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Almost quasi-Yamabe solitons on $(LCS)_n$-manifolds

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Abstract.

The article attempts to study hyper generalized pseudo symmetric $(LCS)_n$ manifolds admitting almost quasi-Yamabe solitons (g, V, p, λ) and almost quasi-Yamabe gradient solitons (g, f, p, λ). Also, we give an example of hyper generalized pseudo symmetric $(LCS)_3$-manifold with (g, V, p, λ) and (g, f, p, λ).

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1. Introduction

A non Einstein pseudo Riemannian manifold (M^n, g) is said to be hyper generalized pseudo symmetric manifold (briefly, $(HGPS)_n$) (see [11]) if the Riemannian curvature tensor R satisfies the relation

$$\begin{aligned} &(\nabla_X R)(Y, U, V, Z) \\ =\ & 2E(X)R(Y, U, V, Z) + 2F(X)(g \wedge S)(Y, U, V, Z) \\ & +E(Y)R(X, U, V, Z) + F(Y)(g \wedge S)(X, U, V, Z) \\ & +E(U)R(Y, X, V, Z) + F(U)(g \wedge S)(Y, X, V, Z) \\ & +E(V)R(Y, U, X, Z) + F(V)(g \wedge S)(Y, U, X, Z) \\ & +E(Z)R(Y, U, V, X) + F(Z)(g \wedge S)(Y, U, V, X), \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} &(g \wedge S)(Y, U, V, Z) \\ =\ & g(Y, Z)S(U, V) + g(U, V)S(Y, Z) \\ & -g(Y, V)S(U, Z) - g(U, Z)S(Y, V), \end{aligned}$$

and E, F are non-zero 1-forms. We note that for $F \equiv 0$, a hyper generalized pseudo symmetric space reduces to Chaki's pseudo symmetric space ([13]).

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It is obvious that an Einstein hyper generalized weakly (pseudo) symmetric space ([3], [8], [18]) reduces to a generalized weakly (pseudo) symmetric space ([10]). Also, we note that such related spaces are studied in ([5], [12] [15]). The study of the Yamabe flow appeared in the work of Hamilton ([16]) as a tool to construct Yamabe metrics on compact Riemannian manifolds. A Yamabe soliton is defined on a Riemannian or pseudo Riemannian manifold (M^n, g) by a vector field V satisfying the equation ([9]).

$$\pounds_V g = 2(r-\lambda)g, \tag{1.2}$$

where $\pounds_V$ denotes the Lie derivative along V, r denotes the scalar curvature and λ is a real number. Moreover, when V is a Killing vector field, the Yamabe soliton is called trivial. Chen ([14]) has investigated almost quasi-Yamabe solitons (g, V, p, λ) on almost cosymplectic manifolds. According to Chen ([14]), a pseudo Riemannian metric is called an almost quasi-Yamabe soliton if there exist a C^∞ function λ, a vector field V and a positive constant p such that

$$\pounds_V g = 2(r-\lambda)g + \frac{2}{p}V^b \otimes V^b. \tag{1.3}$$

An almost quasi-Yamabe soliton is said to be expanding, steady or shrinking when λ is negative, zero and positive, respectively. It is to be mentioned that an almost quasi-Yamabe soliton (g, V, p, λ) reduces to an almost Yamabe soliton when $p \to \infty$. Furthermore, for $V = gradf$ such soliton is called almost quasi-Yamabe gradient soliton and denoted by (g, f, p, λ).
Our paper is structured as follows: Section 2 is concerned with some known results of $(LCS)_n$-manifolds which will be useful later. Section 3 deals with hyper generalized pseudo symmetric $(LCS)_n$-manifolds and such a manifold is η-Einstein, space of quasi-constant curvature and conformally flat. Next, we have observed that if a hyper generalized pseudo symmetric $(LCS)_n$-manifold admits closed almost quasi-Yamabe solitons, then the manifold is Einstein and the sectional curvature is $(\alpha^2 - \rho)$ or the potential vector field of the soliton is pointwise collinear with ξ. Moreover, almost quasi-Yamabe gradient solitons on hyper generalized pseudo symmetric $(LCS)_n$-manifold is discussed in Section 5. Lastly, we construct a non trivial example of Lorentzian concircular structure which is a hyper generalized pseudo symmetric space and admits an almost quasi-Yamabe soliton.

2. Preliminaries

Let (M^n, g) be a Lorentzian manifold admitting an unit time-like concircular vector field ξ, the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \tag{2.2}$$

$$(\nabla_X \eta)Y = \alpha[g(X,Y) + \eta(X)\eta(Y)], \ (\alpha \neq 0), \tag{2.3}$$

$$\nabla_X \alpha = (X\alpha) = \rho\eta(X), \tag{2.4}$$

$$\nabla_X \rho = \beta\eta(X), \tag{2.5}$$

where η is the dual 1-form of ξ, ∇ is the covariant differentiation operator with respect to g, and α, β, ρ are non-zero scalar function.
If we assume

$$\phi X = \frac{1}{\alpha}\nabla_X \xi = X + \eta(X)\xi, \tag{2.6}$$

then ϕ is a symmetric $(1,1)$-tensor field. A manifold with such structure (ϕ, ξ, η, g) and associated scalars α, β, ρ is known as an $(LCS)_n$-manifold [1]. In an $(LCS)_n$-manifold, the following relations hold [7]:

$$\phi \circ \xi = 0, \eta \circ \phi = 0, \tag{2.7}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.8}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.9}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.11}$$

$$\begin{aligned} & (\nabla_X R)(Y,Z)\xi \\ = \ & \alpha(\alpha^2 - \rho)[g(X,Z)Y - g(X,Y)Z] \\ & +(2\alpha\rho - \beta)\eta(X)[\eta(Z)Y - \eta(Y)Z] - \alpha R(Y,Z)X, \end{aligned} \tag{2.12}$$

$$\begin{aligned} & (\nabla_X R)(Y,Z,V,\xi) + \alpha R(Y,Z,V,X) \\ = \ & -\alpha(\alpha^2 - \rho)[g(X,Z)g(Y,V) - g(X,Y)g(Z,V)] \\ & -(2\alpha\rho - \beta)\eta(X)[\eta(Z)g(Y,V) - \eta(Y)g(Z,V)], \end{aligned} \tag{2.13}$$

$$\begin{aligned} & (\nabla_X S)(Y,\xi) \\ = \ & (n-1)[\alpha(\alpha^2 - \rho)g(X,Y) + (2\alpha\rho - \beta)\eta(X)\eta(Y)] - \alpha S(X,Y), \end{aligned} \tag{2.14}$$

for any vector fields X, Y, Z and V in M^n.
In [17] (also in [2], [4]) the authors claimed that the generalized Robertson–Walker (in brief, GRW) spacetime coincides with 4-dimensional Lorentzian concircular structure (in brief $(CS)_4$-spacetime [6], [19], [20]).

3. Hyper generalized pseudo symmetric $(LCS)_n$-manifolds

We first consider an $(LCS)_n$-manifold which is $(HGPS)_n$. Now putting $Z = \xi$ and using (1.1) in (2.14), we find

$$\begin{aligned}
&\{\alpha + E(\xi)\}R(X,U,Y,V) \\
&+[(\alpha^2-\rho)\{\alpha g(V,U) + D(U)\eta(V)\} - F(\xi)S(V,U)]g(X,Y) \\
&+[F(\xi)S(Y,U) - (\alpha^2-\rho)\{\alpha g(Y,U) + D(U)\eta(Y)\}]g(X,V) \\
&-(\alpha^2-\rho)\{D(V)\eta(Y) - D(Y)\eta(V)\}g(X,U) \\
&+[F(\xi)S(X,V) + (\alpha^2-\rho)D(V)\eta(X) \\
&+\{2(\alpha^2-\rho)D(X) + (\beta - 2\alpha\rho)\eta(X)\}\eta(V)]g(Y,U) \\
&-[F(\xi)S(X,Y) + (\alpha^2-\rho)D(Y)\eta(X) \\
&+\{2(\alpha^2-\rho)D(X) + (\beta - 2\alpha\rho)\eta(X)\}\eta(Y)]g(V,U) \\
&+F(U)\eta(V)S(X,Y) - F(U)\eta(Y)S(X,V) \\
&+\{F(Y)\eta(V) - F(V)\eta(Y)\}S(X,U) \\
&+\{F(V)\eta(X) + 2F(X)\eta(V)\}S(Y,U) \\
=\ &\{F(Y)\eta(X) + 2F(X)\eta(Y)\}S(V,U), \qquad (3.1)
\end{aligned}$$

where $D(X) = E(X) + (n-1)F(X)$.
Again putting $V = X = \xi$, we can easily obtain:

$$\begin{aligned}
&S(Y,U) \\
=\ &\frac{\beta - 2\alpha\rho - 4(\alpha^2-\rho)(E(\xi) + (n-1)F(\xi))}{4F(\xi)}g(Y,U) \\
&+\frac{\beta - 2\alpha\rho - 4(\alpha^2-\rho)(E(\xi) + 2(n-1)F(\xi))}{4F(\xi)}\eta(Y)\eta(U). \qquad (3.2)
\end{aligned}$$

Contracting the above equation (3.2), we have

$$E(\xi) = \frac{\beta - 2\alpha\rho}{4(\alpha^2-\rho)} - \frac{r + (n-2)(\alpha^2-\rho)}{(n-1)(\alpha^2-\rho)}F(\xi). \qquad (3.3)$$

Now putting this value of $E(\xi)$ in (3.2), we have

$$\begin{aligned}
&S(Y,U) \\
=\ &\frac{r - (n-1)(\alpha^2-\rho)}{(n-1)}g(Y,U) \\
&+\frac{r - n(n-1)(\alpha^2-\rho)}{(n-1)}\eta(Y)\eta(U). \qquad (3.4)
\end{aligned}$$

Now contracting (3.1) over V and X, we get

$$\begin{aligned}
&3\alpha^2E(\xi)g(Y,U)-3\rho E(\xi)g(Y,U)+E(\xi)S(Y,U)+3\alpha^2nF(\xi)g(Y,U)\\
&-3n\rho F(\xi)g(Y,U)+rF(\xi)g(Y,U)-3\alpha^2F(\xi)g(Y,U)+3\rho F(\xi)g(Y,U)\\
&+nF(\xi)S(Y,U)+F(\xi)S(Y,U)-2\alpha\rho\eta(e)^2g(Y,U)+\beta\eta(e)^2g(Y,U)\\
&+\alpha^3(-n)g(Y,U)+\alpha n\rho g(Y,U)+\alpha^3g(Y,U)-\alpha\rho g(Y,U)\\
&+\alpha S(Y,U)-\alpha^2nE(U)\eta(Y)+n\rho E(U)\eta(Y)-2\alpha^2E(U)\eta(Y)\\
&+2\rho E(U)\eta(Y)-\alpha^2n^2F(U)\eta(Y)+n^2\rho F(U)\eta(Y)\\
&+(n-1)\left(\alpha^2-\rho\right)F(U)\eta(Y)-\alpha^2nF(U)\eta(Y)+n\rho F(U)\eta(Y)\\
&-rF(U)\eta(Y)-3\eta(Y)F(S(U))+2\alpha^2F(U)\eta(Y)-2\rho F(U)\eta(Y)\\
&+2\alpha\rho\eta(Y)\eta(U)-\beta\eta(Y)\eta(U)=0.
\end{aligned}$$

Again, contracting above over Y and U, we get

$$E(\xi)=\frac{\begin{array}{c}-2F(\xi)\left(2\left(\alpha^2-\rho\right)+n\left((n-3)\left(\alpha^2-\rho\right)+r\right)\right)\\ +(n-1)\left(\beta+\alpha^3n-\alpha(n+2)\rho\right)-\alpha r\end{array}}{2(n-1)\left(\alpha^2-\rho\right)+r}. \tag{3.5}$$

Now comparing (3.5) and (3.3), we get

$$F(\xi)=-\frac{(n-1)\begin{pmatrix}2(n-1)\left(\alpha^2-\rho\right)\left(-2\alpha\rho+\beta+2n\left(\alpha^3-\alpha\rho\right)\right)\\ -r\left(4\alpha^3-6\alpha\rho+\beta\right)\end{pmatrix}}{4r\left(r-(n-1)n\left(\alpha^2-\rho\right)\right)}. \tag{3.6}$$

Again, from symmetry of R, we have

$$R(Y,V,U,Z)+R(V,Y,U,Z)=0. \tag{3.7}$$

Now putting the expression of R from (3.1) in (3.7) and taking contraction over V and U and then putting $Z=\xi$, we have

$$\begin{aligned}
&\frac{(3n+2)\left(\alpha^2-\rho\right)}{\alpha+E(\xi)}E(Y)+\frac{(3n+2)\left((n-2)(n-1)\left(\alpha^2-\rho\right)+r\right)}{(n-1)(\alpha+E(\xi))}F(Y)\\
&-\frac{(n-6)\left(\alpha^2-\rho\right)E(\xi)}{\alpha+E(\xi)}\eta(Y)+\frac{(n-1)(\beta-2\alpha\rho)}{\alpha+E(\xi)}\eta(Y)\\
&-\frac{(n-6)F(\xi)\left[(n-2)(n-1)\left(\alpha^2-\rho\right)+r\right]}{(n-1)(\alpha+E(\xi))}\eta(Y)=0.
\end{aligned}$$

Now using (3.3) in above we get

$$E(Y)=-\frac{4F(Y)\left((n-2)(n-1)\left(\alpha^2-\rho\right)+r\right)+(n-1)\eta(Y)(\beta-2\alpha\rho)}{4(n-1)\left(\alpha^2-\rho\right)}. \tag{3.8}$$

Finally, using (3.4), (3.6) and (3.8) in (3.1), we get the following:

$$
\begin{aligned}
& R(Y,V,U,Z) \\
= \quad & \frac{r-2(n-1)\left(\alpha^2-\rho\right)}{(n-2)(n-1)} G(Y,V,U,Z) \\
+ \quad & \frac{r-n(n-1)\left(\alpha^2-\rho\right)}{(n-2)(n-1)} H(Y,V,U,Z),
\end{aligned} \tag{3.9}
$$

where $H = g \wedge (\eta \otimes \eta)$.
Therefore we can state the following:

Theorem 3.1. *Every hyper generalized pseudo symmetric $(LCS)_n$-manifold is a space of quasi-constant curvature and η-Einstein.*

It is to be noted that a hyper generalized pseudo symmetric $(LCS)_n$-manifold reduces to Chaki's pseudo symmetric $(LCS)_n$-manifold ([13]) for $F \equiv 0$. This motivates us to state the following:

Corollary 3.2. *Every pseudo symmetric $(LCS)_n$-manifold is a space of quasi-constant curvature and η-Einstein.*

Next, in view of (3.4) and (3.9) we see that the Weyl conformal curvature tensor

$$
\begin{aligned}
C(X,Y)Z \ &= R(X,Y)Z \ - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\
-g(X,Z)QY &+ \frac{r}{(n-1)}\{g(Y,Z)X - g(X,Z)Y\}].
\end{aligned}
$$

vanishes. Thus we can say that

Theorem 3.3. *Every hyper generalized pseudo symmetric $(LCS)_n$-manifold is conformally flat.*

Corollary 3.4. *Every pseudo symmetric $(LCS)_n$-manifold is conformally flat.*

4. Almost quasi-Yamabe solitons on hyper generalized pseudo symmetric $(LCS)_n$-manifolds

Let the metric of the almost quasi-Yamabe soliton be closed on hyper generalized pseudo symmetric $(LCS)_n$-manifolds. Then the equation (1.3) yields

$$\nabla_X V = (r-\lambda)X + \frac{1}{p} g(X,V)V. \tag{4.1}$$

Taking the covariant derivative along the vector field Y of (4.1), we have

$$\nabla_Y \nabla_X V = (Y(r-\lambda))X + (r-\lambda)\nabla_Y X + \frac{1}{p}\nabla_Y g(X,V)V + \frac{1}{p} g(X,V)\nabla_Y V. \tag{4.2}$$

In view of (4.1), (4.2), we determine

$$\begin{aligned} R(X,Y)V &= (X(r-\lambda))Y - (Y(r-\lambda))X \\ &\quad + \frac{1}{p}(r-\lambda)[g(Y,V)X - g(X,V)Y]. \end{aligned} \tag{4.3}$$

Next, taking the inner product in (4.3) with ξ, we have

$$\begin{aligned} \eta(R(X,Y)V) &= (X(r-\lambda))\eta(Y) - (Y(r-\lambda))\eta(X) \\ &\quad + \frac{1}{p}(r-\lambda)[g(Y,V)\eta(X) - g(X,V)\eta(Y)]. \end{aligned} \tag{4.4}$$

Using (2.9) in (4.4) and then replacing X by ϕX and Y by ξ, we find

$$\phi\, grad(r-\lambda) - \{\frac{1}{p}(r-\lambda) - \big(\alpha^2 - \rho\big)\}\phi V = 0. \tag{4.5}$$

Again, contracting (4.3) and using (3.4), we obtain

$$\begin{aligned} &\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)}g(X,V) + \frac{r-n(n-1)(\alpha^2-\rho)}{(n-1)}\eta(X)\eta(V) \\ = \; & -(n-1)(X(r-\lambda)) + \frac{1}{p}(n-1)(r-\lambda)g(X,V). \end{aligned} \tag{4.6}$$

Next replacing X by ϕX in (4.6), we get

$$\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)}\phi V + (n-1)\phi\, grad(r-\lambda) - \frac{1}{p}(n-1)(r-\lambda)\phi V = 0. \tag{4.7}$$

Now combining (4.5) and (4.7), we obtain

$$[\frac{r}{(n-1)} - n(\alpha^2-\rho)]\phi V = 0. \tag{4.8}$$

From (4.8) two cases may arise: one is $\frac{r}{(n-1)} - n(\alpha^2-\rho) = 0$ and the other is $\phi V = 0$.

- Case-1, if $r = n(n-1)(\alpha^2-\rho)$, then (3.4) and (3.9) together impliy that the manifold is Einstein and the sectional curvature is $(\alpha^2-\rho)$.
- Case-2, if $\phi V = 0$, then from (2.6) we see that $V = -\eta(V)\xi$ i.e., the potential vector field of the almost quasi-Yamabe soliton is pointwise collinear with ξ.

Thus we can state that

Theorem 4.1. *Every hyper generalized pseudo symmetric $(LCS)_n$-manifold admitting closed almost quasi-Yamabe solitons reduces to an Einstein manifold and the sectional curvature is $(\alpha^2-\rho)$ or the potential vector field of the soliton is pointwise collinear with ξ.*

Corollary 4.2. *Every hyper generalized pseudo symmetric $(LCS)_n$-manifold admitting closed almost quasi-Yamabe solitons reduces to a generalized pseudo symmetric $(LCS)_n$-manifold.*

Corollary 4.3. *Every pseudo symmetric $(LCS)_n$-manifold admitting closed almost quasi-Yamabe solitons reduces to an Einstein manifold and the sectional curvature is $(\alpha^2-\rho)$ or the potential vector field of the soliton is pointwise collinear with ξ.*

5. Gradient almost quasi-Yamabe solitons on hyper generalized pseudo symmetric $(LCS)_n$-manifolds

In this section we take the potential vector field V as a gradient of a smooth function $f : M^n \to \mathbb{R}$. Then relation (1.3) becomes

$$\nabla_X \, gradf = (r-\lambda)X + \frac{1}{p} g(X, gradf)\, gradf \tag{5.1}$$

which yields

$$R(X,Y)\, gradf = (X(r-\lambda))Y - (Y(r-\lambda))X + \frac{1}{p}(r-\lambda)[(Yf)X - (Xf)Y]. \tag{5.2}$$

Taking the inner product in (5.2) with ξ and then from (2.9), we obtain

$$(X(r-\lambda))\eta(Y) - (Y(r-\lambda))\eta(X) + \{\frac{(r-\lambda)}{p} - (\alpha^2-\rho)\}[(Yf)\eta(X) - (Xf)\eta(Y)] = 0. \tag{5.3}$$

Now replacing Y by ξ and X by ϕX in (5.3), we obtain

$$[\frac{(r-\lambda)}{p} - (\alpha^2-\rho)]\phi \, gradf = \phi \, grad(r-\lambda). \tag{5.4}$$

Taking the contraction over X in (5.2), we find

$$S(Y, gradf) = -(n-1)(Y(r-\lambda)) + \frac{1}{p}(r-\lambda)(n-1)(Yf). \tag{5.5}$$

Combining (5.5) and (3.4), we get

$$\begin{aligned} &\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)}(Yf) + \frac{r-n(n-1)(\alpha^2-\rho)}{(n-1)}\eta(Y)(\xi f) \\ &= -(n-1)(Y(r-\lambda)) + \frac{1}{p}(r-\lambda)(n-1)(Yf). \end{aligned} \tag{5.6}$$

Replacing Y by ϕY in (5.6) we have

$$\frac{r-(n-1)(\alpha^2-\rho)}{(n-1)}\phi \, gradf = -(n-1)\phi \, grad(r-\lambda) + \frac{1}{p}(r-\lambda)(n-1)\phi \, gradf. \tag{5.7}$$

Combining (5.4) and (5.7), we obtain

$$[\frac{r}{(n-1)} - n(\alpha^2 - \rho)]\phi\, gradf = 0.$$

Thus we can conclude that

Theorem 5.1. *Every hyper generalized pseudo symmetric $(LCS)_n$-manifold admitting gradient almost quasi-Yamabe solitons reduces to an Einstein manifold and the sectional curvature is $(\alpha^2 - \rho)$ or the gradient of the potential function f of the soliton is pointwise collinear with ξ.*

Corollary 5.2. *Every hyper generalized pseudo symmetric $(LCS)_n$-manifold admitting gradient almost quasi-Yamabe solitons reduces to a generalized pseudo symmetric $(LCS)_n$-manifold.*

Corollary 5.3. *Every pseudo symmetric $(LCS)_n$-manifold admitting gradient almost quasi-Yamabe solitons reduces to an Einstein manifold and the sectional curvature is $(\alpha^2 - \rho)$ or the gradient of the potential function f of the soliton is pointwise collinear with ξ.*

Again, $gradf + \xi(f)\xi = 0$ implies that

$$\nabla_X\, gradf = -X(\xi f)\xi - (\xi f)\nabla_X \xi. \tag{5.8}$$

In view of (5.1), equation (5.8) yields

$$(r-\lambda)X + \frac{1}{p}g(X,\, gradf)\, gradf + X(\xi f)\xi + (\xi f)\nabla_X\xi = 0. \tag{5.9}$$

Taking the inner product with ξ, we have

$$(r-\lambda)\eta(X) + \frac{1}{p}(Xf)(\xi f) - X(\xi f) = 0. \tag{5.10}$$

Again contracting (5.9) over X, we get

$$n(r-\lambda) + \xi(\xi f) + (\xi f)\, div\xi = 0. \tag{5.11}$$

Now setting $X = \xi$ in (5.10) and then comparing with (5.11) we obtain

$$(n-1)(r-\lambda) + (\xi f)[\frac{1}{p}(\xi f) + \, div\xi] = 0.$$

Therefore we can state

Theorem 5.4. *Every hyper generalized pseudo symmetric $(LCS)_n$-manifold admitting gradient almost quasi-Yamabe solitons satisfies*

$$\lambda = r + \frac{(\xi f)}{(n-1)}[\frac{1}{p}(\xi f) + \, div\xi].$$

6. Example

Example 6.1. Consider a 3-dimensional manifold $M^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_3 \neq 0$, where (x_1, x_2, x_3) are the standard coordinates in $\mathbb{R}^3\}$. Let $\{e_1, e_2, e_3\}$ be a linearly independent global frame on M^3 given by

$$e_1 = x_3 \frac{\partial}{\partial x_1}, \quad e_2 = x_3 \frac{\partial}{\partial x_2}, \quad e_3 = (x_3)^3 \frac{\partial}{\partial x_3}.$$

Let g be the Lorentzian metric defined by $g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) = g(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}) = \left(\frac{1}{x_3}\right)^2$, $g(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3}) = -\left(\frac{1}{x_3}\right)^6$ and $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = 0$ for $i \neq j = 1, 2, 3$. Let η be the 1-form defined by $\eta(U) = g(U, e_3)$ for any $U \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi \cdot e_1 = e_1$, $\phi \cdot e_2 = e_2$, $\phi \cdot e_3 = 0$. Then using the linearity of ϕ and g we have $\eta(e_3) = -1$, $\phi^2 = I + \eta \otimes e_3$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to g and R be the curvature tensor of g. Then we infer

$$[e_1, e_3] = -(x_3)^2 e_1, \quad [e_2, e_3] = -(x_3)^2 e_2.$$

Taking $e_3 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily bring out

$$\nabla_{e_1} e_3 = -(x_3)^2 e_1, \quad \nabla_{e_2} e_3 = -(x_3)^2 e_2, \quad \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = 0, \qquad \nabla_{e_2} e_2 = -(x_3)^2 e_3, \qquad \nabla_{e_3} e_2 = 0,$$

$$\nabla_{e_1} e_1 = -(x_3)^2 e_3, \qquad \nabla_{e_2} e_1 = 0, \qquad \nabla_{e_3} e_1 = 0.$$

From above, we can say (ϕ, ξ, η, g) is an $(LCS)_3$ structure on M. Consequently, $M^3(\phi, \xi, \eta, g)$ is an $(LCS)_3$-manifold with $\alpha = -(x_3)^2 \neq 0$ and $\rho = 2(x_3)^4$. Using the above relations, we calculate the non-vanishing components of the curvature tensor as follows :

$$\begin{aligned} R(e_2, e_3)e_3 &= -(x_3)^4 e_2, & G(e_2, e_3)e_3 &= -e_2 \\ R(e_1, e_2)e_1 &= -(x_3)^4 e_2, & R(e_1, e_2)e_1 &= -e_2, \\ R(e_1, e_3)e_3 &= -(x_3)^4 e_1, & G(e_1, e_3)e_3 &= -e_1, \\ R(e_1, e_3)e_1 &= (x_4)^4 e_3\epsilon_3, & G(e_1, e_3)e_1 &= -e_3\epsilon_3, \\ R(e_2, e_3)e_2 &= (x_3)^4 e_3\epsilon_3, & G(e_2, e_3)e_2 &= -e_3\epsilon_3, \end{aligned}$$

and the components which can be obtained from these by the symmetric properties. Thus, we have

$$\begin{aligned} R(X, Y, Z, U) &= (\alpha^2 - \rho)\epsilon_b G(X, Y, Z, U), \\ \text{and } S(Y, Z) &= 3(\alpha^2 - \rho)\epsilon_b g(Y, Z), \quad r = 9(\alpha^2 - \rho) \\ \text{where } \epsilon_b &= -1, \text{ for } b = 3, \\ &= 0\ , \ \text{ for } b = 1, 2. \end{aligned}$$

Additionally, it is well known that the scalar curvature r of the space of constant curvature is always constant. Consequently, $(\alpha^2 - \rho)$, that is, $(x_3)^4$ is constant. Hence, the covariant derivatives of the curvature tensor are

$$(\nabla_{e_i} R)(X, Y, Z, U) = 0, \qquad \text{for } i = 1, 2, 3.$$

For the choice of the following 1-forms

$$\begin{aligned} F(e_i) &= -\frac{1}{(\alpha^2 - \rho)}, && \text{for } i = 1, 2, 3, \\ E(e_i) &= -\frac{F(e_i)}{(\alpha^2 - \rho)}, && \text{for } i = 1, 2, 3, \end{aligned}$$

spacetime under consideration is a hyper generalized pseudo symmetric.
For $V = \delta e_3$, we have

$$\begin{aligned} \nabla_X V &= (r - \lambda)X + \frac{1}{p} g(X, V)V. \\ (e_3 \delta) &= (r - \lambda) - \frac{1}{p} \\ \lambda &= r - (e_3 \delta) - \frac{1}{p}. \end{aligned}$$

This motivates us to state that

Theorem 6.2. *The manifold (M^3, g) under consideration is a hyper generalized pseudo symmetric $(LCS)_3$-manifold. Then*

- $(g, V = -5 (x_3)^2 e_3, p, \lambda = (x_3)^4 - \frac{1}{p})$ is an expanding almost quasi-Yamabe soliton
- $(g, V = 2 (x_3)^2 e_3, p, \lambda = -13(x_3)^4 - \frac{1}{p})$ is a shrinking almost quasi-Yamabe soliton

Furthermore, for $f = (x_3)^4$, $V = gradf = 4e_3$, Subsequently, we infer

- $(g, f = (x_3)^4, p, \lambda = -9(x_3)^4 - \frac{1}{p})$ is a shrinking almost quasi-Yamabe gradient soliton.

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