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Study of the implicit relation and $(\mathcal{F}, \mathcal{H})$-contractions on double controlled metric-like space with an application[1]

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Abstract. In this article, utilizing the newly introduced concept of double controlled metric like space, we study the implicit relation along with $(\mathcal{F}, \mathcal{H})$-contractions through orbital admissible mappings. Our results extend, generalize and unify many well known results. Examples are also presented to justify the effectiveness of our new findings. Lastly, one application is presented in a system of non-linear integral equations to discuss the existence of a solution.

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1. Introduction and preliminaries

In the year 2012, the concept of an α-admissible mapping was proposed by Samet et al.[20]. This concept become very popular as it covers theorems in distinct platforms such as fixed point theorems in standard metric space, fixed point theorems in metric space for cyclic mappings via closed subsets, fixed point theorems in metric space endowed with a graph etc. Later in 2014, Popescu [18] introduced the notion of an α-orbital admissible mapping as an improvement of α-admissible mappings. Very recently, the concept of an α-orbital admissible mapping was extended to the (α, β)-orbital cyclic admissible mapping by Alqahtani et al. [5].

On the other side, Czerwik [11] defined the concept of a b-metric space as a generalization of the metric space. Not long ago, Kamran et al.[21] have extended the notion of a b-metric space by introducing the concept of an extended b-metric space. Researchers put their attention on this setup and established many well known classical results of fixed point theorems (see for example [2], [3], [4], [6], [24] and the references cited therein). Further, the notion of an extended b-metric space has been extended into several directions such as dislocated extended b-metric [16], controlled metric type space [14], double controlled metric type space [1], and very recently double controlled metric-like space [15].

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[1] Dedicated to Professor Ravi P. Agarwal for his mathematical contribution.

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The main aim of this paper is to study a new kind of fixed point results involving implicit relation and $(\mathcal{F}, \mathcal{H})$-contraction in the structure of newly introduced double controlled metric-like spaces. Before we present our main results, we first recall some basic definitions, and preliminary results from the existing literature. From now, we write $\mathbb{R}_+$ to mean $[0, \infty)$.

Definition 1.1. [18] Let $\aleph$ be a non-empty set. Let $\mathcal{K} : \aleph \to \aleph$ and $\alpha : \aleph \times \aleph \to \mathbb{R}_+$ be two mappings. Then we say $\mathcal{K}$ is an α-orbital admissible mapping if, for all $\vartheta \in \aleph$, we have

$$\alpha(\vartheta, \mathcal{K}\vartheta) \geqslant 1 \Rightarrow \alpha(\mathcal{K}\vartheta, \mathcal{K}^2\vartheta) \geqslant 1.$$

Definition 1.2. [18] Let $\aleph$ be a non-empty set. Let $\mathcal{K} : \aleph \to \aleph$ and $\alpha : \aleph \times \aleph \to \mathbb{R}_+$ be two mappings. Then we say $\mathcal{K}$ is a triangular α-orbital admissible mapping if $\mathcal{K}$ is α-orbital admissible mapping, and

$$\alpha(\vartheta, u) \geqslant 1,\ \alpha(u, \mathcal{K}u) \geqslant 1 \Rightarrow \alpha(\vartheta, \mathcal{K}u) \geqslant 1, \text{ for all } \vartheta, u \in \aleph.$$

Note: Every triangular α-admissible mapping is a triangular α-orbital admissible mapping but the converse statement is not true (see [18]).

Lemma 1.3. *[18] Let $\aleph$ be a non-empty set. Let $\mathcal{K} : \aleph \to \aleph$ be a triangular α-orbital admissible mapping. Suppose that $\{\vartheta_r\}_{r=1}^{\infty}$ is a sequence in $\aleph$ with $\vartheta_{r+1} = \mathcal{K}\vartheta_r$ and $\alpha(\vartheta_1, \mathcal{K}\vartheta_1) \geqslant 1$. Then we have $\alpha(\vartheta_r, \vartheta_s) \geqslant 1$, for all $r, s \in \mathbb{N}$ with $r < s$.*

Next, we move to the definition of an (α, β)-orbital cyclic admissible pair.

Definition 1.4. [5] Let $\aleph$ be a non-empty set. Let $\mathcal{K}, \mathcal{L} : \aleph \to \aleph$, and $\alpha, \beta : \aleph \times \aleph \to \mathbb{R}_+$ be four given mappings. Then, the pair $(\mathcal{K}, \mathcal{L})$ is said to be an (α, β)-orbital cyclic admissible pair if for any $\vartheta \in \aleph$, we have

$$\alpha(\vartheta, \mathcal{K}\vartheta) \geqslant 1 \Rightarrow \beta(\mathcal{K}\vartheta, \mathcal{L}\mathcal{K}\vartheta) \geqslant 1, \text{ and}$$
$$\beta(\vartheta, \mathcal{L}\vartheta) \geqslant 1 \Rightarrow \alpha(\mathcal{L}\vartheta, \mathcal{K}\mathcal{L}\vartheta) \geqslant 1.$$

Now, we come to the definition of a double controlled metric-like space [15].

Definition 1.5. [15] Let $\aleph$ be a non empty set. Suppose that $\gamma, \lambda : \aleph \times \aleph \to [1, \infty)$ are two given mappings. Suppose that a mapping $\delta_\rho : \aleph \times \aleph \to \mathbb{R}_+$ satisfies the following conditions for all $\vartheta, u, v \in \aleph$

$(\ast C_1)$ $\delta_\rho(\vartheta, u) = 0 \Rightarrow \vartheta = u$;
$(\ast C_2)$ $\delta_\rho(\vartheta, u) = \delta_\rho(u, \vartheta)$;
$(\ast C_3)$ $\delta_\rho(\vartheta, v) \leqslant \gamma(\vartheta, u)\delta_\rho(\vartheta, u) + \lambda(u, v)\delta_\rho(u, v)$.

Then the pair $(\aleph, \delta_\rho)$ is called a double controlled metric-like space (in brief DCMLS). Throughout the paper, we consider δ_ρ as a continuous functional.

We now state some topological concepts, such as Cauchy sequence, convergence and completeness in a DCMLS.

Definition 1.6. [15] Let $(\aleph,\delta_\rho)$ be a DCMLS, and let $\{\vartheta_r\}_{r=1}^{\infty}$ be a sequence in $\aleph$.

(i) $\{\vartheta_r\}$ is a δ_ρ Cauchy sequence if and only if $\lim_{r,s\to\infty}\delta_\rho(\vartheta_r,\vartheta_s)$ exists and is finite.

(ii) $\{\vartheta_r\}$ converges to ϑ^* in $\aleph$ if and only if $\lim_{r\to\infty}\delta_\rho(\vartheta_r,\vartheta^*)=\delta_\rho(\vartheta^*,\vartheta^*)$.

(iii) $(\aleph,\delta_\rho)$ is said to be complete if for each Cauchy sequence $\{\vartheta_r\}$, there is $\vartheta^*\in\aleph$ such that $\lim_{r\to\infty}\delta_\rho(\vartheta_r,\vartheta^*)=\delta_\rho(\vartheta^*,\vartheta^*)=\lim_{r,s\to\infty}\delta_\rho(\vartheta_r,\vartheta_s)$.

Definition 1.7. [15] Let $(\aleph,\delta_\rho)$ be a DCMLS. For $\vartheta\in\aleph$ and $\nu>0$:

(i) An open ball $B(\vartheta,\nu)$ in $(\aleph,\delta_\rho)$ is $B(\vartheta,\nu)=\{u\in\aleph,|\delta_\rho(\vartheta,u)-\delta_\rho(\vartheta,\vartheta)|<\nu\}$.

(ii) The mapping $\mathcal{K}:\aleph\to\aleph$ is said to be continuous at $\vartheta\in\aleph$ if for all $\tau>0$, there exists $\nu>0$, such that $\mathcal{K}(B(\vartheta,\nu))\subseteq B(\mathcal{K}(\vartheta),\tau)$. Thus if $\mathcal{K}$ is continuous at ϑ, then for any sequence $\{\vartheta_r\}$ converging to ϑ, we have $\lim_{r\to\infty}\mathcal{K}\vartheta_r=\mathcal{K}\vartheta$, i.e., $\lim_{r\to\infty}\delta_\rho(\mathcal{K}\vartheta_r,\mathcal{K}\vartheta)=\delta_\rho(\mathcal{K}\vartheta,\mathcal{K}\vartheta)$.

Let Φ denotes the collection of all $\eta:\mathbb{R}_+\to\mathbb{R}_+$ such that

1. η is non-decreasing;
2. $\eta(\tau)<\tau$, for $\tau>0$ with $\eta(0)=0$.

2. Main results

Let $\aleph$ be a non-empty set and $\mathcal{K},\mathcal{L}:\aleph\to\aleph$ be two self mappings. Denote by $\mathcal{C}(\mathcal{K},\mathcal{L})$ and $\mathrm{Fix}(\mathcal{K})$ the collection of all common fixed points of $\mathcal{K}$, $\mathcal{L}$ and the collection of all fixed points of $\mathcal{K}$, respectively, i.e.,

$$\mathcal{C}(\mathcal{K},\mathcal{L})=\{x\in\aleph\,|\,x=\mathcal{K}x=\mathcal{L}x\},$$

and

$$\mathrm{Fix}(\mathcal{K})=\{x\in\aleph\,|\,\mathcal{K}x=x\}.$$

2.1. Implicit relation in the context of DCMLS

In this section, we investigate implicit function in the context of DCMLS using (α,β)-orbital cyclic admissible mappings. To do this, first we consider the following implicit function (readers may also look into [9], [10], [17] [22], [23]).

Definition 2.1. Let Ω be the collection of all functions $\mathfrak{G}(\xi_1,\cdots,\xi_4):\mathbb{R}_+^4\to\mathbb{R}$ which satisfies the following properties:
$(\mathfrak{G}_1)$ $\mathfrak{G}$ is continuous in each variable and non-decreasing in the variable ξ_1;
$(\mathfrak{G}_{2a})$ for all $c,d\geqslant 0$ with $\mathfrak{G}(c,d,d,c)\leqslant 0$ there exists a $\eta\in\Phi$ such that $c\leqslant\eta(d)$;
$(\mathfrak{G}_{2b})$ for all $c,d\geqslant 0$, $\mathfrak{G}(c,d,c,d)\leqslant 0$ implies $c\leqslant\eta(d)$, where $\eta\in\Phi$;
$(\mathfrak{G}_3)$ for all $\tau>0$, we obtain $0<\mathfrak{G}(\tau,\tau,0,0)$.

Now, we provide some examples of $\mathfrak{G}$.

Example 2.2. $\mathfrak{G}(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - k\xi_2 - l\xi_3 - m\xi_4$, where $k, l, m \geqslant 0$ with $k + l + m < 1$.

Example 2.3. $\mathfrak{G}(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - k \max\{\xi_2, \xi_3, \xi_4\}$, where $k \in [0, \frac{1}{2})$.

Example 2.4. $\mathfrak{G}(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1^2 - k\xi_2\xi_3 - l\xi_4^2$, where $k, l \geqslant 0$ with $k + l < 1$.

Example 2.5. $\mathfrak{G}(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - k\xi_2 - l\xi_3 - m \max\{2\xi_4, \xi_1 + \xi_4\}$, where $k, l, m \geqslant 0$ with $k + l + 2m < 1$.

Example 2.6. $\mathfrak{G}(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1^2 - k \max\{\xi_2^2, \xi_3^2, \xi_4^2\} - l \max\{\xi_1\xi_3, \xi_2\xi_4\} - m\xi_3\xi_4$, where $k, l, m \geqslant 0$ with $k + l + m < 1$.

Example 2.7. $\mathfrak{G}(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1^3 - k\xi_1^2\xi_2 - l\xi_1\xi_3\xi_4 - m\xi_2\xi_3^2 - n\xi_3\xi_4^2$, where $k, l, m, n \geqslant 0$ with $k + l + m + n < 1$.

Example 2.8. $\mathfrak{G}(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - k \max\{\xi_2, \frac{\xi_3 + \xi_4}{2}\}$, where $k \in [0, 1)$.

Example 2.9. $\mathfrak{G}(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - (k\xi_2^n + l\xi_3^n + m\xi_4^n)^{\frac{1}{n}}$, where $k, l, m, n > 0$ with $k + l + m < 1$.

Theorem 2.10. *Let $(\aleph, \delta_\rho)$ be a complete DCMLS. Let α, β be two mappings from the cross product of $\aleph$ into $\mathbb{R}_+$. Further, let $\mathcal{K}, \mathcal{L}$ be two mappings from $\aleph$ into itself. Suppose that for all $\vartheta, u \in \aleph$ the following relation holds*

$$\mathfrak{G}(\alpha(\vartheta, \mathcal{K}\vartheta)\beta(u, \mathcal{L}u)\delta_\rho(\mathcal{K}\vartheta, \mathcal{L}u), \delta_\rho(\vartheta, u), \delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{L}u)) \leqslant 0, \tag{2.1}$$

where $\mathfrak{G} \in \Omega$. Also, suppose that the following conditions are satisfied:
(C1) there exists a ϑ_0 such that $\alpha(\vartheta_0, \mathcal{K}\vartheta_0) \geqslant 1$;
(C2) the pair $(\mathcal{K}, \mathcal{L})$ is an (α, β)-orbital cyclic admissible pair;
(C3) for any sequence $\{\vartheta_r\}_{r=0}^{\infty}$ with $\vartheta_{2r+1} = \mathcal{K}\vartheta_{2r}, \vartheta_{2r} = \mathcal{L}\vartheta_{2r-1}$ and $\alpha(\vartheta_{2r}, \vartheta_{2r+1}) \geqslant 1, \beta(\vartheta_{2r+1}, \vartheta_{2r+2}) \geqslant 1$ for all $r \in \mathbb{N}$, we have:

$$\sup_{s\geqslant 1} \lim_{r\to\infty} \frac{\gamma(\vartheta_{r+1}, \vartheta_{r+2})}{\gamma(\vartheta_r, \vartheta_{r+1})} \lambda(\vartheta_{r+1}, \vartheta_{r+s}) \frac{\eta^{r+1}(\delta_\rho(\vartheta_0, \vartheta_1))}{\eta^r(\delta_\rho(\vartheta_0, \vartheta_1))} < 1, \tag{B_*}$$

where $\eta \in \Phi$, and also the sequence $\{\gamma(\vartheta_r, \vartheta_{r+1})\}$ is bounded;
(C4) $\mathcal{K}$ and $\mathcal{L}$ both are continuous mappings or;
$(\widehat{C4})$ if $\{\vartheta_r\}$ is a sequence in $\aleph$ such that $\vartheta_r \to \vartheta^*$, then $\alpha(\vartheta^*, \mathcal{K}\vartheta^*) \geqslant 1, \beta(\vartheta^*, \mathcal{L}\vartheta^*) \geqslant 1$ and also suppose that for any $\vartheta \in \aleph$, we have $\delta_\rho(\vartheta^*, \vartheta) \leqslant \limsup_{r\to\infty} \delta_\rho(\vartheta_r, \vartheta)$.
Then $\mathcal{K}$ and $\mathcal{L}$ have a common fixed point, i.e., $\mathcal{C}(\mathcal{K}, \mathcal{L}) \neq \varnothing$.

Proof. By assumption $(C1)$ there exists a ϑ_0 such that $\alpha(\vartheta_0, \mathcal{K}\vartheta_0) \geqslant 1$. Consider $\vartheta_1 = \mathcal{K}\vartheta_0$ and $\vartheta_2 = \mathcal{L}\vartheta_1$. Using induction, we can easily build up a sequence $\{\vartheta_r\}_{r=0}^{\infty}$ such that $\vartheta_{2r+1} = \mathcal{K}\vartheta_{2r}$ and $\vartheta_{2r} = \mathcal{L}\vartheta_{2r-1}$, for all $r \in \mathbb{N}$. Again, we have $\alpha(\vartheta_0, \vartheta_1) \geqslant 1$ and by the given condition the pair $(\mathcal{K}, \mathcal{L})$ is a $(\alpha,\ \beta)$-orbital cyclic admissible mapping. Thus, we get

$$\alpha(\vartheta_0, \vartheta_1) \geqslant 1 \Rightarrow \beta(\mathcal{K}\vartheta_0, \mathcal{L}\mathcal{K}\vartheta_0) = \beta(\vartheta_1, \vartheta_2) \geqslant 1, \text{ and}$$
$$\beta(\vartheta_1, \vartheta_2) \geqslant 1 \Rightarrow \alpha(\mathcal{L}\vartheta_1, \mathcal{K}\mathcal{L}\vartheta_1) = \alpha(\vartheta_2, \vartheta_3) \geqslant 1.$$

Proceeding in this way, using $(C2)$, we obtain, $\alpha(\vartheta_{2r}, \vartheta_{2r+1}) \geqslant 1, \beta(\vartheta_{2r+1}, \vartheta_{2r+2}) \geqslant 1$, for all $r \in \mathbb{N}$. Without loss of generality, we may assume that $\vartheta_r \neq \vartheta_{r+1}$, for all $r \in \mathbb{N}$. Next, we want to show that if for some r, $\delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}) = 0$, then $\mathcal{K}\vartheta_{2r} = \vartheta_{2r} = \mathcal{L}\vartheta_{2r}$, and if for some r, $\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}) = 0$, then $\mathcal{K}\vartheta_{2r+1} = \vartheta_{2r+1} = \mathcal{L}\vartheta_{2r+1}$. Suppose that for some r, $\delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}) = 0$. Then we will show that $\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}) = 0$. We establish this fact by using the method of contradiction. Suppose $\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}) > 0$. Now take $\vartheta = \vartheta_{2r}, u = \vartheta_{2r+1}$ in (2.1), then we have

$$\begin{aligned}
&\mathfrak{G}(\alpha(\vartheta_{2r}, \mathcal{K}\vartheta_{2r})\beta(\vartheta_{2r+1}, \mathcal{L}\vartheta_{2r+1})\delta_\rho(\mathcal{K}\vartheta_{2r}, \mathcal{L}\vartheta_{2r+1}), \delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}), \delta_\rho(\vartheta_{2r}, \mathcal{K}\vartheta_{2r}),\\
&\delta_\rho(\vartheta_{2r+1}, \mathcal{L}\vartheta_{2r+1})) \leqslant 0,\\
&\Rightarrow \mathfrak{G}(\alpha(\vartheta_{2r}, \vartheta_{2r+1})\beta(\vartheta_{2r+1}, \vartheta_{2r+2})\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}), \delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}), \delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}),\\
&\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2})) \leqslant 0.
\end{aligned}$$

By $(\mathfrak{G}_1)$, we have

$$\mathfrak{G}(\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}), \delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}), \delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}), \delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2})) \leqslant 0.$$

Now, by applying $(\mathfrak{G}_{2a})$, we obtain

$$\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}) \leqslant \eta(\delta_\rho(\vartheta_{2r}, \vartheta_{2r+1})), \quad \text{where } \eta \in \Phi. \tag{2.2}$$

But we have $\delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}) = 0$ which implies $\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}) = 0$. Again, by using $(\mathfrak{G}_{2a})$, we can show that $\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}) = 0$ implies $\delta_\rho(\vartheta_{2r+2}, \vartheta_{2r+3}) = 0$. Consequently, we have $\vartheta_{2r+1} = \vartheta_{2r+2} = \vartheta_{2r+3}$ which implies

$$\begin{aligned}
\vartheta_{2r+1} &= \vartheta_{2r+2} = \mathcal{L}\vartheta_{2r+1} \text{ and}\\
\vartheta_{2r+1} &= \vartheta_{2r+3} = \mathcal{K}\vartheta_{2r+2} = \mathcal{K}\mathcal{L}\vartheta_{2r+1} = \mathcal{K}\vartheta_{2r+1}.
\end{aligned}$$

Thus, ϑ_{2r+1} becomes a common fixed point of $\mathcal{K}$ and $\mathcal{L}$. Hence if $\delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}) = 0$ or $\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}) = 0$, then our proof is completed. So, from now we assume that $\delta_\rho(\vartheta_r, \vartheta_{r+1}) > 0$ for all $r \in \{0\} \cup \mathbb{N}$. Next, we show that $\{\vartheta_r\}_{r=0}^{\infty}$ is a Cauchy sequence. To show this consider $\vartheta = \vartheta_{2r}, u = \vartheta_{2r+1}$ in (2.1). Then, we have

$$\begin{aligned}
&\mathfrak{G}(\alpha(\vartheta_{2r}, \vartheta_{2r+1})\beta(\vartheta_{2r+1}, \vartheta_{2r+2})\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}), \delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}), \delta_\rho(\vartheta_{2r}, \vartheta_{2r+1}),\\
&\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2})) \leqslant 0.
\end{aligned}$$

Now, using $(\mathfrak{G}_1)$ and $(\mathfrak{G}_{2a})$, we have

$$\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2}) \leqslant \eta(\delta_\rho(\vartheta_{2r}, \vartheta_{2r+1})), \text{ for all } r \in \{0\} \cup \mathbb{N}, \quad \text{where } \eta \in \Phi.$$

In a similar way, by taking $\vartheta = \vartheta_{2r+2}, u = \vartheta_{2r+1}$ in (2.1), we have

$$\delta_\rho(\vartheta_{2r+2}, \vartheta_{2r+3}) \leqslant \eta(\delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2})), \text{ for all } r \in \{0\} \cup \mathbb{N}, \quad \text{where } \eta \in \Phi.$$

Thus, for all $r \in \{0\} \cup \mathbb{N}$, we get

$$\delta_\rho(\vartheta_r, \vartheta_{r+1}) \leqslant \eta(\delta_\rho(\vartheta_{r-1}, \vartheta_r)), \text{ for all } r \in \{0\} \cup \mathbb{N}. \tag{2.3}$$

Repeatedly applying (2.3), we have

$$\delta_\rho(\vartheta_r, \vartheta_{r+1}) \leqslant \eta^r(\delta_\rho(\vartheta_0, \vartheta_1)), \text{ for all } r \in \{0\} \cup \mathbb{N}.$$

Now for $r, s \in \{0\} \cup \mathbb{N}$ with $s \geqslant 1$, we have

$$\begin{aligned}
&\delta_\rho(\vartheta_r, \vartheta_{r+s}) \\
&\leqslant \gamma(\vartheta_r, \vartheta_{r+1})\delta_\rho(\vartheta_r, \vartheta_{r+1}) + \lambda(\vartheta_{r+1}, \vartheta_{r+s})\delta_\rho(\vartheta_{r+1}, \vartheta_{r+s}) \\
&\leqslant \gamma(\vartheta_r, \vartheta_{r+1})\delta_\rho(\vartheta_r, \vartheta_{r+1}) + \lambda(\vartheta_{r+1}, \vartheta_{r+s})\gamma(\vartheta_{r+1}, \vartheta_{r+2})\delta_\rho(\vartheta_{r+1}, \vartheta_{r+2}) \\
&+ \lambda(\vartheta_{r+1}, \vartheta_{r+s})\lambda(\vartheta_{r+2}, \vartheta_{r+s})\delta_\rho(\vartheta_{r+2}, \vartheta_{r+s}) \\
&\leqslant \gamma(\vartheta_r, \vartheta_{r+1})\delta_\rho(\vartheta_r, \vartheta_{r+1}) + \lambda(\vartheta_{r+1}, \vartheta_{r+s})\gamma(\vartheta_{r+1}, \vartheta_{r+2})\delta_\rho(\vartheta_{r+1}, \vartheta_{r+2}) \\
&+ \lambda(\vartheta_{r+1}, \vartheta_{r+s})\lambda(\vartheta_{r+2}, \vartheta_{r+s})\gamma(\vartheta_{r+2}, \vartheta_{r+3})\delta_\rho(\vartheta_{r+2}, \vartheta_{r+3}) \\
&+ \lambda(\vartheta_{r+1}, \vartheta_{r+s})\lambda(\vartheta_{r+2}, \vartheta_{r+s})\lambda(\vartheta_{r+3}, \vartheta_{r+s})\delta_\rho(\vartheta_{r+3}, \vartheta_{r+s}) \\
&\leqslant \\
&\vdots \\
&\leqslant \gamma(\vartheta_r, \vartheta_{r+1})\delta_\rho(\vartheta_r, \vartheta_{r+1}) + \sum_{t=r+1}^{r+s-2}\Big(\prod_{q=r+1}^{t}\lambda(\vartheta_q, \vartheta_{r+s})\Big)\gamma(\vartheta_t, \vartheta_{t+1})\delta_\rho(\vartheta_t, \vartheta_{t+1}) \\
&+ \Big(\prod_{t=r+1}^{r+s-1}\lambda(\vartheta_t, \vartheta_{r+s})\Big)\delta_\rho(\vartheta_{r+s-1}, \vartheta_{r+s}) \\
&\leqslant \gamma(\vartheta_r, \vartheta_{r+1})\delta_\rho(\vartheta_r, \vartheta_{r+1}) + \sum_{t=r+1}^{r+s-2}\Big(\prod_{q=r+1}^{t}\lambda(\vartheta_q, \vartheta_{r+s})\Big)\gamma(\vartheta_t, \vartheta_{t+1})\delta_\rho(\vartheta_t, \vartheta_{t+1}) \\
&+ \Big(\prod_{t=r+1}^{r+s-1}\lambda(\vartheta_t, \vartheta_{r+s})\Big)\gamma(\vartheta_{r+s-1}, \vartheta_{r+s})\delta_\rho(\vartheta_{r+s-1}, \vartheta_{r+s}) \quad [\text{since } \gamma(\vartheta, u) \geqslant 1, \forall\ \vartheta, u \in \aleph] \\
&\leqslant \gamma(\vartheta_r, \vartheta_{r+1})\eta^r(\delta_\rho(\vartheta_0, \vartheta_1)) + \sum_{t=r+1}^{r+s-2}\Big(\prod_{q=r+1}^{t}\lambda(\vartheta_q, \vartheta_{r+s})\Big)\gamma(\vartheta_t, \vartheta_{t+1})\eta^t(\delta_\rho(\vartheta_0, \vartheta_1)) \\
&+ \Big(\prod_{t=r+1}^{r+s-1}\lambda(\vartheta_t, \vartheta_{r+s})\Big)\gamma(\vartheta_{r+s-1}, \vartheta_{r+s})\eta^{r+s-1}(\delta_\rho(\vartheta_0, \vartheta_1)) \\
&= \gamma(\vartheta_r, \vartheta_{r+1})\eta^r(\delta_\rho(\vartheta_0, \vartheta_1)) + \sum_{t=r+1}^{r+s-1}\Big(\prod_{q=r+1}^{t}\lambda(\vartheta_q, \vartheta_{r+s})\Big)\gamma(\vartheta_t, \vartheta_{t+1})\eta^t(\delta_\rho(\vartheta_0, \vartheta_1)) \\
&\leqslant \gamma(\vartheta_r, \vartheta_{r+1})\eta^r(\delta_\rho(\vartheta_0, \vartheta_1)) + \sum_{t=r+1}^{r+s-1}\Big(\prod_{q=0}^{t}\lambda(\vartheta_q, \vartheta_{r+s})\Big)\gamma(\vartheta_t, \vartheta_{t+1})\eta^t(\delta_\rho(\vartheta_0, \vartheta_1)) \qquad (A_1).
\end{aligned}$$

Let us put $\Delta_i = \sum_{t=0}^{i}\Big(\prod_{q=0}^{t}\lambda(\vartheta_q, \vartheta_{r+s})\Big)\gamma(\vartheta_t, \vartheta_{t+1})\eta^t(\delta_\rho(\vartheta_0, \vartheta_1))$. Then, inequality (A_1) can be written as

$$\delta_\rho(\vartheta_r, \vartheta_{r+s}) \leqslant \gamma(\vartheta_r, \vartheta_{r+1})\eta^r(\delta_\rho(\vartheta_0, \vartheta_1)) + [\Delta_{r+s-1} - \Delta_r]. \tag{2.4}$$

Again, by condition $(C3)$, we have

$$\sup_{s\geqslant 1}\lim_{r\to\infty}\frac{\gamma(\vartheta_{r+1}, \vartheta_{r+2})}{\gamma(\vartheta_r, \vartheta_{r+1})}\lambda(\vartheta_{r+1}, \vartheta_{r+s})\frac{\eta^{r+1}(\delta_\rho(\vartheta_0, \vartheta_1))}{\eta^r(\delta_\rho(\vartheta_0, \vartheta_1))} < 1.$$

Consequently, we obtain $\lim_{r\to\infty} \frac{\Delta_{r+1}}{\Delta_r} < 1$. Clearly, ratio test guarantees that the series $\sum_{i=0}^{\infty} \Delta_i$ is convergent. Again, observe that $\eta^i(\delta_\rho(\vartheta_0, \vartheta_1)) \leqslant \Delta_i$, since $\gamma(\vartheta, u) \geqslant 1, \lambda(\vartheta, u) \geqslant 1$ for all $\vartheta, u \in \aleph$. Hence $\sum_{i=0}^{\infty} \Delta_i < \infty$ implies $\sum_{i=0}^{\infty} \eta^i(\delta_\rho(\vartheta_0, \vartheta_1)) < \infty$. Thus, we obtain $\eta^i(\delta_\rho(\vartheta_0, \vartheta_1)) \to 0$ as $i \to \infty$. Thus, using boundedness of $\{\gamma(\vartheta_r, \vartheta_{r+1})\}_{r=0}^{\infty}$, from (2.4), we obtain $\delta_\rho(\vartheta_r, \vartheta_{r+s}) \to 0$ as $r, s \to \infty$. Hence, $\{\vartheta_r\}_{r=0}^{\infty}$ is a Cauchy sequence. Now by the completeness of $(\aleph, \delta_\rho)$, there exists a point $\vartheta^* \in \aleph$ such that

$$\lim_{r\to\infty} \delta_\rho(\vartheta_r, \vartheta^*) = \delta_\rho(\vartheta^*, \vartheta^*) = \lim_{r,s\to\infty} \delta_\rho(\vartheta_r, \vartheta_s) = 0.$$

Clearly $\delta_\rho(\vartheta^*, \vartheta^*) = 0$. Also, we can write $\vartheta_{2r} \to \vartheta^*$ and $\vartheta_{2r-1} \to \vartheta^*$ as $r \to \infty$, since $\vartheta_r \to \vartheta^*$ as $r \to \infty$. First, we suppose that $\mathcal{K}, \mathcal{L}$ are continuous mappings.

$$\begin{aligned}
\vartheta^* &= \lim_{r\to\infty} \vartheta_{2r+1} = \lim_{r\to\infty} \mathcal{K}\vartheta_{2r} = \mathcal{K} \lim_{r\to\infty} \vartheta_{2r} = \mathcal{K}\vartheta^*, \text{ and} \\
\vartheta^* &= \lim_{r\to\infty} \vartheta_{2r+2} = \lim_{r\to\infty} \mathcal{L}\vartheta_{2r+1} = \mathcal{L} \lim_{r\to\infty} \vartheta_{2r+1} = \mathcal{L}\vartheta^*.
\end{aligned}$$

Now, let us consider the condition $\widehat{(C4)}$ (in place of continuity). Taking $\vartheta = \vartheta^*, u = \vartheta_{2r+1}$ in (2.1), we have

$$\begin{aligned}
&\mathfrak{G}(\alpha(\vartheta^*, \mathcal{K}\vartheta^*)\beta(\vartheta_{2r+1}, \mathcal{L}\vartheta_{2r+1})\delta_\rho(\mathcal{K}\vartheta^*, \mathcal{L}\vartheta_{2r+1}), \delta_\rho(\vartheta^*, \vartheta_{2r+1}), \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*), \\
&\delta_\rho(\vartheta_{2r+1}, \mathcal{L}\vartheta_{2r+1})) \leqslant 0.
\end{aligned}$$

Since $\mathfrak{G}$ is non-decreasing in the first coordinate, thus we have

$$\mathfrak{G}(\delta_\rho(\mathcal{K}\vartheta^*, \vartheta_{2r+2}), \delta_\rho(\vartheta^*, \vartheta_{2r+1}), \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*), \delta_\rho(\vartheta_{2r+1}, \vartheta_{2r+2})) \leqslant 0.$$

Now, using the continuity of $\mathfrak{G}$, we obtain

$$\begin{aligned}
&\mathfrak{G}(\delta_\rho(\mathcal{K}\vartheta^*, \vartheta^*), 0, \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*), 0) \leqslant 0 \\
&\Rightarrow \delta_\rho(\mathcal{K}\vartheta^*, \vartheta^*) \leqslant \eta(0), \ [\text{by } (\mathfrak{G}_{2b}), \text{ where } \eta \in \Phi] \\
&\Rightarrow \delta_\rho(\mathcal{K}\vartheta^*, \vartheta^*) = 0 \\
&\Rightarrow \vartheta^* = \mathcal{K}\vartheta^*.
\end{aligned}$$

In a similar way, by considering $\vartheta = \vartheta_{2r}, u = \vartheta^*$ in (2.1), we can show that $\vartheta^* = \mathcal{L}\vartheta^*$. Thus $\mathcal{K}$ and $\mathcal{L}$ have a common fixed point, i.e., $\mathcal{C}(\mathcal{K}, \mathcal{L}) \neq \varnothing$. Hence, our proof is completed. □

In our next theorem, we deal with the uniqueness of fixed point for the operators $\mathcal{K}, \mathcal{L}$. In order to obtain unique common fixed point, here we consider the following hypothesis.

$(\mathbf{U})$: For all $\vartheta_1, \vartheta_2 \in \mathcal{C}(\mathcal{K}, \mathcal{L})$, suppose that $\alpha(\vartheta_1, \mathcal{K}\vartheta_1) \geqslant 1$, $\beta(\vartheta_2, \mathcal{L}\vartheta_2) \geqslant 1$.

Theorem 2.11. *Suppose that all the hypothesis of Theorem 2.10 are satisfied together with hypotheses ($\boldsymbol{U}$). Then the mappings $\mathcal{K}$ and $\mathcal{L}$ have a unique common fixed point.*

Proof. First of all observe that if $\vartheta \in \mathcal{C}(\mathcal{K},\mathcal{L})$, then $\delta_\rho(\vartheta,\vartheta) = 0$. Now, let us suppose that ϑ_1, ϑ_2 are two common fixed points of $\mathcal{K}$ and $\mathcal{L}$, i.e., $\vartheta_1 = \mathcal{K}\vartheta_1 = \mathcal{L}\vartheta_1, \vartheta_2 = \mathcal{K}\vartheta_2 = \mathcal{L}\vartheta_2$. By considering $\vartheta = \vartheta_1, u = \vartheta_2$ in (2.1), we obtain

$$\begin{aligned}&\mathfrak{G}(\alpha(\vartheta_1,\mathcal{K}\vartheta_1)\beta(\vartheta_2,\mathcal{L}\vartheta_2)\delta_\rho(\mathcal{K}\vartheta_1,\mathcal{L}\vartheta_2),\delta_\rho(\vartheta_1,\vartheta_2),\delta_\rho(\vartheta_1,\mathcal{K}\vartheta_1),\delta_\rho(\vartheta_2,\mathcal{L}\vartheta_2)) \leqslant 0\\ &\Rightarrow \mathfrak{G}(\delta_\rho(\vartheta_1,\vartheta_2),\delta_\rho(\vartheta_1,\vartheta_2),0,0) \leqslant 0.\end{aligned}$$

Clearly, we arrive at a contradiction by $(\mathfrak{G}_3)$ if we assume $\delta_\rho(\vartheta_1,\vartheta_2) > 0$. Thus, we must have $\delta_\rho(\vartheta_1,\vartheta_2) = 0 \Rightarrow \vartheta_1 = \vartheta_2$. Consequently, our proof is completed. □

Remark 1: In Theorem 2.10, we have used implicit relation to discuss our new fixed point result. One can obtain various types of contractions by choosing different types of implicit relations. Now, if one chooses a particular type of implicit relation, i.e., $\mathfrak{G}(\xi_1,\xi_2,\xi_3,\xi_4) = \xi_1 - \eta(\max\{\xi_2,\xi_3,\xi_4\})$, where $\eta(t) \in \Phi$, satisfying the condition (B_*) given in $(C3)$ of Theorem 2.10, then our first main result improves, unifies and extends several well known fixed point results which we have mentioned below. Also, since the concept of DCMLS is more general than the concept of the usual metric space, b-metric space, b-metric like space, extended b-metric space, controlled metric type space etc, so our first main result can be viewed as an extension and generalization of the following results:

- The classical Banach contraction principle [8], Kannan's fixed point theorem [12], Reich's fixed point theorem [19] in the context of metric space;
- Samet et al.'s fixed point theorem [20];
- Shatanawi et al.'s fixed point theorem [24];
- Mlaiki's fixed point theorem [15].

2.2. $(\mathcal{F},\mathcal{H})$-contraction in the context of DCMLS

In this section, motivated by the paper of Popescu [18], first we introduce the following definition.

Definition 2.12. Let $\aleph$ be a non-empty set. Let $\mathcal{K} : \aleph \to \aleph$ and $\mu : \aleph\times\aleph \to \mathbb{R}_+$ be two mappings. Then, we say that $\mathcal{K}$ is a μ-suborbital admissible mapping if, for all $\vartheta \in \aleph$, we have

$$\mu(\vartheta,\mathcal{K}\vartheta) \leqslant 1 \Rightarrow \mu(\mathcal{K}\vartheta,\mathcal{K}^2\vartheta) \leqslant 1.$$

Definition 2.13. Let $\aleph$ be a non-empty set. Let $\mathcal{K} : \aleph \to \aleph$ and $\mu : \aleph\times\aleph \to \mathbb{R}_+$ be two mappings. Then, we say that $\mathcal{K}$ is a triangular μ-suborbital admissible mapping if $\mathcal{K}$ is μ-suborbital admissible mapping, and

$$\mu(\vartheta,u) \leqslant 1,\ \mu(u,\mathcal{K}u) \leqslant 1 \Rightarrow \mu(\vartheta,\mathcal{K}u) \leqslant 1,\ \text{for all } \vartheta,u \in \aleph.$$

Lemma 2.14. *Let $\aleph$ be a non-empty set. Let $\mathcal{K} : \aleph \to \aleph$ be a triangular μ-suborbital admissible mapping. Suppose that $\{\vartheta_r\}_{r=1}^{\infty}$ is a sequence in $\aleph$ such that $\vartheta_{r+1} = \mathcal{K}\vartheta_r$ with $\mu(\vartheta_1,\mathcal{K}\vartheta_1) \leqslant 1$. Then we have $\mu(\vartheta_r,\vartheta_s) \leqslant 1$, for all $r,s \in \mathbb{N}$ with $r < s$.*

Proof. Similar to Lemma 1.3. □

Next, we move to the definition of $(\mathcal{F},\mathcal{H})$-contraction.

Definition 2.15. [7] A function $\mathcal{H}:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is said to be a subclass of type-I if

$$u\geqslant 1\Rightarrow \mathcal{H}(1,v)\leqslant \mathcal{H}(u,v) \text{ for all } u,v\in\mathbb{R}_+.$$

Definition 2.16. [7] Let $\mathcal{H},\mathbb{F}:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be two given functions. Then the pair $(\mathcal{F},\mathcal{H})$ is said to be an upper class of type-I if $\mathcal{F}$ is a function, $\mathcal{H}$ is a subclass of type-I, and

$$0<u<1\Rightarrow \mathcal{F}(u,v)\leqslant \mathcal{F}(1,v),$$

$$\mathcal{H}(1,y)\leqslant \mathcal{F}(x,v)\Rightarrow y\leqslant xv, \text{ for all } u,v,x,y\in\mathbb{R}_+.$$

Definition 2.17. [7] Let $\mathcal{H},\mathbb{F}:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be two given functions. Then the pair $(\mathcal{F},\mathcal{H})$ is said to be a special upper class of type-I if $\mathcal{F}$ is a function, $\mathcal{H}$ is a subclass of type-I, and

$$0<u<1\Rightarrow \mathcal{F}(u,v)\leqslant \mathcal{F}(1,v),$$

$$\mathcal{H}(1,y)\leqslant \mathcal{F}(1,v)\Rightarrow y\leqslant v, \text{ for all } u,v,y\in\mathbb{R}_+.$$

Next, we state the definition of an altering distance function.

Definition 2.18. [13] A function $\theta:\mathbb{R}_+\to\mathbb{R}_+$ is said to be an altering distance function if the following two conditions are satisfied:
(i) θ is a strictly monotone non-decreasing and continuous;
(ii) $\theta(\tau)=0$ if and only if $\tau=0$.

From now we write Θ to denote the collection of all altering distance functions. Now, we introduce the following definition in the context of DCMLS.

Definition 2.19. Let $(\aleph,\delta_\rho)$ be a DCMLS. Let $\mathcal{F},\mathcal{H}:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be two functions such that the pair $(\mathcal{F},\mathcal{H})$ is a special upper class of type-I. An operator $\mathcal{K}:\aleph\to\aleph$ is said to be $(\alpha,\mu,\mathcal{F},\mathcal{H})-(\theta,\xi)$ weakly contractive if there exist two mappings $\alpha,\mu:\aleph\times\aleph\to\mathbb{R}_+$ such that

$$\begin{aligned}&\mathcal{H}(\alpha(\vartheta,u),\theta(\delta_\rho(\mathcal{K}\vartheta,\mathcal{K}u)))\\ &\leqslant \mathcal{F}(\mu(\vartheta,u),\theta(\frac{\delta_\rho(\vartheta,\mathcal{K}\vartheta)+\delta_\rho(u,\mathcal{K}u)}{2})-\xi(\delta_\rho(\vartheta,\mathcal{K}\vartheta),\delta_\rho(u,\mathcal{K}u))),\end{aligned}\tag{2.5}$$

for all $\vartheta,u\in\aleph$ with $\alpha(\vartheta,u)\geqslant 1,\mu(\vartheta,u)\leqslant 1$ where $\theta\in\Theta$ and $\xi:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function such that $\xi(c,d)=0\Leftrightarrow c=d=0$.

Now we state and prove our second main result.

Theorem 2.20. *Let $(\aleph,\delta_\rho)$ be a complete DCMLS. Let $\mathcal{K}:\aleph\to\aleph$ be a $(\alpha,\mu,\mathcal{F},\mathcal{H})-(\theta,\xi)$ weakly contractive mapping satisfying the following conditions:*
(D1) there exists a $\vartheta_0\in\aleph$ such that $\alpha(\vartheta_0,\mathcal{K}\vartheta_0)\geqslant 1$, $\mu(\vartheta_0,\mathcal{K}\vartheta_0)\leqslant 1$;

(D2) $\mathcal{K}$ is triangular α-orbital and triangular μ-suborbital admissible mapping;
(D3) $\mathcal{K}$ is a continuous mapping or;
$\widehat{(D3)}$ if $\{\vartheta_r\}_{r=0}^{\infty}$ is a sequence in $\aleph$ such that $\alpha(\vartheta_r, \vartheta_{r+1}) \geqslant 1, \mu(\vartheta_r, \vartheta_{r+1}) \leqslant 1$ for all $r \in \{0\} \cup \mathbb{N}$ and $\vartheta_r \to u^*$ as $r \to \infty$, then there exists a sub-sequence $\{\vartheta_{r(n)}\}$ of $\{\vartheta_r\}$ such that $\alpha(\vartheta_{r(n)}, u^*) \geqslant 1, \mu(\vartheta_{r(n)}, u^*) \leqslant 1$ for all $n \in \{0\} \cup \mathbb{N}$ together with $\delta_\rho(u^*, u) \leqslant \limsup_{r\to\infty} \delta_\rho(\vartheta_r, u)$ for all $u \in \aleph$.
Then, $\mathcal{K}$ has a fixed point $\vartheta^* \in \aleph$ with $\delta_\rho(\vartheta^*, \vartheta^*) = 0$.

Proof. By assumption $(D1)$ there exists a ϑ_0 such that $\alpha(\vartheta_0, \mathcal{K}\vartheta_0) \geqslant 1$, $\mu(\vartheta_0, \mathcal{K}\vartheta_0)$
$\leqslant 1$. Consider $\vartheta_1 = \mathcal{K}\vartheta_0$ and $\vartheta_2 = \mathcal{K}\vartheta_1$. Using induction, we can easily build up a sequence $\{\vartheta_r\}_{r=0}^{\infty}$ such that $\vartheta_{r+1} = \mathcal{K}\vartheta_r$ for all $r \in \{0\} \cup \mathbb{N}$. Again, we have $\alpha(\vartheta_0, \vartheta_1) \geqslant 1$ and by the given condition (D2) the mapping $\mathcal{K}$ is a triangular α-orbital admissible mapping. Thus, we get

$$\begin{aligned}
&\alpha(\vartheta_0, \vartheta_1) \geqslant 1 \Rightarrow \alpha(\mathcal{K}\vartheta_0, \mathcal{K}^2\vartheta_0) = \alpha(\vartheta_1, \vartheta_2) \geqslant 1, \text{ and}\\
&\alpha(\vartheta_1, \vartheta_2) \geqslant 1 \Rightarrow \alpha(\mathcal{K}\vartheta_1, \mathcal{K}^2\vartheta_1) = \alpha(\vartheta_2, \vartheta_3) \geqslant 1.
\end{aligned}$$

Proceeding in this way, using $(D2)$, we obtain, $\alpha(\vartheta_r, \vartheta_{r+1}) \geqslant 1$ for all $r \in \{0\} \cup \mathbb{N}$. In a similar way, using triangular μ-suborbital admissible property we can show that $\mu(\vartheta_r, \vartheta_{r+1}) \leqslant 1$ for all $r \in \{0\} \cup \mathbb{N}$. Without loss of generality, we may assume that $\vartheta_r \neq \vartheta_{r+1}$, $\forall r \in \{0\} \cup \mathbb{N}$. We can assume that $\delta_\rho(\vartheta_r, \vartheta_{r+1}) > 0$ for $r \in \{0\} \cup \mathbb{N}$. Suppose, on the contrary, that for some r, $\delta_\rho(\vartheta_r, \vartheta_{r+1}) = 0$, then $\vartheta_{r+1} = \vartheta_r$ implies $\mathcal{K}\vartheta_r = \vartheta_r$. Consequently, our proof would be over. Next, we want to show that $\delta_\rho(\vartheta_r, \vartheta_{r+1}) \to 0$ as $r \to \infty$. Now putting $\vartheta = \vartheta_{r-1}, u = \vartheta_r$ in (2.5) and keeping in mind $\alpha(\vartheta_r, \vartheta_{r+1}) \geqslant 1, \mu(\vartheta_r, \vartheta_{r+1}) \leqslant 1$ for all r, we have

$$\begin{aligned}
&\mathcal{H}(1, \theta(\delta_\rho(\vartheta_r, \vartheta_{r+1})))\\
&= \mathcal{H}(1, \theta(\delta_\rho(\mathcal{K}\vartheta_{r-1}, \mathcal{K}\vartheta_r)))\\
&\leqslant \mathcal{H}(\alpha(\vartheta_{r-1}, \vartheta_r), \theta(\delta_\rho(\mathcal{K}\vartheta_{r-1}, \mathcal{K}\vartheta_r)))\\
&\leqslant \mathcal{F}(\mu(\vartheta_{r-1}, \vartheta_r), \theta(\frac{\delta_\rho(\vartheta_{r-1}, \mathcal{K}\vartheta_{r-1}) + \delta_\rho(\vartheta_r, \mathcal{K}\vartheta_r)}{2})\\
&\quad - \xi(\delta_\rho(\vartheta_{r-1}, \mathcal{K}\vartheta_{r-1}), \delta_\rho(\vartheta_r, \mathcal{K}\vartheta_r)))\\
&\leqslant \mathcal{F}(1, \theta(\frac{\delta_\rho(\vartheta_{r-1}, \mathcal{K}\vartheta_{r-1}) + \delta_\rho(\vartheta_r, \mathcal{K}\vartheta_r)}{2}) - \xi(\delta_\rho(\vartheta_{r-1}, \mathcal{K}\vartheta_{r-1}), \delta_\rho(\vartheta_r, \mathcal{K}\vartheta_r))).
\end{aligned}$$

Since the pair $(\mathcal{F}, \mathcal{H})$ is a special upper class of type-I, consequently we have

$$\begin{aligned}
&\theta(\delta_\rho(\mathcal{K}\vartheta_{r-1}, \mathcal{K}\vartheta_r))\\
&\leqslant \theta(\frac{\delta_\rho(\vartheta_{r-1}, \mathcal{K}\vartheta_{r-1}) + \delta_\rho(\vartheta_r, \mathcal{K}\vartheta_r)}{2})\\
&\quad - \xi(\delta_\rho(\vartheta_{r-1}, \mathcal{K}\vartheta_{r-1}), \delta_\rho(\vartheta_r, \mathcal{K}\vartheta_r))\\
&\Rightarrow \theta(\delta_\rho(\vartheta_r, \vartheta_{r+1})) \leqslant \theta(\frac{\delta_\rho(\vartheta_{r-1}, \vartheta_r) + \delta_\rho(\vartheta_r, \vartheta_{r+1})}{2})\\
&\quad - \xi(\delta_\rho(\vartheta_{r-1}, \vartheta_r), \delta_\rho(\vartheta_r, \vartheta_{r+1})).
\end{aligned} \tag{2.6}$$

Since $\delta_\rho(\vartheta_r,\vartheta_{r+1})>0$ for all $r\in\{0\}\cup\mathbb{N}$, and θ is a strictly non-decreasing function, thus we have

$$\begin{aligned}\delta_\rho(\vartheta_r,\vartheta_{r+1}) &\leqslant \frac{\delta_\rho(\vartheta_{r-1},\vartheta_r)+\delta_\rho(\vartheta_r,\vartheta_{r+1})}{2}\\ &\Rightarrow \delta_\rho(\vartheta_r,\vartheta_{r+1})\leqslant \delta_\rho(\vartheta_{r-1},\vartheta_r).\end{aligned}$$

Thus $\{\delta_\rho(\vartheta_r,\vartheta_{r+1})\}_{r=0}^{\infty}$ becomes a non-increasing sequence. Hence, it will converge to some non negative real number, say $a^*\in[0,\infty)$. We claim $a^*=0$. Suppose not, i.e., $a^*>0$. By considering the limit as $r\to\infty$ in (2.6), and using continuity of θ,ξ, we obtain

$$\begin{aligned}\theta(a^*) &\leqslant \theta(\frac{a^*+a^*}{2})-\xi(a^*,a^*)\\ &\Rightarrow \theta(a^*)\leqslant\theta(a^*)-\xi(a^*,a^*),\end{aligned}$$

a contradiction, since we have assumed $a^*>0$. Thus, we have $\lim_{r\to\infty}\delta_\rho(\vartheta_r,\vartheta_{r+1})=0$. Our next goal is to show that $\lim_{s>r,r\to\infty}\delta_\rho(\vartheta_r,\vartheta_s)=0$. We now show this by using the method of contradiction, i.e., suppose that $\lim_{s>r,r\to\infty}\delta_\rho(\vartheta_r,\vartheta_s)\neq 0$. Then for each $\tau>0$, there exist two sub-sequences $\{r(n)\}_{n=1}^{\infty}$, $\{s(n)\}_{n=1}^{\infty}$ such that $\delta_\rho(\vartheta_{r(n)},\vartheta_{s(n)})\geqslant\tau,\delta_\rho(\vartheta_{r(n)},\vartheta_{s(n)-1})<\tau$, where $r(n)<s(n)$. Again, by Lemma 1.3 and Lemma 2.14, we have $\alpha(\vartheta_r,\vartheta_s)\geqslant 1,\mu(\vartheta_r,\vartheta_s)\leqslant 1$ for all $r,s\in\{0\}\cup\mathbb{N}$ with $r<s$. Considering $\vartheta=\vartheta_{r(n)-1},u=\vartheta_{s(n)-1}$ in (2.5), we have

$$\begin{aligned}&\mathcal{H}(1,\theta(\delta_\rho(\vartheta_{r(n)},\vartheta_{s(n)})))\\ &=\mathcal{H}(1,\theta(\delta_\rho(\mathcal{K}\vartheta_{r(n)-1},\mathcal{K}\vartheta_{s(n)-1})))\\ &\leqslant\mathcal{H}(\alpha(\vartheta_{r(n)-1},\vartheta_{s(n)-1}),\theta(\delta_\rho(\mathcal{K}\vartheta_{r(n)-1},\mathcal{K}\vartheta_{s(n)-1})))\\ &\leqslant\mathcal{F}(\mu(\vartheta_{r(n)-1},\vartheta_{s(n)-1}),\theta(\frac{\delta_\rho(\vartheta_{r(n)-1},\mathcal{K}\vartheta_{r(n)-1})+\delta_\rho(\vartheta_{s(n)-1},\mathcal{K}\vartheta_{s(n)-1})}{2})\\ &-\xi(\delta_\rho(\vartheta_{r(n)-1},\mathcal{K}\vartheta_{r(n)-1}),\delta_\rho(\vartheta_{s(n)-1},\mathcal{K}\vartheta_{s(n)-1})))\\ &\leqslant\mathcal{F}(1,\theta(\frac{\delta_\rho(\vartheta_{r(n)-1},\mathcal{K}\vartheta_{r(n)-1})+\delta_\rho(\vartheta_{s(n)-1},\mathcal{K}\vartheta_{s(n)-1})}{2})\\ &-\xi(\delta_\rho(\vartheta_{r(n)-1},\mathcal{K}\vartheta_{r(n)-1}),\delta_\rho(\vartheta_{s(n)-1},\mathcal{K}\vartheta_{s(n)-1}))).\end{aligned}$$

Since the pair $(\mathcal{F},\mathcal{H})$ is a special upper class of type-I, consequently we have

(2.7)

$$\begin{aligned}&\theta(\delta_\rho(\mathcal{K}\vartheta_{r(n)-1},\mathcal{K}\vartheta_{s(n)-1}))\\ &\leqslant\theta(\frac{\delta_\rho(\vartheta_{r(n)-1},\mathcal{K}\vartheta_{r(n)-1})+\delta_\rho(\vartheta_{s(n)-1},\mathcal{K}\vartheta_{s(n)-1})}{2})\\ &-\xi(\delta_\rho(\vartheta_{r(n)-1},\mathcal{K}\vartheta_{r(n)-1}),\delta_\rho(\vartheta_{s(n)-1},\mathcal{K}\vartheta_{s(n)-1}))\\ &\Rightarrow\theta(\delta_\rho(\vartheta_{r(n)},\vartheta_{s(n)}))\\ &\leqslant\theta(\frac{\delta_\rho(\vartheta_{r(n)-1},\vartheta_{r(n)})+\delta_\rho(\vartheta_{s(n)-1},\vartheta_{s(n)})}{2})-\xi(\delta_\rho(\vartheta_{r(n)-1},\vartheta_{r(n)}),\delta_\rho(\vartheta_{s(n)-1},\vartheta_{s(n)})).\end{aligned}$$

Now, for sufficiently large value of n, we can arrive at a contradiction, since L.H.S of (2.7) is always greater than or equal to $\theta(\tau)(>0)$ whereas in R.H.S

$\theta(\frac{\delta_\rho(\vartheta_{r(n)-1},\vartheta_{r(n)})+\delta_\rho(\vartheta_{s(n)-1},\vartheta_{s(n)})}{2}) \to 0$ as $n \to \infty$ and $\xi(\delta_\rho(\vartheta_{r(n)-1},\vartheta_{r(n)}),$ $\delta_\rho(\vartheta_{s(n)-1},\vartheta_{s(n)})) > 0$ for all $n \in \{0\} \cup \mathbb{N}$. Thus, we must have $\tau = 0$. Hence, we obtain $\lim_{s>r, r\to\infty} \delta_\rho(\vartheta_r,\vartheta_s) = 0$. Now, by completeness there exists a point $\vartheta^* \in \aleph$ such that $\lim_{r\to\infty} \delta_\rho(\vartheta_r,\vartheta^*) = \delta_\rho(\vartheta^*,\vartheta^*) = \lim_{s>r,r\to\infty} \delta_\rho(\vartheta_r,\vartheta_s) = 0$. Next, we wish to show that $\vartheta^* = \mathcal{K}\vartheta^*$. To do this, first we suppose that $\mathcal{K}$ is continuous. Then,

$$\vartheta^* = \lim_{r\to\infty} \vartheta_{r+1} = \lim_{r\to\infty} \mathcal{K}\vartheta_r = \mathcal{K} \lim_{r\to\infty} \vartheta_r = \mathcal{K}\vartheta^*.$$

Now we suppose that $\mathcal{K}$ is not a continuous mapping, and hence we consider condition $\widehat{(D3)}$. Putting $\vartheta = \vartheta_{r(n)}, u = \vartheta^*$ in (2.5), we obtain

$$\begin{aligned}
&\mathcal{H}(1, \theta(\delta_\rho(\vartheta_{r(n)+1}, \mathcal{K}\vartheta^*)))\\
&= \mathcal{H}(1, \theta(\delta_\rho(\mathcal{K}\vartheta_{r(n)}, \mathcal{K}\vartheta^*)))\\
&\leqslant \mathcal{H}(\alpha(\vartheta_{r(n)}, \vartheta^*), \theta(\delta_\rho(\mathcal{K}\vartheta_{r(n)}, \mathcal{K}\vartheta^*)))\\
&\leqslant \mathcal{F}(\mu(\vartheta_{r(n)}, \vartheta^*), \theta(\frac{\delta_\rho(\vartheta_{r(n)}, \mathcal{K}\vartheta_{r(n)}) + \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*)}{2})\\
&\quad - \xi(\delta_\rho(\vartheta_{r(n)}, \mathcal{K}\vartheta_{r(n)}), \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*)))\\
&\leqslant \mathcal{F}(1, \theta(\frac{\delta_\rho(\vartheta_{r(n)}, \mathcal{K}\vartheta_{r(n)}) + \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*)}{2})\\
&\quad - \xi(\delta_\rho(\vartheta_{r(n)}, \mathcal{K}\vartheta_{r(n)}), \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*))).
\end{aligned}$$

Since the pair $(\mathcal{F}, \mathcal{H})$ is a special upper class of type-I, consequently we have

$$\begin{aligned}
&\theta(\delta_\rho(\mathcal{K}\vartheta_{r(n)}, \mathcal{K}\vartheta^*))\\
&\leqslant \theta(\frac{\delta_\rho(\vartheta_{r(n)}, \mathcal{K}\vartheta_{r(n)}) + \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*)}{2}) - \xi(\delta_\rho(\vartheta_{r(n)}, \mathcal{K}\vartheta_{r(n)}), \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*))\\
&\Rightarrow \theta(\delta_\rho(\vartheta_{r(n)+1}, \mathcal{K}\vartheta^*))\\
&\leqslant \theta(\frac{\delta_\rho(\vartheta_{r(n)}, \vartheta_{r(n)+1}) + \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*)}{2}) - \xi(\delta_\rho(\vartheta_{r(n)}, \vartheta_{r(n)+1}), \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*)).
\end{aligned} \tag{2.8}$$

Taking limit as $n \to \infty$ in (2.8) and using the fact $\delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*) \leqslant \limsup_{r\to\infty} \delta_\rho(\vartheta_r, \mathcal{K}\vartheta^*)$, we have

$$\theta(\delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*)) \leqslant \theta(\frac{\delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*)}{2}) - \xi(0, \delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*)),$$

a contradiction. Thus, we must have $\delta_\rho(\vartheta^*, \mathcal{K}\vartheta^*) = 0 \Rightarrow \vartheta^* = \mathcal{K}\vartheta^*$. □

Our next theorem deals with the uniqueness of fixed point. First, we state hypothesis $(\mathbf{U}_1)$.

$(\mathbf{U}_1)$: For all $\vartheta_1^*, \vartheta_2^* \in \text{Fix}(\mathcal{K})$, $\alpha(\vartheta_1^*, \vartheta_2^*) \geqslant 1, \mu(\vartheta_1^*, \vartheta_2^*) \leqslant 1$.

Theorem 2.21. *Suppose that all the hypotheses of Theorem 2.20 are satisfied together with* $(\boldsymbol{U}_1)$*. Then* $\mathcal{K}$ *has a unique fixed point.*

Proof. First of all, observe that if ϑ^* is a fixed point of $\mathcal{K}$, then we can easily show that $\delta_\rho(\vartheta^*,\vartheta^*)=0$. Now, let us suppose that $\vartheta_1^*,\vartheta_2^*$ are two fixed points of $\mathcal{K}$. Hence, by hypothesis $(\mathbf{U}_1)$ we have $\alpha(\vartheta_1^*,\vartheta_2^*)\geqslant 1,\mu(\vartheta_1^*,\vartheta_2^*)\leqslant 1$. Now, putting $\vartheta=\vartheta_1^*,u=\vartheta_2^*$ in (2.5), we have

$$\begin{aligned}
&\mathcal{H}(1,\theta(\delta_\rho(\vartheta_1^*,\vartheta_2^*)))\\
&=\mathcal{H}(\alpha(\vartheta_1^*,\vartheta_2^*),\theta(\delta_\rho(\mathcal{K}\vartheta_1^*,\mathcal{K}\vartheta_2^*)))\\
&\leqslant \mathcal{F}(\mu(\vartheta_1^*,\vartheta_2^*),\theta(\frac{\delta_\rho(\vartheta_1^*,\mathcal{K}\vartheta_1^*)+\delta_\rho(\vartheta_2^*,\mathcal{K}\vartheta_2^*)}{2})-\xi(\delta_\rho(\vartheta_1^*,\mathcal{K}\vartheta_1^*),\delta_\rho(\vartheta_2^*,\mathcal{K}\vartheta_2^*)))\\
&\leqslant \mathcal{F}(1,\theta(\frac{\delta_\rho(\vartheta_1^*,\vartheta_1^*)+\delta_\rho(\vartheta_2^*,\vartheta_2^*)}{2})-\xi(\delta_\rho(\vartheta_1^*,\vartheta_1^*),\delta_\rho(\vartheta_2^*,\vartheta_2^*))).
\end{aligned}$$

Since the pair $(\mathcal{F},\mathcal{H})$ is a special upper class of type-I, consequently we have

$$\begin{aligned}
&\theta(\delta_\rho(\vartheta_1^*,\vartheta_2^*))\leqslant\theta(\frac{\delta_\rho(\vartheta_1^*,\vartheta_1^*)+\delta_\rho(\vartheta_2^*,\vartheta_2^*)}{2})-\xi(\delta_\rho(\vartheta_1^*,\vartheta_1^*),\delta_\rho(\vartheta_2^*,\vartheta_2^*))\\
&\Rightarrow\theta(\delta_\rho(\vartheta_1^*,\vartheta_2^*))\leqslant\theta(0)-\xi(0,0).
\end{aligned}$$

Clearly, we arrive at a contradiction if we assume $\delta_\rho(\vartheta_1^*,\vartheta_2^*)>0$. Thus we have $\delta_\rho(\vartheta_1^*,\vartheta_2^*)=0\Rightarrow\vartheta_1^*=\vartheta_2^*$. Hence our proof is completed. □

Remark 2: In Definition 2.19 if one consider $\mathcal{H}(s,t)=t,\mathcal{F}(s,t)=t$ with $\theta(\tau)=\tau,\xi(c,d)=\left(\frac{1}{2}-\varrho\right)(c+d)$, where $\varrho\in[0,\frac{1}{2})$ then the contraction becomes famous Kannan-type [12] contraction in setting of DCMLS.

Example 2.22. Let $\aleph=\{-1\}\cup\mathbb{R}_+$. Let $\delta_\rho:\aleph\times\aleph\to\mathbb{R}_+$ be a mapping, defined as

$$\delta_\rho(\vartheta,\vartheta)=\vartheta,\ \text{if } \vartheta\in\mathbb{R}_+,$$
$$\delta_\rho(-1,-1)=2,\delta_\rho(\vartheta,-1)=\delta_\rho(-1,\vartheta)=1,$$
$$\delta_\rho(\vartheta,u)=\max\{\vartheta,u\}\text{ if }\vartheta,u\in\mathbb{R}_+.$$

Here δ_ρ is a continuous complete DCMLS. Let $\gamma,\lambda:\aleph\times\aleph\to[1,\infty)$ be two mappings given by $\gamma(\vartheta,u)=1+|\vartheta|+|u|$, $\lambda(\vartheta,u)=1+|\vartheta||u|$, for all $\vartheta,u\in\aleph$. Let $\alpha,\mu:\aleph\times\aleph\to\mathbb{R}_+$ be two mappings defined by

$$\alpha(\vartheta,u)=\begin{cases}1, & \text{if } u\leqslant\vartheta \text{ and } u,v\in[0,1],\\ 0, & \text{otherwise.}\end{cases}$$

$$\beta(\vartheta,u)=\begin{cases}2, & \text{if } \vartheta u\geqslant 0 \text{ and } u,v\in[0,1],\\ 0, & \text{otherwise.}\end{cases}$$

We now define two mappings $\mathcal{K},\mathcal{L}:\aleph\to\aleph$ as

$$\mathcal{K}(\vartheta)=\begin{cases}\vartheta, & \text{if } \vartheta=-1,\\ log\left(1+\frac{\tau\vartheta}{6}\right), & \text{if } \vartheta\in[0,1],\\ log\left(1+\frac{\tau}{6}\right)+(\vartheta-1), & \text{if } \vartheta\in(1,\infty).\end{cases}$$

$$\mathcal{L}(\vartheta) = \begin{cases} \vartheta, & \text{if } \vartheta = -1, \\ log\Big(1 + \frac{\tau\vartheta}{6}\Big), & \text{if } \vartheta \in [0,1], \\ log\Big(1 + \frac{\tau}{6}\Big) + log(\vartheta), & \text{if } \vartheta \in (1,\infty). \end{cases}$$

Clearly, $\mathcal{K}, \mathcal{L}$ are both continuous mappings. Let us consider a particular form of $\mathfrak{G}$, i.e., $\mathfrak{G}(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - \eta(\max\{\xi_2, \xi_3, \xi_4\})$, where $\eta(t) = \frac{\tau t}{2}, \tau \in (0,1)$. Thus, we have to show

$$\alpha(\vartheta, \mathcal{K}\vartheta)\beta(u, \mathcal{L}u)\delta_\rho(\mathcal{K}\vartheta, \mathcal{L}u) \leqslant \frac{\tau}{2}\max\{\delta_\rho(\vartheta, u), \delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{L}u)\}, \tag{2.9}$$

for all $\vartheta, u \in \aleph$. Observe that inequality (2.9) holds trivially for all $\vartheta, u \in \aleph$ for which either $\alpha(\vartheta, \mathcal{K}\vartheta) = 0$ or $\beta(u, \mathcal{L}u) = 0$. Thus, we have to verify inequality (2.9) for all those $\vartheta, u \in \aleph$ for which both $\alpha(\vartheta, \mathcal{K}\vartheta) \neq 0$ and $\beta(u, \mathcal{L}u) \neq 0$ implies $\vartheta, u \in [0,1]$. Consequently, we have the following

$$\begin{aligned} &\alpha(\vartheta, \mathcal{K}\vartheta)\beta(u, \mathcal{L}u)\delta_\rho(\mathcal{K}\vartheta, \mathcal{L}u) \\ &= 1 \cdot 2 \cdot \max\{log\Big(1 + \frac{\tau\vartheta}{6}\Big), log\Big(1 + \frac{\tau u}{6}\Big)\} \\ &\leqslant 2 \cdot \max\{\frac{\tau\vartheta}{6}, \frac{\tau u}{6}\} \\ &= 2 \cdot \frac{\tau}{6} \cdot \max\{\vartheta, u\} \\ &\leqslant \frac{\tau}{2}\max\{\delta_\rho(\vartheta, u), \delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{L}u)\}. \end{aligned}$$

Taking $\vartheta_0 = 1$ then $\alpha(1, \mathcal{K}1) = \alpha(1, log(1 + \frac{\tau}{6})) \geqslant 1$. Now, starting from ϑ_0, one can easily construct a sequence $\{\vartheta_r\}$ such that $\vartheta_{2r+1} = \mathcal{K}\vartheta_{2r}$ and $\vartheta_{2r+2} = \mathcal{L}\vartheta_{2r+1}$ with $\alpha(\vartheta_{2r}, \vartheta_{2r+1}) \geqslant 1, \beta(\vartheta_{2r+1}, \vartheta_{2r+2}) \geqslant 1$ for all $r \in \mathbb{N}$, where ϑ_r is given by

$$\begin{aligned} &\vartheta_r = log\Big(1 + \frac{\tau}{6}\vartheta_{r-1}\Big) \\ &\Rightarrow \vartheta_r = log\Big(1 + \frac{\tau}{6}log\Big(1 + \frac{\tau}{6}\vartheta_{r-2}\Big)\Big) \\ &\Rightarrow \vartheta_r = log\Big(1 + \frac{\tau}{6}log\Big(1 + \frac{\tau}{6}log\Big(1 + \frac{\tau}{6}\vartheta_{r-3}\Big)\Big)\Big) \\ &\vdots \\ &\Rightarrow \vartheta_r = log\Big(1 + \underbrace{\frac{\tau}{6}log\Big(1 + \frac{\tau}{6}log\Big(1 + \cdots + \frac{\tau}{6}log\Big(1 + \frac{\tau\vartheta_0}{6}\Big)\cdots\Big)\Big)}_{\text{(r-1)}}\Big). \end{aligned}$$

Again, we have

$$\begin{aligned}
&\lim_{r\to\infty}\frac{\gamma(\vartheta_{r+1},\vartheta_{r+2})}{\gamma(\vartheta_r,\vartheta_{r+1})}\lambda(\vartheta_{r+1},\vartheta_{r+s})\frac{\eta^{r+1}(\delta_\rho(\vartheta_0,\vartheta_1))}{\eta^r(\delta_\rho(\vartheta_0,\vartheta_1))}\\
&=\lim_{r\to\infty}1\cdot 1\cdot\frac{\tau^{r+1}}{2^{r+1}}\cdot\frac{2^r}{\tau^r}\\
&=\frac{\tau}{2}<1.
\end{aligned}$$

Also, it can be easily verified that $\{\gamma(\vartheta_r,\vartheta_{r+1})\}$ is a bounded sequence. Here all the conditions of Theorem 2.10 are satisfied and 0 is a fixed point of $\mathcal{K}$ and $\mathcal{L}$. Observe that the fixed point is not unique as -1 is another fixed point. This happens since $\alpha((-1),\mathcal{K}(-1))\not\geq 1,\beta((-1),\mathcal{L}(-1))\not\geq 1$. Next, we give numerical iteration for $\tau=0.87$ and 0.32 with initial point $\vartheta_0=0.3,0.5,0.7,1$.

Table 1: Iteration of u_n for $\tau=0.87$

Iterate for $\tau=0.87$	u_0=0.3	u_0=0.5	u_0=0.7	u_0=1
u_1	0.0426	0.0700	0.0967	0.1354
u_2	0.0062	0.0101	0.0139	0.0194
u_3	8.9210e-04	0.0015	0.0020	0.0028
u_4	1.2935e-04	2.1213e-04	2.9234e-04	4.0814e-04
u_5	1.8755e-05	3.0758e-05	4.2388e-05	5.9178e-05
u_6	2.7195e-06	4.4599e-06	6.1462e-06	8.5808e-06
u_7	3.9433e-07	6.4668e-07	8.9120e-07	1.2442e-06

Table 2: Iteration of u_n for $\tau=0.32$

Iterate for $\tau=0.32$	u_0=0.3	u_0=0.5	u_0=0.7	u_0=1
u_1	0.0159	0.0263	0.0367	0.0520
u_2	8.4622e-04	0.0014	0.0020	0.0028
u_3	4.5131e-05	7.4803e-05	1.0415e-04	1.4758e-04
u_4	2.4070e-06	3.9895e-06	5.5547e-06	7.8710e-06
u_5	1.2837e-07	2.1277e-07	2.9625e-07	4.1979e-07
u_6	6.8465e-09	1.1348e-08	1.5800e-08	2.2389e-08
u_7	3.6515e-10	6.0522e-10	8.4267e-10	1.1941e-09

Example 2.23. Let $\aleph=\{e_1,e_2,e_3,e_4\}$. Let $\delta_\rho:\aleph\times\aleph\to\mathbb{R}_+$ be a mapping, defined as

$$\delta_\rho(e_1,e_1)=0,\delta_\rho(e_2,e_2)=2,\delta_\rho(e_3,e_3)=4,\delta_\rho(e_4,e_4)=5,$$

$$\delta_\rho(e_1,e_2)=\delta_\rho(e_2,e_1)=8,\delta_\rho(e_1,e_3)=\delta_\rho(e_3,e_1)=3,\delta_\rho(e_1,e_4)=\delta_\rho(e_4,e_1)=6,$$
$$\delta_\rho(e_2,e_3)=\delta_\rho(e_3,e_2)=9,\delta_\rho(e_2,e_4)=\delta_\rho(e_4,e_2)=1,\delta_\rho(e_3,e_4)=\delta_\rho(e_4,e_3)=2.$$

Now we define a mapping $\mathcal{K}:\aleph\to\aleph$ as

$$\mathcal{K}(e_1)=e_1,\mathcal{K}(e_2)=e_3,\mathcal{K}(e_3)=e_1,\mathcal{K}(e_4)=e_4.$$

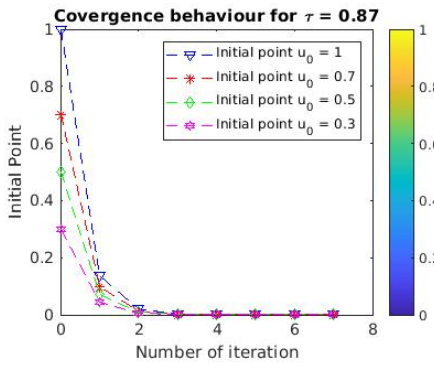

Figure 1: Interpolation for Data 1

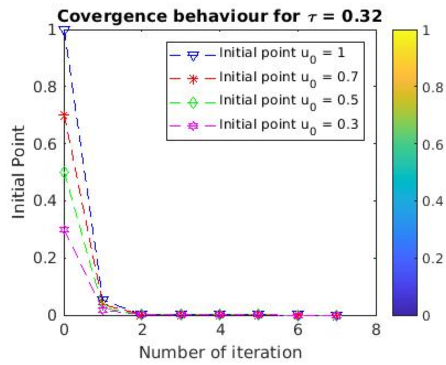

Figure 2: Interpolation for Data 2

Next, we define two mappings $\alpha, \mu : \aleph \times \aleph \to \mathbb{R}_+$ as

$$\alpha(\vartheta, u) = \begin{cases} 1, & \text{if } \vartheta, u \in \{e_1, e_2, e_3\}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu(\vartheta, u) = \begin{cases} 0.9, & \text{if } \vartheta, u \in \{e_1, e_2, e_3\}, \\ 2, & \text{otherwise.} \end{cases}$$

Finally, we define two mappings $\gamma, \lambda : \aleph \times \aleph \to [1, \infty)$ as $\gamma(e_i, e_i) = \lambda(e_i, e_i) = 1$, for all $i \in \{1, 2, 3, 4\}$,

$$\gamma(e_1, e_2) = \gamma(e_2, e_1) = 2, \gamma(e_1, e_3) = \gamma(e_3, e_1) = 3, \gamma(e_1, e_4) = \gamma(e_4, e_1) = 4,$$

$$\gamma(e_2, e_3) = \gamma(e_3, e_2) = 3, \gamma(e_2, e_4) = \gamma(e_4, e_2) = 7, \gamma(e_3, e_4) = \gamma(e_4, e_3) = 6,$$

and

$$\lambda(e_1, e_2) = \lambda(e_2, e_1) = 1, \lambda(e_1, e_3) = \lambda(e_3, e_1) = 5, \lambda(e_1, e_4) = \lambda(e_4, e_1) = 2,$$

$$\lambda(e_2, e_3) = \lambda(e_3, e_2) = 3, \lambda(e_2, e_4) = \lambda(e_4, e_2) = 4, \lambda(e_3, e_4) = \lambda(e_4, e_3) = 3.$$

The reader can check that δ_ρ is a continuous complete DCMLS, which is not a partial metric space (since $\delta_\rho(e_4, e_4) \nleqslant \delta_\rho(e_4, e_2)$) as well as not a metric like space (since $\delta_\rho(e_1, e_4) \nleqslant \delta_\rho(e_1, e_3) + \delta_\rho(e_3, e_4)$). Now we verify that $\mathcal{K}$ is a $(\alpha, \mu, \mathcal{F}, \mathcal{H}) - (\theta, \xi)$ weakly contractive mapping for $\theta(t) = 2t, \xi(c, d) = \frac{1}{7}\max\{c, d\}$ with $\mathcal{H}(\tau, s) = s, \mathcal{F}(\tau, s) = s$. Basically, we want to show that the weak contraction is valid for all $\vartheta, u \in \aleph$ with $\alpha(\vartheta, u) \geqslant 1, \mu(\vartheta, u) \leqslant 1$. Clearly, $\alpha(\vartheta, u) \geqslant 1, \mu(\vartheta, u) \leqslant 1$ implies $\vartheta, u \in \{e_1, e_2, e_3\}$. Now we have the following cases.

Case I: If $\{\vartheta, u\} \subset \{e_1, e_3\}$, then $\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u) = \delta_\rho(e_1, e_1) = 0$, and inequality (2.5) holds trivially.

Case II: If $\vartheta = e_2, u = e_1$, then $\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u) = \delta_\rho(e_3, e_1) = 3, \delta_\rho(\vartheta, \mathcal{K}\vartheta) = \delta_\rho(e_2, e_3) = 9, \delta_\rho(u, \mathcal{K}u) = \delta_\rho(e_1, e_1) = 0$. Now $\frac{\delta_\rho(\vartheta, \mathcal{K}\vartheta) + \delta_\rho(u, \mathcal{K}u)}{2} = \frac{9}{2}$. Hence,

$\theta(\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u)) = 6$, $\theta(\frac{\delta_\rho(\vartheta,\mathcal{K}\vartheta)+\delta_\rho(u,\mathcal{K}u)}{2}) = 9$, and $\xi(\delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{K}u)) = \frac{9}{7}$. Consequently, we have

$$\theta(\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u)) \leqslant \theta(\frac{\delta_\rho(\vartheta, \mathcal{K}\vartheta) + \delta_\rho(u, \mathcal{K}u)}{2}) - \xi(\delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{K}u)).$$

Case III: If $\vartheta = e_2, u = e_2$, then $\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u) = \delta_\rho(e_3, e_3) = 4, \delta_\rho(\vartheta, \mathcal{K}\vartheta) = \delta_\rho(e_2, e_3) = 9, \delta_\rho(u, \mathcal{K}u) = \delta_\rho(e_2, e_3) = 9$. Now $\frac{\delta_\rho(\vartheta,\mathcal{K}\vartheta)+\delta_\rho(u,\mathcal{K}u)}{2} = 9$. Hence, $\theta(\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u)) = 8$, $\theta(\frac{\delta_\rho(\vartheta,\mathcal{K}\vartheta)+\delta_\rho(u,\mathcal{K}u)}{2}) = 18$, and $\xi(\delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{K}u)) = \frac{9}{7}$. Consequently, we have

$$\theta(\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u)) \leqslant \theta(\frac{\delta_\rho(\vartheta, \mathcal{K}\vartheta) + \delta_\rho(u, \mathcal{K}u)}{2}) - \xi(\delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{K}u)).$$

Case IV: If $\vartheta = e_2, u = e_3$, then $\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u) = \delta_\rho(e_3, e_1) = 3, \delta_\rho(\vartheta, \mathcal{K}\vartheta) = \delta_\rho(e_2, e_3) = 9, \delta_\rho(u, \mathcal{K}u) = \delta_\rho(e_3, e_1) = 3$. Now $\frac{\delta_\rho(\vartheta,\mathcal{K}\vartheta)+\delta_\rho(u,\mathcal{K}u)}{2} = 6$. Hence, $\theta(\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u)) = 6$, $\theta(\frac{\delta_\rho(\vartheta,\mathcal{K}\vartheta)+\delta_\rho(u,\mathcal{K}u)}{2}) = 12$, and $\xi(\delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{K}u)) = \frac{9}{7}$. Consequently, we have

$$\theta(\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u)) \leqslant \theta(\frac{\delta_\rho(\vartheta, \mathcal{K}\vartheta) + \delta_\rho(u, \mathcal{K}u)}{2}) - \xi(\delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{K}u)).$$

Thus, $\mathcal{K}$ is a $(\alpha, \mu, \mathcal{F}, \mathcal{H}) - (\theta, \xi)$ weakly contractive mapping. Observe that in this example the inequality given by (2.5) does not hold if one does not consider $\vartheta, u \in \aleph$ with $\alpha(\vartheta, u) \geqslant 1, \mu(\vartheta, u) \leqslant 1$. To check this consider $\vartheta = e_1, u = e_4$. Then $\alpha(e_1, e_4) \ngeqslant 1, \mu(e_1, e_4) \nleqslant 1$, and we have

$$\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u) = \delta_\rho(e_1, e_4) = 6, \delta_\rho(\vartheta, \mathcal{K}\vartheta) = \delta_\rho(e_1, e_1) = 0, \delta_\rho(u, \mathcal{K}u) = \delta_\rho(e_4, e_4) = 5.$$

Now $\frac{\delta_\rho(\vartheta,\mathcal{K}\vartheta)+\delta_\rho(u,\mathcal{K}u)}{2} = \frac{5}{2}$. Hence, $\theta(\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u)) = 12$, $\theta(\frac{\delta_\rho(\vartheta,\mathcal{K}\vartheta)+\delta_\rho(u,\mathcal{K}u)}{2}) = 5$, and $\xi(\delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{K}u)) = \frac{5}{7}$. Consequently, we have

$$\theta(\delta_\rho(\mathcal{K}\vartheta, \mathcal{K}u)) \nleqslant \theta(\frac{\delta_\rho(\vartheta, \mathcal{K}\vartheta) + \delta_\rho(u, \mathcal{K}u)}{2}) - \xi(\delta_\rho(\vartheta, \mathcal{K}\vartheta), \delta_\rho(u, \mathcal{K}u)).$$

Clearly, $\mathcal{K}$ is a continuous function and all the conditions of Theorem 2.20 are satisfied with $\vartheta_0 = e_2$. Here, e_1 is a fixed point of $\mathcal{K}$ with $\delta_\rho(e_1, e_1) = 0$.

Note: In our main results, we explore the idea of orbital admissible mappings. Consequently, the following fixed point results can be derived as a corollaries from our main results.

- fixed point results in DCMLS endowed with a graph;
- fixed point results in DCMLS endowed with a partial order;
- fixed point results in DCMLS endowed with a binary relation;
- fixed point results for cyclic mappings via closed subsets of a DCMLS.

3. Application

To show the applicability of our obtained results, here we consider the following integral equations.

$$(3.1)\qquad \begin{aligned} u(r) &= \int_c^d A(r,s)\Omega_1(s,u(s))ds;\\ u(r) &= \int_c^d A(r,s)\Omega_2(s,u(s))ds, \end{aligned}$$

where $\Omega_1, \Omega_2 : [0,1] \times \mathbb{R} \to \mathbb{R}_+$ are continuous functions and A is measurable at $r \in [0,1]$ for every $s \in [0,1]$ and $A \in C\Big([0,1]^2, \mathbb{R}_+\Big)(= \aleph)$. Next, we define two operators $\mathcal{K}, \mathcal{L} : \aleph \to \aleph$ as

$$(3.2)\qquad \begin{aligned} \mathcal{K}u(r) &= \int_c^d A(r,s)\Omega_1(s,u(s))ds;\\ \mathcal{L}u(r) &= \int_c^d A(r,s)\Omega_2(s,u(s))ds, \end{aligned}$$

with DCMLS as

$$\delta_\rho(u,v) = \sup_{r\in[c,d]} (u(r)+v(r))^2, \gamma(u,v) = 2 + \sup_{r\in[c,d]} (|u(r)|^2 + |v(r)|^2),$$
$$\lambda(u,v) = 2 + \frac{\sup_{r\in[c,d]}(|u(r)||v(r)|)}{1+\sup_{r\in[c,d]}(|u(r)||v(r)|)}.$$

It can be easily verified that $(\aleph, \delta_\rho)$ is a complete continuous DCMLS. Suppose $\aleph$ is endowed with a partial order σ defined by

$$u \;\sigma\; v \Leftrightarrow v(r) \leqslant u(r), \text{ for all } r \in [c,d].$$

Theorem 3.1. *Let $\mathcal{K}, \mathcal{L}$ be two operators defined by (3.2). Assume that the operators satisfy the following properties:*
(P_1) Ω_1, Ω_2 are two continuous functions from $[c,d] \times \mathbb{R} \to \mathbb{R}_+$ such that

$$[\Omega_1(r,s) + \Omega_2(r,t)]^2 \leqslant \zeta(r)\phi([s+t]^2),$$

where $\zeta : [c,d] \to \mathbb{R}_+$ is a continuous function that satisfies the following condition

$$\sup_{r\in[c,d]} \Big(\int_c^d \zeta(r)dr\Big) < \frac{1}{\omega},$$

where $\omega \in (0,\infty)$, and ϕ is a continuous function such that $\phi \in \Phi$;
(P_2) $A : [c,d]^2 \to \mathbb{R}_+$ is measurable as well as continuous at $r \in [c,d]$ for every $s \in [c,d]$ such that

$$\int_c^d A(r,s)^2 ds < \omega;$$

(P_3) *for each* $s,t \in [c,d]$

(P_{3_i}) $\int_c^d A(s,t)\Omega_2(s,\int_c^d A(s,t)\Omega_1(t,u(t))dt)ds \leqslant \int_c^d A(s,t)\Omega_1(t,u(t))dt$ *whenever* $u\ \sigma\ \mathcal{K}u$, *and*

$(P_{3_{ii}})$ $\int_c^d A(s,t)\Omega_1(s,\int_c^d A(s,t)\Omega_2(t,u(t))dt)ds \leqslant \int_c^d A(s,t)\Omega_2(t,u(t))dt$ *whenever* $u\ \sigma\ \mathcal{L}u$;

(P_4) *suppose there exists a* $u_0 \in \aleph$ *such that* $u_0\sigma\mathcal{K}u_0$ *with* $\lim_{i\to\infty}\frac{\phi^{i+1}(\delta_\rho(u_0,\mathcal{K}u_0))}{\phi^i(\delta_\rho(u_0,\mathcal{K}u_0))} < \frac{1}{3}$.

Then, the system of integral equations (3.1) has a solution in $\aleph$.

Proof. First, we define a mapping $\alpha : \aleph^2 \to \mathbb{R}_+$ as

$$\alpha(u,v) = \begin{cases} 1, & \text{if } u\ \sigma\ v, \\ 0, & \text{otherwise.} \end{cases}$$

Here we assume $\alpha = \beta$. To show the pair $(\mathcal{K},\mathcal{L})$ is a (α,α)-orbital cyclic admissible pair, let $\alpha(u,\mathcal{K}u) \geqslant 1 \Rightarrow u\ \sigma\ \mathcal{K}u$. Now we show that $\mathcal{L}\mathcal{K}u(t) \leqslant \mathcal{K}u(t)$, for all $t \in [c,d]$.

$$\begin{aligned}\mathcal{K}u(r) &= \int_c^d A(r,s)\Omega_1(s,u(s))ds \\ &\geqslant \int_c^d A(r,s)\Omega_2(s,\int_c^d A(s,t)\Omega_1(t,u(t))dt)ds \text{ [since } u\ \sigma\ \mathcal{K}u] \\ &= \int_c^d A(r,s)\Omega_2(s,\mathcal{K}u(s))dt = \mathcal{L}\mathcal{K}u(r) \Rightarrow \alpha(\mathcal{K}u,\mathcal{L}\mathcal{K}u) \geqslant 1.\end{aligned}$$

In a similar way, using $(P_{3_{ii}})$, we can show that $\alpha(u,\mathcal{L}u) \geqslant 1 \Rightarrow \alpha(\mathcal{L}u,\mathcal{K}\mathcal{L}u) \geqslant 1$. Thus, the pair $(\mathcal{K},\mathcal{L})$ is an (α,α)-orbital cyclic admissible pair. By (P_4) there exists a $u_0 \in \aleph$ such that $u_0\sigma\mathcal{K}u_0$, i.e., $\alpha(u_0,\mathcal{K}u_0) \geqslant 1$. Next, using $u_0\sigma\mathcal{K}u_0$, we can construct a sequence $\{u_r\}_{r=1}^\infty$ such that $u_{2r+1} = \mathcal{K}u_{2r}, u_{2r+2} = \mathcal{L}u_{2r+1}$with $u_0\sigma u_1\sigma u_2\sigma\cdots$, i.e., the sequence $\{u_r\}$ becomes a decreasing sequence. Also, observe that $\alpha(u_r,u_{r+1}) \geqslant 1$ for all $r \in \mathbb{N}$. Since, $\{u_r\}_{r=1}^\infty$ is a decreasing sequence, consequently it will converge to some $u^* \in \aleph$. Let us take $m^* = \sup_{r\in[c,d]} u^*(r)$. Now, we have

$$\begin{aligned}&\lim_{i\to\infty}\frac{\gamma(u_{i+1},u_{i+2})}{\gamma(u_i,u_{i+1})}\lambda(u_{i+1},u_{i+s})\frac{\phi^{i+1}(\delta_\rho(u_0,u_1))}{\phi^i(\delta_\rho(u_0,u_1))} \\ &= \lim_{i\to\infty}\frac{2(1+{m^*}^2)}{2(1+{m^*}^2)}\cdot\left(2+\frac{{m^*}^2}{1+{m^*}^2}\right)\cdot\frac{\phi^{i+1}(\delta_\rho(u_0,u_1))}{\phi^i(\delta_\rho(u_0,u_1))} < 1.\end{aligned}$$

Next, we verify the following inequality

$$\alpha(u,\mathcal{K}u)\beta(v,\mathcal{L}v)\delta_\rho(\mathcal{K}u,\mathcal{L}v) \leqslant \phi(\delta_\rho(u,v)).$$

Suppose that either u is not related to $\mathcal{K}u$ or v is not related to $\mathcal{L}v$ by the relation σ, then we are done, by the definition of α. Next, we suppose that $u\sigma\mathcal{K}u$ and $v\sigma\mathcal{L}v$ then it will be sufficient to show that $\delta_\rho(\mathcal{K}u,\mathcal{L}v) \leqslant \phi(\delta_\rho(u,v))$. Now, using the Cauchy−Schwarz inequality, we have

$$\begin{aligned}
[\mathcal{K}u(r)+\mathcal{L}v(r)]^2 &= \Big[\int_c^d A(r,s)\Omega_1(s,u(s))ds + \int_c^d A(r,s)\Omega_2(s,v(s))ds\Big]^2 \\
&= \Big(\int_c^d A(r,s)\Big[\Omega_1(s,u(s))+\Omega_2(s,v(s))\Big]ds\Big)^2 \\
&\leqslant \Big[\Big(\int_c^d A(r,s)^2ds\Big)^{\frac{1}{2}}\Big(\int_c^d \Big(\Omega_1(s,u(s))+\Omega_2(s,v(s))\Big)^2ds\Big)^{\frac{1}{2}}\Big]^2 \\
&= \Big(\int_c^d A(r,s)^2ds\Big)\Big(\int_c^d \Big(\Omega_1(s,u(s))+\Omega_2(s,v(s))\Big)^2ds\Big) \\
&\leqslant \omega\cdot\Big(\int_c^d \zeta(s)\cdot\phi([u(s)+v(s)]^2)ds\Big) \\
&\leqslant \omega\cdot\phi(\delta_\rho(u,v))\cdot\Big(\int_c^d \zeta(s)ds\Big) \leqslant \omega\cdot\frac{1}{\omega}\cdot\phi(\delta_\rho(u,v)) \leqslant \phi(\delta_\rho(u,v)).
\end{aligned}$$

Considering supremum over $r \in [c,d]$, we have

$$\delta_\rho(\mathcal{K}u,\mathcal{L}v) \leqslant \phi(\delta_\rho(u,v)) \leqslant \phi(\max\{\delta_\rho(u,v),\delta_\rho(u,\mathcal{K}u),\delta_\rho(v,\mathcal{L}v)\}).$$

Also, $\mathcal{K},\mathcal{L}$ are both continuous functions. In fact, it can be easily checked that all the conditions of Theorem 2.10 are satisfied with $\mathfrak{G}(\xi_1,\xi_2,\xi_3,\xi_4) = \xi_1 - \phi(\max\{\xi_2,\xi_3,\xi_4\})$. Consequently, we obtain a common fixed point, i.e., the system (3.1) has a solution. □

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References

[1] T. Abdeljawad, N. Mlaiki, H. Aydi, and N. Souayah. Double controlled metric type spaces and some fixed point results. *Mathematics*, pages Paper No. 320, 6, 2018.

[2] Thabet Abdeljawad, Erdal Karapı nar, Sumati Kumari Panda, and Nabil Mlaiki. Solutions of boundary value problems on extended-Branciari b-distance. *J. Inequal. Appl.*, pages Paper No. 103, 16, 2020. doi:10.1186/s13660-020-02373-1.

[3] N. Alamgir, Q. Kiran, H. Aydi, and A. Mukheimer. A mizoguchi-takahashi type fixed point theorem in complete extended b-metric spaces. *Mathematics*, pages Paper No. 478, 7, 2019.

[4] B. Alqahtani, A. Fulga, and E. Karapinar. Non-unique fixed point results in extended b-metric space. *Mathematics*, pages Paper No. 478, 7, 2018.

[5] Badr Alqahtani, Andreea Fulga, and Erdal Karapı nar. Common fixed point results on an extended b-metric space. *J. Inequal. Appl.*, pages Paper No. 158, 15, 2018. `doi:10.1186/s13660-018-1745-4`.

[6] Badr Alqahtani, Erdal Karapı nar, and Ali Öztürk. On (α, ψ)-K-contractions in the extended b-metric space. *Filomat*, 32(15):5337–5345, 2018. `doi:10.2298/fil1815337a`.

[7] Arslan H. Ansari, Liliana Guran, and Abdul Latif. Fixed point problems concerning contractive type operators on KST-spaces. *Carpathian J. Math.*, 34(3):287–294, 2018.

[8] Stefan Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.*, 3:133–181, 1922. `doi:10.4064/fm-3-1-133-181`.

[9] I. Beg and A. R. Butt. Fixed points for weakly compatible mappings satisfying an implicit relation in partially ordered metric spaces. *Carpathian J. Math.*, 25(1):1–12, 2009.

[10] Vasile Berinde and Francesca Vetro. Common fixed points of mappings satisfying implicit contractive conditions. *Fixed Point Theory Appl.*, pages 2012:105, 8, 2012. `doi:10.1186/1687-1812-2012-105`.

[11] S. Czerwik. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostraviensis*, 1:5–11, 1993.

[12] R. Kannan. Some results on fixed points. *Bull. Calcutta Math. Soc.*, 60:71–76, 1968.

[13] M. S. Khan, M. Swaleh, and S. Sessa. Fixed point theorems by altering distances between the points. *Bull. Austral. Math. Soc.*, 30(1):1–9, 1984. `doi:10.1017/S0004972700001659`.

[14] N. Mlaiki, H. Aydi, N. Souayah, and T. Abdeljawad. Controlled metric type spaces and the related contraction principle. *Mathematics*, pages Paper No. 194, 6, 2018.

[15] Nabil Mlaiki. Double controlled metric-like spaces. *J. Inequal. Appl.*, pages Paper No. 189, 12, 2020. `doi:10.1186/s13660-020-02456-z`.

[16] S.K. Panda, E. Karapinar, and A. Atangana. A numerical schemes and comparisons for fixed point results with applications to the solutions of volterra integral equations in dislocated extended b-metric space. *Alex. Eng. J.*, 59(2):815–827, 2020.

[17] Valeriu Popa. Fixed point theorems for implicit contractive mappings. *Stud. Cercet. Ştiinţ. Ser. Mat. Univ. Bacău*, (7):127–133 (1999), 1997.

[18] Ovidiu Popescu. Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces. *Fixed Point Theory Appl.*, pages 2014:190, 12, 2014. `doi:10.1186/1687-1812-2014-190`.

[19] Simeon Reich. Some remarks concerning contraction mappings. *Canad. Math. Bull.*, 14:121–124, 1971. `doi:10.4153/CMB-1971-024-9`.

[20] Bessem Samet, Calogero Vetro, and Pasquale Vetro. Fixed point theorems for α-ψ-contractive type mappings. *Nonlinear Anal.*, 75(4):2154–2165, 2012. doi:10.1016/j.na.2011.10.014.

[21] Maria Samreen, Khansa Waheed, and Quanita Kiran. Multivalued φ contractions and fixed point theorems. *Filomat*, 32(4):1209–1220, 2018. doi:10.2298/fil1804209s.

[22] S. Sedghi, I. Altun, and N. Shobe. A fixed point theorem for multi-maps satisfying an implicit relation on metric spaces. *Appl. Anal. Discrete Math.*, 2(2):189–196, 2008. doi:10.2298/AADM0802189S.

[23] S. Sedghi, K. P. R. Rao, and N. Shobe. A general common fixed point theorem for multi-maps satisfying an implicit relation on fuzzy metric spaces. *Filomat*, 22(1):1–11, 2008. doi:10.2298/FIL0801001S.

[24] Wasfi Shatanawi, Kamaleldin Abodayeh, and Aiman Mukheimer. Some fixed point theorems in extended b-metric spaces. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, 80(4):71–78, 2018.

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