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Weakly 2-absorbing δ-primary elements

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Abstract. We prove some results on weakly δ-primary elements and weakly 2-absorbing δ-primary elements in a multiplicative lattice. A sufficient condition for a weakly δ-primary element (weakly 2-absorbing δ-primary element) to be a δ-primary element (2-absorbing δ-primary element) is proved. The concept of an expansion function of elements is introduced on a product of a finite number of lattices. Some results about weakly 2-absorbing δ-primary elements in a product of lattices are proved.

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1. Introduction

As a generalization of the concept of a prime ideal in a commutative ring with identity, Badawi [1] introduced the concept of a 2-absorbing ideal. The concept of a primary ideal was generalized by Badawi et. al. [2] and [3] to that of a 2-absorbing primary ideal and a weakly 2-absorbing primary ideal. Jayaram, et. al. [6] introduced the concepts of a 2-absorbing and a weakly 2-absorbing element in a multiplicative lattice, which generalize a prime and a weakly prime element in a multiplicative lattice. Calliap, et. al. [4] introduced the concepts of a 2-absorbing primary element and a weakly 2-absorbing primary element in a multiplicative lattice, which are generalizations of a primary and a weakly primary element.

Zhao [10] introduced and investigated the notions of an expansion of ideals and δ-primary ideals of a commutative ring, where δ is a mapping with some additional properties. Manjarekar and Bingi [7] introduced and investigated the notions of an expansion of elements and a δ-primary element in a multiplicative lattice. Fahid and Zhao [5] defined a 2-absorbing δ-primary ideal in a commutative ring. Nimbhorkar and Nehete [9], introduced the concepts of a weakly δ-primary element and a δ-twin zero in a multiplicative lattice and studied some result related to these concepts. Also Nimbhorkar and Nehete [8], introduced the concept of a 2-absorbing δ-primary element and a weakly 2-absorbing δ-primary element in a multiplicative lattice. They defined a δ-triple-zero in a multiplicative lattice.

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In this paper, we study some properties of a weakly δ-primary element and a weakly 2-absorbing δ-primary element in a multiplicative lattice. The concept of a weakly 2-absorbing δ-primary element unifies the concepts of a weakly 2-absorbing elements and weakly 2-absorbing primary elements under one frame. We also prove a characterization for weakly δ-primary elements and weakly 2-absorbing δ-primary elements in a multiplicative lattice. We also investigate some properties of a weakly 2-absorbing δ-primary element with respect to a homomorphism. The concept of an expansion function of elements namely, $\delta_\times$, is introduced and studied in a product of lattices. Some characterizations for a weakly 2-absorbing δ-primary element in a product of lattices are proved.

Throughout this paper, L denotes a compactly generated multiplicative lattice, with 1 compact, in which every finite product of compact elements is compact.

2. Definitions

We recall some concepts from multiplicative lattices.

Definition 2.1. A multiplicative lattice L is a complete lattice with commutative, associative and join distributive multiplication defined on it, in which the largest element 1 acts as a multiplicative identity.

Definition 2.2. An element $a \in L$ is said to be proper if $a < 1$.

Definition 2.3. A proper element $p \in L$ is called a prime element if $ab \leq p$ implies $a \leq p$ or $b \leq p$ where $a, b \in L$.

Definition 2.4. A proper element $p \in L$ is called a weakly prime element if $0 \neq ab \leq p$ implies $a \leq p$ or $b \leq p$ where $a, b \in L$.

Definition 2.5. The radical of $a \in L$ is defined as, $\sqrt{a} = \vee\{x \in L_* | x^n \leq a \text{ for some, } n \in Z_+\}$, where L_* is a set of compact elements of multiplicative lattice L.

Definition 2.6. A proper element $p \in L$ is called a primary element if $ab \leq p$ implies $a \leq p$ or $b \leq \sqrt{p}$ where $a, b \in L$.

Definition 2.7. A proper element p of L is called a 2-absorbing element of L if $a, b, c \in L$ and $abc \leq p$ implies $ab \leq p$ or $bc \leq p$ or $ac \leq p$.

Definition 2.8. A proper element p of L is called a 2-absorbing primary element of L if $a, b, c \in L$ and $abc \leq p$ implies $ab \leq p$ or $bc \leq \sqrt{p}$ or $ac \leq \sqrt{p}$.

Definition 2.9. A proper element p of L is called a weakly 2-absorbing element of L if $a, b, c \in L$ and $0 \neq abc \leq p$ implies $ab \leq p$ or $bc \leq p$ or $ac \leq p$.

3. Properties of 2-absorbing δ-primary elements

The following definitions are from Manjarekar and Bingi [7].

Definition 3.1. An expansion of elements, or an expansion function, is a function $\delta : L \to L$, such that the following conditions are satisfied:
(i) $a \leq \delta(a)$ for all $a \in L$ (ii) $a \leq b$ implies $\delta(a) \leq \delta(b)$ for all $a, b \in L$.

Example 3.2.

(1) The identity function $\delta_0 : L \to L$, where $\delta_0(a) = a$ for every $a \in L$, is an expansion of elements.

(2) $\mathbf{M} : L \to L$, where $\mathbf{M}(a) = \wedge\{m \in L | a \leq m,\ m \text{ is a maximal element }\}$, where a is a proper element of L and $\mathbf{M}(1) = 1$. Then $\mathbf{M}$ is an expansion of elements.

(3) $\delta_1 : L \to L$ define $\delta_1(a) = \sqrt{a}$, the radical of a. Then δ_1 is an expansion of elements.

Definition 3.3. For an expansion of elements δ, an element p of L is called δ-primary if $ab \leq p$ implies either $a \leq p$ or $b \leq \delta(p)$ for all $a, b \in L$.

Definition 3.4. (Nimbhorkar and Nehete [9]) For an expansion of elements δ, an element p of L is called weakly δ-primary if $0 \neq ab \leq p$ implies either $a \leq p$ or $b \leq \delta(p)$ for all $a, b \in L$.

Definition 3.5. (Nimbhorkar and Nehete [8]) A proper element p of L is called a 2-absorbing δ-primary element of L if $a, b, c \in L$ and $abc \leq p$ implies $ab \leq p$ or $bc \leq \delta(p)$ or $ac \leq \delta(p)$.

Definition 3.6. (Nimbhorkar and Nehete [8]) A proper element p of L is called a weakly 2-absorbing δ-primary element of L if $a, b, c \in L$ and $0 \neq abc \leq p$ implies $ab \leq p$ or $bc \leq \delta(p)$ or $ac \leq \delta(p)$.

The following result gives a relationship between a weakly δ-primary element and a weakly 2-absorbing δ-primary element.

Lemma 3.7. *If w is a weakly δ-primary element of L, then w is a weakly 2-absorbing δ-primary element of L.*

Proof. Suppose that w is a weakly δ-primary element of L. Let $x, y, z \in L$ be such that $0 \neq xyz \leq w$. Assume that $xy \not\leq w$. Since w is a weakly δ-primary element, we conclude that $z \leq \delta(w)$. Thus $xz \leq \delta(w)$. Hence w is a weakly 2-absorbing δ-primary element of L. □

Lemma 3.8. *Let $w \in L$. Let δ and δ_1 be expansions of elements of L such that $\delta(w) \leq \delta_1(w)$.*

(i) If w is a weakly δ-primary element, then w is a weakly δ_1-primary element.

(ii) If w is a weakly 2-absorbing δ-primary element, then w is a weakly 2-absorbing δ_1-primary element.

Proof. (i) Let $x, y \in L$ be such that $0 \neq xy \leq w$. As w is a weakly δ-primary element implies that $x \leq w$ or $y \leq \delta(w)$. Since $\delta(w) \leq \delta_1(w)$ we get either $x \leq w$ or $y \leq \delta_1(w)$. Thus w is a weakly δ_1-primary element.
(ii) Let $0 \neq xyz \leq w$ for some $x, y, z \in L$. As w is a weakly 2-absorbing δ-primary element, we get $xy \leq w$ or $yz \leq \delta(w)$ or $xz \leq \delta(w)$.

Since $\delta(w) \leq \delta_1(w)$, we conclude that either $xy \leq w$ or $yz \leq \delta_1(w)$ or $xz \leq \delta_1(w)$. Thus w is a weakly 2-absorbing δ_1-primary element. □

Theorem 3.9. *Let δ be an expansion of elements of L. If $\delta(w)$ is a weakly prime element of L, then w is a weakly 2-absorbing δ-primary element of L.*

Proof. Suppose that $\delta(w)$ is a weakly prime element of L. Assume that for some $x, y, z \in L$, $0 \neq xyz \leq w$ and $xy \nleq w$.

We have two cases (i)$xy \nleq \delta(w)$ and (ii) $xy \leq \delta(w)$.
Case (1): Suppose that $xy \nleq \delta(w)$. Since $\delta(w)$ is a weakly prime element of L, from $xyz \leq w \leq \delta(w)$ and $xy \nleq \delta(w)$, we have $z \leq \delta(w)$. Thus $xz \leq \delta(w)$.
Case (2): Suppose that $xy \leq \delta(w)$. Since $0 \neq xyz$, we have $0 \neq xy \leq \delta(w)$. Since $\delta(w)$ is a weakly prime element of L, we get $x \leq \delta(w)$ or $y \leq \delta(w)$. Thus $xz \leq \delta(w)$ or $yz \leq \delta(w)$. Hence w is a weakly 2-absorbing δ-primary element of L. □

Theorem 3.10. *Let δ be an expansion of elements of L. Let $x, w \in L$ be proper elements such that (i) $x \leq w$ and (ii) w is a weakly δ-primary element satisfying $\delta(x) = \delta(w)$. If $y \in L$ is such that $x \leq y \leq w$, then y is a weakly 2-absorbing δ-primary element.*

Proof. By hypothesis, $x \leq y \leq w$. Hence we have $\delta(x) \leq \delta(y) \leq \delta(w)$. As $\delta(x) = \delta(w)$, we get $\delta(x) = \delta(y) = \delta(w)$. Suppose that for some $p, q, r \in L$, $0 \neq pqr \leq y$. Assume that $pq \nleq y$. Since $y \leq w$, we get $0 \neq pqr \leq w$. We have two cases: (i)$pq \nleq w$ and (ii) $pq \leq w$.
Case (i): Suppose that $pq \nleq w$. Since w is a weakly δ-primary element, we get $r \leq \delta(w) = \delta(y)$. Thus $pr \leq \delta(y)$.
Case (ii): Suppose that $pq \leq w$. Clearly, $pq \neq 0$. Since w is weakly δ-primary, we get $p \leq w \leq \delta(w)$ or $q \leq \delta(w)$. Thus $pr \leq \delta(y)$ or $qr \leq \delta(y)$.

Hence y is a weakly 2-absorbing δ-primary element of L. □

Theorem 3.11. *Let δ be an expansion of elements of L such that $\delta(\delta(0)) = \delta(0)$. Let $\delta(0)$ be a δ-primary element of L. Then*

(i) $\delta(0)$ is a prime element.

(ii) If w is a weakly δ-primary element which is not δ-primary, then $\delta(w) = \delta(0)$ is a prime element of L.

(iii) If w is a weakly 2-absorbing δ-primary element which is not 2-absorbing δ-primary, then $\delta(w) = \delta(0)$ is a prime element.

Proof. (i) Let $p, q \in L$ be such that $pq \leq \delta(0)$ and $p \nleq \delta(0)$. Since $\delta(0)$ is a δ-primary element and $p \nleq \delta(0)$, we get $q \leq \delta(\delta(0)) = \delta(0)$. Thus $\delta(0)$ is a prime element.

(ii) Clearly $\delta(0) \leq \delta(w)$. By [9, Lemma 3.13], $w^2 = 0$ and as $\delta(0)$ is a prime element, we get $w \leq \delta(0)$. Hence $\delta(w) \leq \delta(\delta(0)) = \delta(0)$. Thus $\delta(w) = \delta(0)$ is a prime element.

(iii) Clearly $\delta(0) \leq \delta(w)$. By [8, Theorem 5.13], $w^3 = 0$ and as $\delta(0)$ is a prime element, we get $w \leq \delta(0)$. Hence $\delta(w) \leq \delta(\delta(0)) = \delta(0)$. Thus $\delta(w) = \delta(0)$ is a prime element. □

Theorem 3.12. *Let δ be an expansion of elements of L. Suppose that $\delta(0)$ is a prime element of L and $\delta(\delta(0)) = \delta(0)$. Let p be a weakly δ-primary element which is not δ-primary. Let $q \in L$ be such that $q \leq p$. Then q is a weakly 2-absorbing δ-primary element. In particular, if r is an element of L then $a = p \wedge r$ and $b = pr$ are weakly 2-absorbing δ-primary elements.*

Proof. Since p is a weakly δ-primary element that is not a δ-primary element, it follows from Theorem 3.11 that $\delta(p) = \delta(0)$. Since $0 \leq q \leq p$, we get $\delta(p) = \delta(0) = \delta(q)$. Let $0 \neq xyz \leq q \leq p$.
Case (i): Suppose that $xy \nleq p$. Since p is weakly δ-primary, we get
$z \leq \delta(p) = \delta(0) = \delta(q)$. Thus $xz \leq \delta(q)$.
Case (ii): Suppose that $xy \leq p$. Clearly, $0 \neq xy$. Since p is weakly δ-primary, we get $x \leq p \leq \delta(p)$ or $y \leq \delta(p) = \delta(0) = \delta(q)$. Hence either $xz \leq \delta(q)$ or $yz \leq \delta(q)$. Thus q is a weakly 2-absorbing δ-primary element. Since $a = p \wedge r \leq p$ and $b = pr \leq p$, we conclude that a, b are weakly 2-absorbing δ-primary elements. □

Definition 3.13. A multiplicative lattice L is said to be δ-reduced if $\delta(0) = 0$.

In the following theorem, we give characterizations for a proper element to be weakly δ-primary (weakly 2-absorbing δ-primary).

Theorem 3.14. *Let δ be an expansion of elements of L. Suppose that $\delta(0)$ is a δ-primary element and that L is δ-reduced. Then*

(i) A proper element p is weakly δ-primary iff p is δ-primary element.

(ii) A proper element p is weakly 2-absorbing δ-primary iff p is 2-absorbing δ-primary element.

Proof. As L is δ-reduced, we have $\delta(\delta(0)) = \delta(0) = 0$. Since $\delta(0)$ is a δ-primary element, it follows from Theorem 3.11 that $\delta(0) = 0$ is a prime element.
(i): Suppose that $p \neq 0$ and p is weakly δ-primary. If p is not δ-primary, then by Theorem, 3.11, $\delta(p) = \delta(0) = 0$. Thus $p = 0$, a contradiction. Hence p is δ-primary. The converse is obvious.
(ii): Suppose that $p \neq 0$ and p is weakly 2-absorbing δ-primary. If p is not 2-absorbing δ-primary, then by Theorem, 3.11, $\delta(p) = \delta(0) = 0$. Thus $p = 0$, a contradiction. Hence p is 2-absorbing δ-primary. The converse is obvious. □

Proposition 3.15. *Assume that δ_1 and δ_2 are expansions of elements of L. Let $\delta : L \to L$ be such that $\delta(p) = (\delta_1 \circ \delta_2)(p) = \delta_1(\delta_2(p))$. Then δ is an expansion function of elements of L.*

Theorem 3.16. *Let δ and γ be expansions of elements of L. Suppose that $\delta(0)$ is a γ-primary element of L. A proper element p is weakly 2-absorbing $\gamma \circ \delta$-primary iff p is 2-absorbing $\gamma \circ \delta$-primary element.*

Proof. Suppose that p is a weakly 2-absorbing $\gamma \circ \delta$-primary element. Assume that for some $x, y, z \in L$, $xyz \leq p$. We have two cases (i) $xyz = 0$ and (ii) $xyz \neq 0$.
(i): If $xyz = 0 \leq \delta(0)$, then as $\delta(0)$ is γ-primary, we conclude that $x \leq \delta(0)$ or $y \leq \gamma(\delta(0))$ or $z \leq \gamma(\delta(0))$. Since $\gamma(\delta(0)) \leq \gamma(\delta(p))$, we conclude that $xz \leq \gamma(\delta(0)) \leq \gamma(\delta(p))$ or $yz \leq \gamma(\delta(0)) \leq \gamma(\delta(p))$.
(ii): If $0 \neq xyz \leq p$, then as p is a weakly 2-absorbing $\gamma \circ \delta$-primary element, we have $xy \leq p$ or $xz \leq \gamma(\delta(p))$ or $yz \leq \gamma(\delta(p))$.

Thus p is a 2-absorbing $\gamma \circ \delta$-primary element. The converse is clear. □

Theorem 3.17. *Let δ and γ be expansions of elements of L. If $\gamma(p)$ is a weakly δ-primary element of L, then p is a weakly 2-absorbing $\delta \circ \gamma$-primary element.*

Proof. Suppose that for some $x, y, z \in L$, $0 \neq xyz$ and $xy \nleq p$. Assume that $xy \nleq \gamma(p)$. Since $\gamma(p)$ is weakly δ-primary, we have $z \leq \delta(\gamma(p))$. Thus $xz \leq \delta(\gamma(p))$. Suppose that $xy \leq \gamma(p)$. Since $0 \neq xyz \leq p$ and $0 \neq xy \leq \gamma(p)$ hold and as $\gamma(p)$ is a weakly δ-primary element, we conclude that $x \leq \delta(\gamma(p))$ or $y \leq \delta(\gamma(p))$. Thus $xz \leq \delta(\gamma(p))$ or $yz \leq \delta(\gamma(p))$. Hence p is a weakly 2-absorbing $\delta \circ \gamma$-primary element. □

Theorem 3.18. *Let δ be a meet preserving expansion of elements of L such that $\delta(\delta(0)) = \delta(0)$. Suppose that $\delta(0)$ is a δ-primary element of L. Let $p_1, p_2, \ldots, p_n$ be weakly 2-absorbing δ-primary elements of L such that no p_i, $i = 1, 2, \ldots, n$ is a 2-absorbing δ-primary element. Then $p = \bigwedge_{i=1}^{n} p_i$ is a weakly 2-absorbing δ-primary element.*

Proof. From Theorem 3.11 we note that $\delta(p_i) = \delta(0)$ for each $1 \leq i \leq n$. Since δ is meet preserving, we conclude that $\delta(p) = \delta(0)$. Suppose that $x, y, z \in L$ with $0 \neq xyz \leq p$ and $xy \nleq p$. Then $xy \nleq p_k$ for some k, $1 \leq k \leq n$. Hence $yz \leq \delta(p_k) = \delta(0) = \delta(p)$ or $xz \leq \delta(p_k) = \delta(0) = \delta(p)$. Thus p is a weakly 2-absorbing δ-primary element. □

The following definition is from Nimbhorkar and Nehete [8].

Definition 3.19. Let L_1 and L_2 be multiplicative lattices. Let γ and δ be element expansions of L_1 and L_2, respectively. We say that a multiplicative lattice homomorphism $f : L_1 \to L_2$ is a $\gamma\delta$-multiplicative lattice homomorphism if

$$\gamma(f^{-1}(a)) = f^{-1}(\delta(a)) \text{ for all } a \in L_2.$$

If f is an isomorphism, then we call it a $\gamma\delta$-lattice isomorphism. In this case $f(\gamma(a)) = \delta(f(a))$ for every element $a \in L_1$.

In the following, we prove that the inverse image of a weakly 2-absorbing δ-primary element of L under a multiplicative lattice homomorphism is again a weakly 2-absorbing δ-primary element.

Lemma 3.20. *Let L_1 and L_2 be multiplicative lattices. Let $f : L_1 \to L_2$ be a $\gamma\delta$-multiplicative lattice isomorphism. Then for any weakly 2-absorbing δ-primary element $p \in L_2$, $f^{-1}(p)$ is a weakly 2-absorbing γ-primary element of L_1.*

Proof. Let $x, y, z \in L_1$ with $0 \neq xyz \leq f^{-1}(p)$. This implies that

$$f(0) \neq f(xyz) = f(x)f(y)f(z) \leq p.$$

As p is a weakly 2-absorbing δ-primary element, either

$$f(x)f(y) \leq p \text{ or } f(x)f(z) \leq \delta(p) \text{ or } f(y)f(z) \in \delta(p).$$

This implies that either

$$xy \leq f^{-1}(p) \text{ or } xz \leq f^{-1}(\delta(p)) = \gamma(f^{-1}(p)) \text{ or } xy \in f^{-1}(\delta(p)) = \gamma(f^{-1}(p)).$$

Hence $f^{-1}(p)$ is a weakly 2-absorbing γ-primary element of L_1. □

The following result gives a characterization for a weakly 2-absorbing δ-primary element.

Lemma 3.21. *Let L_1 and L_2 be multiplicative lattices. Let $f : L_1 \to L_2$ be a $\delta\gamma$-multiplicative lattice isomorphism. An element $p \in L_1$ is a weakly 2-absorbing δ-primary element if and only if $f(p)$ is a weakly 2-absorbing γ-primary element of L_2.*

Proof. Suppose that $f(p)$ is a weakly 2-absorbing γ-primary element. Since f is an isomorphism, we have $f^{-1}(f(p)) = p$. Hence by Lemma 3.20, p is a weakly 2-absorbing δ-primary element of L_1.

Conversely, suppose that p is a weakly 2-absorbing δ-primary element of L_1. Let $a, b, c \in L_2$ be such that $0_{L_2} \neq abc \leq f(p)$. Since f is an isomorphism, there exist $x, y, z \in L_1$ such that $0_{L_2} = f(0)$, $f(x) = a$ and $f(y) = b$, and $f(z) = c$. Then

$$f(0) \neq f(xyz) = f(x)f(y)f(z) = abc \leq f(p)$$

implies that $xyz \leq f^{-1}(f(p)) = p$. As p is a weakly 2-absorbing δ-primary element of L_1, we get either

$$xy \leq p \text{ or } xz \leq \delta(p) \text{ or } yz \leq \delta(p).$$

As f is an $\delta\gamma$-lattice isomorphism, we have $\gamma(f(a)) = f(\delta(a))$. Hence we get either

$$xy \leq p \text{ or } xz \leq \delta(p) = f^{-1}(\gamma(f(p))) \text{ or } yz \leq \delta(p) = f^{-1}(\gamma(f(p)))$$

which implies that either

$$ab \leq f(p) \text{ or } ac \leq \gamma(f(p)) \text{ or } bc \leq \gamma(f(p)).$$

Thus $f(p)$ is a weakly 2-absorbing γ-primary element of L_2. □

Theorem 3.22. *Let δ be an expansion function of elements of L. Let p, q be proper elements of L. Suppose that p, q are δ-primary elements satisfying $\delta(p \wedge q) = \delta(p) \wedge \delta(q)$. Then $r = p \wedge q$ is a 2-absorbing δ-primary element of L.*

Proof. Suppose that for some $x, y, z \in L$, $xyz \leq r$ and assume that $xy \nleq r$. We consider three cases.
Case 1: Suppose that $xy \leq p$ and $xy \nleq q$. Since q is a δ-primary element, we have $z \leq \delta(q)$. Since p is a δ-primary element, we have $x \leq \delta(p)$ or $y \leq \delta(p)$. Thus

$$xz \leq \delta(p \wedge q) = \delta(p) \wedge \delta(q) \text{ or } yz \leq \delta(p \wedge q) = \delta(p) \wedge \delta(q).$$

Case 2: Suppose that $xy \nleq p$ and $xy \leq q$. By a similar argument as in case 1, we conclude that

$$xz \leq \delta(p \wedge q) = \delta(p) \wedge \delta(q) \text{ or } yz \leq \delta(p \wedge q) = \delta(p) \wedge \delta(q).$$

Case 3: Suppose that $xy \nleq p$ and $xy \nleq q$. Since p, q are δ-primary elements, we conclude that $z \leq \delta(p) \wedge \delta(q)$. Thus

$$xz \leq \delta(p \wedge q) = \delta(p) \wedge \delta(q) \text{ or } yz \leq \delta(p \wedge q) = \delta(p) \wedge \delta(q).$$

Hence $r = p \wedge q$ is a 2-absorbing δ-primary element of L. □

The following result can be proved by using similar arguments as in the proof of Theorem 3.22.

Theorem 3.23. *Let δ be an expansion function of elements of L and let p, q be proper elements of L. Suppose that p, q are weakly δ-primary elements of L such that $\delta(p \wedge q) = \delta(p) \wedge \delta(q)$. Then $r = p \wedge q$ is a weakly 2-absorbing δ-primary element of L.*

4. Weakly 2-absorbing δ-primary elements in a finite product of lattices

Let $L_1, L_2, \ldots, L_n$, be multiplicative lattices and $L = L_1 \times L_2 \times \ldots \times L_n$. Assume that $\delta_1, \delta_2, \ldots, \delta_n$ are expansion functions of elements of $L_1, L_2, \ldots, L_n$, respectively.

Define a function $\delta_\times : L \to L$ by

$$\delta_\times((a_1, a_2, \ldots, a_n)) = (\delta_1(a_1), \delta_2(a_2), \ldots, \delta_n(a_n))$$

for $a_i \in L_i$, $1 \leq i \leq n$. Clearly, $\delta_\times$ is an expansion function of elements of L.

The proof of the following theorem follows from the definition of a 2-absorbing (weakly 2-absorbing) $\delta_\times$-primary element.

Theorem 4.1. *Let $L_1, L_2, \ldots, L_n$ be multiplicative lattices with $1 \neq 0$. Let $L = L_1 \times L_2 \times \ldots \times L_n$ and δ_i, $1 \leq i \leq n$ and $\delta_\times$ be expansion functions of elements of L_i and L, respectively. If $(p_1, \ldots, p_i, \ldots, p_n)$ is a 2-absorbing (weakly 2-absorbing) $\delta_\times$-primary element of L, then $p_i \in L_i$ is a 2-absorbing (weakly 2-absorbing) δ_i-primary element.*

The following example shows that the converse of Theorem 4.1 does not hold.

Example 4.2. Consider the multiplicative lattices L_1 and L_2 shown in Figure 1 with multiplication as meet.

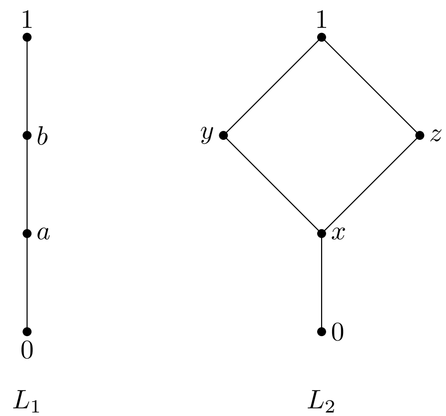

Figure 1

We note that the elements $a \in L_1$ and $x \in L_2$ are 2-absorbing (weakly 2-absorbing) δ_i-primary elements, where δ_i is the identity expansion function on L_i.

Consider the element $(a, x) \in L_1 \times L_2$. We have $(1, y)(0, 1)(1, z) = (0, x) \leq (a, x)$. But neither $(1, y)(0, 1) = (0, y) \leq (a, x)$ nor $(0, 1)(1, z) = (0, z) \leq (a, x)$ nor $(1, y)(1, z) = (1, x) \leq (a, x)$. Thus (a, x) is not a 2-absorbing (weakly 2-absorbing) $\delta_\times$-primary element in $L_1 \times L_2$.

However, we have the following theorem.

Theorem 4.3. *Let $L_1, L_2, \ldots, L_n$ be multiplicative lattices with $1 \neq 0$. Let $L = L_1 \times L_2 \times \ldots \times L_n$ and δ_i, $1 \leq i \leq n$ and $\delta_\times$ be expansion functions of elements of L_i and L, respectively. Then $p_i \in L_i$ is a 2-absorbing δ_i-primary element if and only if $(1, 1, \ldots, p_i, 1, \ldots, 1)$ is a 2-absorbing $\delta_\times$-primary element of L.*

Proof. Suppose that p_i is a 2-absorbing δ_i-primary element in L_i. Let $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$, $(c_1, \ldots, c_n) \in L$ be such that

$$(a_1, \ldots, a_i, \ldots, a_n)(b_1, \ldots, b_i, \ldots, b_n)(c_1, \ldots, c_i, \ldots, c_n) \leq (1, \ldots, p_i, 1, \ldots, 1).$$

Then we get $a_i b_i c_i \leq p_i$.

As p_i is a 2-absorbing δ_i-primary element in L_i either

$$a_i b_i \leq p_i \text{ or } b_i c_i \leq \delta_i(p_i) \text{ or } a_i c_i \leq \delta_i(p_i).$$

Hence either

$$(a_1, \ldots, a_i, \ldots, a_n)(b_1, \ldots, b_i, \ldots, b_n) \leq (1, \ldots, p_i, 1, \ldots, 1)$$

or

$$\begin{aligned} &(b_1, \ldots, b_i, \ldots, b_n)(c_1, \ldots, c_i, \ldots, c_n) \\ &\leq (\delta_1(1), \ldots, \delta_{i-1}(1), \delta_i(p_i), \delta_{i+1}(1), \ldots, \delta_n(1)) \\ &= \delta_\times((1, \ldots, 1, p_i, 1, \ldots, 1)) \end{aligned}$$

or

$$\begin{aligned} &(a_1, \ldots, a_i, \ldots, a_n)(c_1, \ldots, c_i, \ldots, c_n) \\ &\leq (\delta_1(1), \ldots, \delta_{i-1}(1), \delta_i(p_i), \delta_{i+1}(1), \ldots, \delta_n(1)) \\ &= \delta_\times((1, \ldots, 1, p_i, 1, \ldots, 1)) \end{aligned}$$

Thus $(1, 1, \ldots, p_i, 1, \ldots, 1)$ is a 2-absorbing $\delta_\times$-primary element of L.

We get the converse by observing that, if $a_1 b_b c_i \leq p_i$, then

$$\begin{aligned} &(1, 1, \ldots, a_i, 1, \ldots, 1)(1, 1, \ldots, b_i, 1, \ldots, 1)(1, 1, \ldots, c_i, 1, \ldots, 1) \\ &\leq (1, 1, \ldots, p_i, 1, \ldots, 1) \end{aligned}$$

and using the fact that $(1, 1, \ldots, p_i, 1, \ldots, 1)$ is a 2-absorbing $\delta_\times$-primary element of L. □

Theorem 4.4. *Let $L_1, L_2, \ldots, L_n$ be multiplicative lattices with $1 \neq 0$. Let $L = L_1 \times L_2 \times \ldots \times L_n$ and δ_i, $1 \leq i \leq n$ and $\delta_\times$ be expansion functions of elements of L_i and L, respectively. If $(1, 1, \ldots, p_i, 1, \ldots, 1)$ is a weakly 2-absorbing $\delta_\times$-primary element of L, then $p_i \in L_i$ is a weakly 2-absorbing δ_i-primary element.*

Proof. We note that if $0 \neq a_i b_i c_i \leq p_i$, then

$$\begin{aligned} &(1, 1, \ldots, a_i, 1, \ldots, 1)(1, 1, \ldots, b_i, 1, \ldots, 1)(1, 1, \ldots, c_i, 1, \ldots, 1) \\ &\leq (1, 1, \ldots, p_i, 1, \ldots, 1) \end{aligned}$$

Now the result follows from the fact that $(1, 1, \ldots, p_i, 1, \ldots, 1)$ is a weakly 2-absorbing $\delta_\times$-primary element of L. □

The following example shows that the converse of Theorem 4.4 does not hold.

Example 4.5. Consider the multiplicative lattices L_1 and L_2 shown in Figure 2 with multiplication as meet.

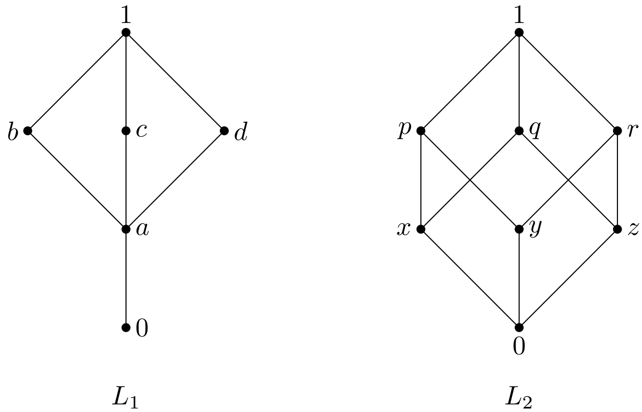

Figure 2

We note that the elements $x \in L_2$ are weakly 2-absorbing δ_i-primary element, where δ_i is the identity expansion function on L_i but the element $(1, x)$ is not a weakly 2-absorbing $\delta_\times$-primary element in $L_1 \times L_2$.

Theorem 4.6. *Let $L_1, L_2, \ldots, L_n$ be multiplicative lattices with $1 \neq 0$. Let $L = L_1 \times L_2 \times \ldots \times L_n$ and δ_i, $1 \leq i \leq n$ and $\delta_\times$ be expansion functions of elements of L_i and L, respectively. Let $p_i \in L_i$ be a nonzero proper element. The following statements are equivalent.*

(i) $(1, 1, \ldots, p_i, 1, \ldots, 1)$ *is a weakly 2-absorbing $\delta_\times$-primary element of L.*

(ii) $(1, 1, \ldots, p_i, 1, \ldots, 1)$ *is a 2-absorbing $\delta_\times$-primary element of L.*

(iii) p_i *is a 2-absorbing δ_i-primary element of L_i.*

Proof. $(i) \Rightarrow (ii)$: Let $q = (1, 1, \ldots, p_i, 1, \ldots, 1)$ be a proper element of L. If q is a weakly 2-absorbing $\delta_\times$-primary element, which is not a 2-absorbing $\delta_\times$-primary element, then by [8, Theorem 5.13], $q^3 = 0$.

But $q^3 = (1, 1, \ldots, p_i^3, 1, \ldots, 1) \neq (0, 0, \ldots, 0)$. Hence q is a 2-absorbing $\delta_\times$-primary element of L.

$(ii) \Rightarrow (iii)$: Follows from Theorem 4.3.

$(iii) \Rightarrow (i)$: Let p_i be a 2-absorbing δ_i-primary element of L_i. By Theorem 4.3, $(1, 1, \ldots, p_i, 1, \ldots, 1)$ is a 2-absorbing $\delta_\times$-primary element of L. As every 2-absorbing δ-primary element of L for any expansion function of elements δ of L is a weakly 2-absorbing δ-primary element, it follows that $(1, 1, \ldots, p_i, 1, \ldots, 1)$ is a weakly 2-absorbing $\delta_\times$-primary element of L. □

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