\in\mathbb{N}} \alpha_i e_i$, then $< \sum_{i\in\mathbb{N}} \alpha_i e_i, e_n >= \alpha_n$, absurd because $\alpha_n \neq 0$. Consequently $\mu_n + a_n$ is an eigenvalue of T.

Corollary 3.5. *For every $T_\mu \in C_Y$, we have $\sigma_e(D + T_\mu) = \sigma_e(D)$, where D is a bounded diagonal operator in c_0.*

Corollary 3.6. *The spectrum of $T = D + T_\mu$ is*

$$\sigma(T) = \{\mu_n + a_n : n \in \mathbb{N}\} \cup (\overline{\Lambda}\backslash\Lambda^*) \cup (\Lambda^* \cap \Lambda').$$

Proposition 3.7. *Let $T = D + T_\mu$ where D is diagonal operator and T_μ is compact and self-adjoint, then $\sigma_p(T) \cap \sigma_p(D) = \emptyset$.*

Proof. Suppose $\lambda \in \sigma_p(T)$, thus there exists $u \neq 0$, $u \in c_0$ such that $Tu = \lambda u$. Equivalently,

$$(\lambda I - D)u = T_\mu u = \sum_{i=1}^{\infty} \mu_i \frac{< u, y_i >}{< y_i, y_i >} y_i.$$

Clearly, all expressions $< u, y_i > \neq 0$ for $i \in \mathbb{N}$. If not, we will get $(\lambda I - D)u = 0$ with $u \neq 0$. That is, $\lambda \in \sigma_p(D)$ and hence there exists $j_0 \in \mathbb{N}$ such that $\lambda = \lambda_{j_0}$,

$u = ae_{j_0}$, $a \in \mathbb{K}\backslash\{0\}$. Then for $i = j_0$, we have: $0 =< u, y_{j_0} >=< ae_{j_0}, y_{j_0} >= a < e_{j_0}, y_{j_0} >\neq 0$. Absurd, consequently λ doesn't belong to $\sigma_p(D)$.

Conversely, suppose that $\lambda \in \sigma_p(D)$. Thus there exists $u \neq 0$, $u \in c_0$ such that $Du = \lambda u$, hence there exists $i_0 \in \mathbb{N}$ and $\alpha_{i_0} \in \mathbb{K}\backslash\{0\}$ such that $\lambda = \lambda_{i_0}$ and $u = \alpha_{i_0} e_{i_0}$. On the other hand, we have

$$Tu = Du + T_\mu u = \lambda u + \mu_{i_0}\alpha_{i_0}\frac{< y_{i_0}, e_{i_0} >}{< y_{i_0}, y_{i_0} >} y_{i_0}.$$

Then $\lambda u - Tu = -\mu_{i_0}\alpha_{i_0}\frac{<y_{i_0},e_{i_0}>}{<y_{i_0},y_{i_0}>} y_{i_0} \neq 0$, if not we will have $\mu_{i_0} y_{i_0} = 0 = T_\mu(y_{i_0})$, absurd. Then λ doesn't belong to $\sigma_p(T)$. □

Remark 3.8. T and T_μ have the same eigenvectors corresponding to $\mu_i + a_i$ and μ_i, respectively.

References

[1] José Aguayo, Miguel Nova, and Khodr Shamseddine. Characterization of compact and self-adjoint operators on free Banach spaces of countable type over the complex Levi-Civita field. *J. Math. Phys.*, 54(2):023503, 19, 2013. doi:10.1063/1.4789541.

[2] J. Araujo, C. Perez-Garcia, and S. Vega. Preservation of the index of p-adic linear operators under compact perturbations. *Compositio Math.*, 118(3):291–303, 1999. doi:10.1023/A:1001561127279.

[3] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry. doi:10.1007/978-3-642-52229-1.

[4] T. Diagana, R. Kerby, TeyLama H. Miabey, and F. Ramaroson. Spectral analysis for finite rank perturbations of diagonal operators in non-archimedean Hilbert space. *p-Adic Numbers Ultrametric Anal. Appl.*, 6(3):171–187, 2014. doi:10.1134/S2070046614030017.

[5] Laurent Gruson. Théorie de Fredholm p-adique. *Bull. Soc. Math. France*, 94:67–95, 1966. URL: http://www.numdam.org/item?id=BSMF_1966__94__67_0.

[6] Lawrence Narici and Edward Beckenstein. A non-Archimedean inner product. In *Ultrametric functional analysis*, volume 384 of *Contemp. Math.*, pages 187–202. Amer. Math. Soc., Providence, RI, 2005. doi:10.1090/conm/384/07136.

[7] C. Perez-Garcia and W. H. Schikhof. *Locally convex spaces over non-Archimedean valued fields*, volume 119 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. doi:10.1017/CBO9780511729959.

[8] W. H. Schikhof. *On p-adic Compact Operators*. Report 8911. Nijmegen, Katholieke Universiteit, Department of Mathematics, 1989. URL: https://hdl.handle.net/2066/57062.

[9] Jean-Pierre Serre. Endomorphismes complètement continus des espaces de Banach p-adiques. *Inst. Hautes Études Sci. Publ. Math.*, (12):69–85, 1962. URL: http://www.numdam.org/item?id=PMIHES_1962__12__69_0.

[10] W. Śliwa. On Fredholm operators between non-Archimedean Fréchet spaces. *Compositio Math.*, 139(1):113–118, 2003. doi:10.1023/B:COMP.0000005075.84696.f8.

[11] A. C. M. van Rooij. *Non-Archimedean functional analysis*, volume 51 of *Monographs and Textbooks in Pure and Applied Math.* Marcel Dekker, Inc., New York, 1978.

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