Novi Sad J. Math. Vol. 54, No. 2, 2024, 27-37
https://doi.org/10.30755/NSJOM.13225

(m, ρ)-quasi Einstein solitons on paracontact geometry

Krishnendu De[1,2] and **Uday Chand De**[3]

Abstract. We set the goal to investigate the geometrical properties of (m, ρ)-*quasi Einstein solitons* within the context of paracontact metric manifolds (especially in para-Sasakian, para-cosymplectic and para-Kenmotsu manifolds). Also, we consider a non-trivial example and validate a result of our paper.

AMS Mathematics Subject Classification (2010): 53C15; 53C50; 53C80

Key words and phrases: 3-dimensional para-Sasakian; para-Kenmotsu; para-cosymplectic manifolds; Gradient ρ-Einstein solitons; (m, ρ)-quasi Einstein metric.

1. Introduction

The investigation of paracontact geometry is performing a crucial role in the development of modern differential geometry. It has many connections with other areas of mathematics and mathematical physics. Due to its broad applications, it became popular among eminent researchers. In 1985, the topic of paracontact metric structures was introduced in [13]. Since then, the properties of paracontact metric manifolds have been studied by many investigators. The significance of paracontact geometry has been indicated minutely in the previous years by various articles ([14], [16], [15]). We may mention ([1], [6], [8], [7], [9], [13]) and the references contained in those for more information about paracontact metric geometry.

The interesting notion, called as *generalized quasi-Einstein metric* is introduced by Catino [3], for studying harmonic Weyl tensor and defined as follows (see [3]):

Let a C^∞ manifold $\mathcal{N}^n$, $n > 2$, admit a Riemannian metric g, then the metric g satisfying

$$S + \nabla^2\gamma = \alpha d\gamma \otimes d\gamma + \beta g$$

is called a *generalized quasi-Einstein metric* for some C^∞ functions α, γ and β, where S, ∇^2, d and $\otimes$ denote the Ricci tensor, Hessian operator, exterior derivative of g and tensor product, respectively. In the current article, we consider an (m, ρ)-*quasi Einstein metric* in the 3-dimensional normal almost paracontact metric (briefly, *apm*) manifolds, which is a particular case of *generalized quasi-Einstein metric*, and study its geometrical properties.

[1]Assistant Professor of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan, Bhadreswar, P.O.-Angus, Hooghly, Pin.712221, West Bengal, India,
e-mail: krishnendu.de67@gmail.com, ORCID iD: orcid.org/0000-0001-6520-4520
[2]Corresponding author
[3]Department of Pure Mathematics, University of Calcutta, 35, B. C. Road, Kol- 700019, West Bengal, India, e-mail: uc_de@yahoo.com, ORCID iD: orcid.org/0000-0002-8990-4609

Analogous to Hung and Wei [12], a semi-Riemannian metric g of a semi-Riemannian manifold $\mathcal{N}$ is called as (m,ρ)-*quasi Einstein soliton* if there exists a C^∞ function $\gamma : \mathcal{N}^n \to \mathbb{R}$ such that

$$S + \nabla^2\gamma - \frac{1}{m} d\gamma \otimes d\gamma = \beta g = (\rho r + \lambda)g, \tag{1.1}$$

where ρ, λ and m $(0 < m \leq \infty)$ are real constants, $\mathbb{R}$ is the set of real numbers and r represents the scalar curvature of g. The expression $S + \nabla^2\gamma - \frac{1}{m} d\gamma \otimes d\gamma$ is the m-Bakry-Emery Ricci tensor, which is proportional to the metric g and the constant λ [19]. The metric g with constant potential function γ is trivial and hence the manifold is an Einstein manifold. Furthermore, when $m = \infty$, the foregoing equation yields the gradient ρ-Einstein soliton. This notion was presented in [4], and recently Venkatesha et al. [18] studied ρ- Einstein metrics on almost Kenmotsu manifolds. In this connection, the properties of (m,ρ)-quasi Einstein solitons in different geometrical structures have been studied in detail by ([11], [12]) and others. In [11], the author proved that if a complete K-contact manifold obeys an (m,ρ)-quasi Einstein metric, then the potential function is constant for $m \neq 1$. Now, an intrinsic question arises: Is the result true for a paracontact metric?

Motivated by the above discussion, we contribute to investigate (m,ρ)-quasi Einstein metric in paracontact geometry and it is proved that the result is also true for paracontact metric but the underlying condition is different.

The current article is constructed as follows: In Section 2, we recall a few fundamental facts and formulas of normal almost paracontact manifolds, which are used throughout the manuscript. Starting from Section 3, we will state our theorems and provide their proofs. Also, we consider a non-trivial example to verify the result of our article.

2. Preliminaries

In this section, we accumulate the fundamental facts and formulas of the paracontact manifold which will be needed in later sections.

Let $\mathcal{N}$ be a $(2n+1)$-dimensional smooth differentiable manifold endowed with a vector field ξ, a $(1,1)$ tensor field ϕ, and a 1-form η such that

$$\phi^2 U = U - \eta(U)\xi, \quad \eta(\phi U) = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

hold for all vector fields U on $\mathcal{N}$, and the almost paracomplex structure on each fibre of $\mathcal{D} = ker\,\eta$ is induced by the tensor field ϕ. In other words, the eigendistributions $\mathcal{D}^+_\phi$ and $\mathcal{D}^-_\phi$ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have equal dimension. Then the triplet (ϕ,ξ,η) satisfying (2.1) is named as an almost paracontact structure and the manifold $\mathcal{N}$ is an almost paracontact manifold. In addition, if a semi-Riemannian metric g of $\mathcal{N}$ satisfies

$$g(\xi, U) = \eta(U), \quad g(U,V) + g(\phi U, \phi V) = \eta(U)\eta(V) \tag{2.2}$$

for all vector fields U and V on $\mathcal{N}$, then the quadruple (ϕ,ξ,η,g) is claimed to be an *apm*-structure and $\mathcal{N}$ an *apm*-manifold [17].

The Nijenhuis torsion is defined by

$$[\phi,\phi](U,V)=[\phi U,\phi V]+\phi^2[U,V]-\phi[U,\phi V]-\phi[\phi U,V]$$

for all $U,\ V\in\mathfrak{X}(\mathcal{N})$, where $\mathfrak{X}(\mathcal{N})$ denotes the collection of all smooth vector fields of $\mathcal{N}$. The almost paracontact manifold is called normal if $N_\phi=[\phi,\phi]-2d\eta\otimes\xi$ vanishes. The fundamental 2-form of the *apm*-manifold is defined by $\Phi(U,V)=g(U,\phi V)$. If $d\eta(U,V)=g(U,\phi V)$, then the manifold $\mathcal{N}$ endowed with structure (ϕ,ξ,η,g) is known as a paracontact metric manifold.

A symmetric trace-free operator $h=\frac{1}{2}\pounds_\xi\phi$ in a paracontact manifold satisfies $h\xi=0$ and

$$\nabla_U\xi=-\phi U+\phi hU,\ \ \forall\ U\in\mathfrak{X}(\mathcal{N}).$$

It is to be noted that ξ being Killing is equivalent to the condition $h=0$ and (ϕ,ξ,η,g) is called K-paracontact structure. If the normality condition is satisfied in a paracontact metric manifold, then it is called a para-Sasakian manifold. It is well circulated that every para-Sasakian manifold is necessarily K-paracontact. The converse is not true, in general, but it holds when the manifold is of dimension three [2].

In a para-Sasakian manifold the subsequent relations hold :

$$R(U,V)\xi=\eta(U)V-\eta(V)U, \tag{2.3}$$

$$(\nabla_U\phi)V=-g(U,V)\xi+\eta(V)U, \tag{2.4}$$

$$\nabla_U\xi=-\phi U, \tag{2.5}$$

$$R(U,\xi)V=g(U,V)\xi-\eta(V)U, \tag{2.6}$$

$$S(U,\xi)=-(n-1)\eta(U),\ Q\xi=-(n-1)\xi \tag{2.7}$$

for all $U,\ V\in\mathfrak{X}(\mathcal{N})$, where the Ricci operator Q of the manifold $\mathcal{N}$ is defined by $g(QU,V)=S(U,V)$.

It is well-known that a 3-dimensional semi-Riemannian manifold $\mathcal{N}$ assumes the following curvature form

$$\begin{aligned} R(U,V)Z &= g(V,Z)QU-g(U,Z)QV+S(V,Z)U-S(U,Z)V \\ &\quad -\frac{r}{2}[g(V,Z)U-g(U,Z)V] \end{aligned} \tag{2.8}$$

for all $U,\ V,\ Z\in\mathfrak{X}(\mathcal{N})$. Replacing $V=Z=\xi$ in the preceding equation and utilizing (2.3) and (2.7), we obtain (see [15])

$$QU=\frac{1}{2}[(r+2)U-(r+6)\eta(U)\xi]. \tag{2.9}$$

In view of (2.9) the Ricci tensor is written as

$$2S=[(r+2)g-(r+6)\eta\otimes\eta]. \tag{2.10}$$

3. (m,ρ)-quasi Einstein solitons on 3-dimensional para-Sasakian manifolds

This section investigates the 3-dimensional para-Sasakian manifold $\mathcal{N}^3$ with (m,ρ)-quasi Einstein metric. Now, before introducing the detailed proof of our prime theorem, we first state the following result [15]:

Lemma 3.1. *On $\mathcal{N}^3$, $\xi r = 0$.*

Lemma 3.2. *Every $\mathcal{N}^3$ satisfies*

$$\begin{aligned} R(U,V)D\gamma &= (\nabla_V Q)U - (\nabla_U Q)V + \frac{\beta}{m}\{(V\gamma)U - (U\gamma)V\} \\ &\quad + \frac{1}{m}\{(U\gamma)QV - (V\gamma)QU\} + \{(U\beta)V - (V\beta)U\}, \end{aligned} \tag{3.1}$$

for all U, $V \in \mathfrak{X}(\mathcal{N})$.

Proof of Lemma 3.2. Let us assume that the semi-Riemannian metric g of $\mathcal{N}^3$ is a (m,ρ)-quasi Einstein metric. Then the equation (1.1) may be expressed as

$$\nabla_U D\gamma + QU = \frac{1}{m} g(U, D\gamma)D\gamma + \beta U. \tag{3.2}$$

After executing covariant derivative of (3.2) along V, we get

$$\begin{aligned} \nabla_V \nabla_U D\gamma &= -\nabla_V QU + \frac{1}{m}\nabla_V g(U, D\gamma)D\gamma \\ &\quad + \frac{1}{m} g(U, D\gamma)\nabla_V D\gamma + \beta\nabla_V U + (V\beta)U. \end{aligned} \tag{3.3}$$

Exchanging U and V in (3.3), we acquire

$$\begin{aligned} \nabla_U \nabla_V D\gamma &= -\nabla_U QV + \frac{1}{m}\nabla_U g(V, D\gamma)D\gamma \\ &\quad + \frac{1}{m} g(V, D\gamma)\nabla_U D\gamma + \beta\nabla_U V + (U\beta)V \end{aligned} \tag{3.4}$$

and

$$\nabla_{[U,V]} D\gamma = -Q[U,V] + \frac{1}{m} g([U,V], D\gamma)D\gamma + \beta[U,V]. \tag{3.5}$$

Using (3.2)-(3.5) and the symmetric property of the Levi-Civita connection together with $R(U,V)D\gamma = \nabla_U\nabla_V D\gamma - \nabla_V\nabla_U D\gamma - \nabla_{[U,V]}D\gamma$, we obtain

$$\begin{aligned} R(U,V)D\gamma &= (\nabla_V Q)U - (\nabla_U Q)V + \frac{\beta}{m}\{(V\gamma)U - (U\gamma)V\} \\ &\quad + \frac{1}{m}\{(U\gamma)QV - (V\gamma)QU\} + \{(U\beta)V - (V\beta)U\}. \end{aligned}$$

□

Theorem 3.3. *The potential function of the (m,ρ)-quasi Einstein soliton on $\mathcal{N}^3$ is constant, provided $\beta=-2$.*

Proof of Theorem 3.3. Executing the inner product operation of (3.1) with ξ and using (2.9), we have

$$\begin{aligned} & g(R(U,V)D\gamma,\xi) \\ &= \frac{\beta}{m}\{(V\gamma)\eta(U)-(U\gamma)\eta(V)\} \\ &+\frac{1}{m}\{(U\gamma)\eta(QV)-(V\gamma)\eta(QU)\}+\{(U\beta)\eta(V)-(V\beta)\eta(U)\}. \end{aligned} \tag{3.6}$$

Again, by a simple calculation we get from(2.3) that

$$g(R(U,V)D\gamma,\xi)=-\{(V\gamma)\eta(U)-(U\gamma)\eta(V)\}. \tag{3.7}$$

Combining equations (3.6) and (3.7) reveals that

$$\begin{aligned} -\{(V\gamma)\eta(U)-(U\gamma)\eta(V)\} &= \frac{\beta}{m}\{(V\gamma)\eta(U)-(U\gamma)\eta(V)\} \\ &+\frac{1}{m}\{(U\gamma)\eta(QV)-(V\gamma)\eta(QU)\} \\ &+\{(U\beta)\eta(V)-(V\beta)\eta(U)\}. \end{aligned} \tag{3.8}$$

Replacing V by ξ in the foregoing equation we infer

$$d(\gamma-\beta)=\xi(\gamma-\beta)\eta, \tag{3.9}$$

in which the exterior differentiation is indicated by d, provided $\beta=-2$. This concludes that $\gamma-\beta$ is invariant along the distribution $\mathcal{D}$, which is defined by $\mathcal{D}=\ker\eta$. In other words, $U(\gamma-\beta)=0$ for any vector field $U\in\mathcal{D}$. Taking into account the above fact and using $\beta=-2$, we infer

$$(U\gamma)=(U\beta)=0. \tag{3.10}$$

Since $(U\gamma)=0$, then we get $\gamma=$ *constant*.

This completes the proof. □

Using $\gamma=$ *constant*, we get from (3.2) that the manifold is an Einstein manifold. Since the manifold under consideration is of dimension 3, hence $\mathcal{N}^3$ has a constant sectional curvature.

Corollary 3.4. *An $\mathcal{N}^3$ endowed with a (m,ρ)-quasi Einstein metric possesses a constant sectional curvature, provided $\beta=-2$.*

We know that when $m=\infty$, the (m,ρ)-quasi Einstein soliton gives the gradient ρ-Einstein soliton. Putting the value $m=\infty$ in (3.8) and by a straightforward calculation, we find that the manifold is of constant sectional curvature. Thus, we can state:

Corollary 3.5. *Let $\mathcal{N}^3$ admit a gradient ρ-Einstein metric. Then it has a constant sectional curvature.*

4. (m,ρ)-quasi Einstein solitons on 3-dimensional para-cosymplectic manifolds

Throughout the section, we suppose that the 3-dimensional para-cosymplectic manifold $\mathcal{N}^3$ admits a (m,ρ)-quasi Einstein metric.

Assume that α can be a function or constant on $\mathcal{N}$. If the 1-form η and 2-form Φ of an *apm*-manifold $\mathcal{N}$ satisfy the following:

$$d\eta = 0 \text{ and } d\Phi = 2\alpha\eta \wedge \Phi, \tag{4.1}$$

then the manifold $\mathcal{N}$ is said to be an almost α-para-cosymplectic manifold. Particularly, if we take $\alpha = 0$ in (4.1), then we get almost para-cosymplectic manifolds. The manifold is said to be para-cosymplectic if further the normality condition is satisfied. For instance (see [5],[14]) and references contained in those for more interesting facts about para-cosymplectic manifolds. Noted that $\mathcal{N}^3$ satisfies the following relations:

$$R(U,V)\xi = 0, \tag{4.2}$$

$$(\nabla_U \phi)V = 0, \tag{4.3}$$

$$\nabla_U \xi = 0, \tag{4.4}$$

$$S(U,\xi) = 0 \Longleftrightarrow Q\xi = 0 \tag{4.5}$$

for all $U, V \in \mathfrak{X}(\mathcal{N})$.

In $\mathcal{N}^3$, utilizing (4.2) and (4.5) in (2.8), we obtain

$$QU = \frac{r}{2}[U - \eta(U)\xi]. \tag{4.6}$$

In view of equation (4.6), the Ricci tensor is written as

$$S(U,V) = \frac{r}{2}[g(U,V) - \eta(U)\eta(V)]. \tag{4.7}$$

In a 3-dimensional para-cosymplectic manifold, Lemma 3.1 and Lemma 3.2 are also valid.

Theorem 4.1. *Let the (m,ρ)-quasi Einstein metric be a semi-Riemannian metric of $\mathcal{N}^3$. Then, $\mathcal{N}^3$ possesses a constant sectional curvature, provided $\beta = m - 2$.*

Proof of Theorem 4.1. Using the Lemma 3.2, equation (4.6) and maintaining the same procedure as in the proof of Theorem 3.1, we can easily obtain

$$d(\gamma - \beta) = \xi(\gamma - \beta)\eta, \tag{4.8}$$

provided $\beta = m - 2$ and hence

$$(U\gamma) = (U\beta) = 0. \tag{4.9}$$

If $(U\gamma) = 0$, then we get $\gamma = constant$.

This finishes the proof. □

The forthcoming corollary can be found by following the same logic used in Corollary 3.4.

Corollary 4.2. *If $\mathcal{N}^3$ admits a (m, ρ)-quasi Einstein soliton, then $\mathcal{N}^3$ is of constant sectional curvature, provided $\beta = m - 2$.*

Remark 4.3. In a 3-dimensional para-cosymplectic manifold, the Corollary 3.5 is also true.

5. (m, ρ)-quasi Einstein solitons on 3-dimensional para-Kenmotsu manifolds

This section deals with the study of 3-dimensional para-Kenmotsu manifold $\mathcal{N}^3$ equipped with a (m, ρ)-quasi Einstein metric.

If we consider the function $\alpha = 1$ in (4.1) we get almost para-Kenmotsu manifolds. The manifolds are called para-Kenmotsu if, additionally, the normality condition is satisfied. For more details, we cite ([10], [14]) and references contained in that for more fascinating facts about para-Kenmotsu manifolds. In a para-Kenmotsu manifold the following relations hold :

$$R(U, V)\xi = \eta(U)V - \eta(V)U, \tag{5.1}$$

$$(\nabla_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U, \tag{5.2}$$

$$\nabla_U \xi = U - \eta(U)\xi, \tag{5.3}$$

$$R(U, \xi)V = g(U, V)\xi - \eta(V)U, \tag{5.4}$$

$$S(U, \xi) = -(n-1)\eta(U), \; Q\xi = -(n-1)\xi, \tag{5.5}$$

for all $U, V \in \mathfrak{X}(\mathcal{N})$. By straightforward calculations, we can easily show that Lemma 3.2 also holds on $\mathcal{N}^3$.

It is noticed that in $\mathcal{N}^3$ the following relation holds (see [15]):

$$QU = \frac{1}{2}[(r+2)U - (r+6)\eta(U)\xi]. \tag{5.6}$$

In view of (5.6), it is observed that the Ricci tensor S of $\mathcal{N}^3$ satisfies the equation (4.7).

Now, before introducing the detailed proof of our key theorem, we first state the following result [15]:

Lemma 5.1. *On $\mathcal{N}^3$, we have*

$$\xi r = -2(r+6). \tag{5.7}$$

Theorem 5.2. *If $\mathcal{N}^3$ admits a (m,ρ)-quasi Einstein soliton, then it is a manifold of constant sectional curvature, provided $r \neq -(\lambda+2)$.*

Proof of Theorem 5.2. Using Lemma 3.2, equation (5.6) and maintaining the same procedure as in the proof of Theorem 3.1, we can easily obtain

$$d(\gamma-\beta)=\xi(\gamma-\beta)\eta \tag{5.8}$$

and

$$(\xi\gamma)=(\xi\beta)=\rho(\xi r)=-2\rho(r+6). \tag{5.9}$$

The contraction of the equation (3.1) along U and applying Lemma 5.1, we get

$$\begin{aligned} S(V,D\gamma) &= \frac{1}{2}(Vr)+\frac{2\beta}{m}(V\gamma) \\ &\quad -\frac{1}{m}\{r(V\gamma)-g(QV,D\gamma)\}-2(V\beta). \end{aligned} \tag{5.10}$$

Clearly, comparing the above equation with (4.7) yields

$$\begin{aligned} &-(Vr)-\frac{4\beta}{m}(V\gamma)+\frac{2}{m}\{r(V\gamma)-g(QV,D\gamma)\} \\ &+4(V\beta)+(r+2)(Vf)-(r+6)\eta(V)\xi\gamma=0. \end{aligned} \tag{5.11}$$

By a straightforward calculation, replacing V by ξ in (5.11) and using (5.9), we can easily obtain

$$(\xi\gamma)=\frac{2m(r+6)}{4\beta-2r-4}. \tag{5.12}$$

Comparing the antecedent relation with (5.9) gives

$$2(r+6)\{\rho+\frac{m}{4\beta-2r-4}\}=0. \tag{5.13}$$

This shows that either $r=-6$ or $\rho+\frac{m}{4\beta-2r-4}=0$.

Case (i): If $r=-6$, then from (4.7) we find that g is an Einstein metric, i.e., $S=-2g$. Hence, by using (2.8) we say that $\mathcal{N}^3$ is of constant sectional curvature -1.

Case (ii): If $\rho+\frac{m}{4\beta-2r-4}=0$, then by a simple calculation we get

$$r=\frac{4\lambda-4-4\rho\lambda-4\rho+m}{2(2\rho^2-3\rho+1)}=\textit{constant}, \tag{5.14}$$

provided $\rho \neq 1, \frac{1}{2}$. Hence by applying Lemma 5.1 we can easily get $r=-6$. Therefore, from Case (i) we see that the manifold is of constant sectional curvature -1. After combining the two conditions namely, $\beta=-2$ and $\rho \neq 1, \frac{1}{2}$, we can write $r \neq -(\lambda+2)$.

This completes the proof. □

Putting the value $m = \infty$ in (3.8) and by a straightforward calculation we find that the manifold is of constant sectional curvature -1. Thus, we can state:

Corollary 5.3. *An $\mathcal{N}^3$ equipped with a gradient ρ-Einstein metric possesses a constant sectional curvature -1.*

6. Example

Here we consider an example of the paper [15]. In this paper, the author considers a 3-dimensional manifold $\mathcal{N} = \{(u, v, w) \in \mathbb{R}^3, w \neq 0\}$ and the vector fields

$$\delta_1 = \frac{\partial}{\partial u}, \quad \phi\delta_1 = \delta_2 = \frac{\partial}{\partial v}, \quad \xi = (u + 2v)\frac{\partial}{\partial u} + (2u + v)\frac{\partial}{\partial v} + \frac{\partial}{\partial w}.$$

and shows that the manifold is a para-Kenmotsu manifold. Further, Koszul's formula yields

$$\nabla_{\delta_1}\delta_3 = \delta_1, \quad \nabla_{\delta_1}\delta_2 = 0, \quad \nabla_{\delta_1}\delta_1 = -\delta_3,$$

$$\nabla_{\delta_2}\delta_3 = \delta_2, \quad \nabla_{\delta_2}\delta_2 = \delta_3, \quad \nabla_{\delta_2}\delta_1 = 0,$$

$$\nabla_{\delta_3}\delta_3 = 0, \quad \nabla_{\delta_3}\delta_2 = -2\delta_1, \quad \nabla_{\delta_3}\delta_1 = -2\delta_2. \tag{6.1}$$

Also, the author has obtained the expressions of the curvature tensor and the Ricci tensor, respectively, as follows:

$$R(\delta_1, \delta_2)\xi = 0, \quad R(\delta_2, \xi)\xi = -\delta_2, \quad R(\delta_1, \xi)\xi = -\delta_1,$$

$$R(\delta_1, \delta_2)\delta_2 = \delta_1, \quad R(\delta_2, \xi)\delta_2 = -\xi, \quad R(\delta_1, \xi)\delta_2 = 0,$$

$$R(\delta_1, \delta_2)\delta_1 = \delta_2, \quad R(\delta_2, \xi)\delta_1 = 0, \quad R(\delta_1, \xi)\delta_1 = \xi$$

and

$$\begin{aligned} S(\delta_1, \delta_1) &= -g(R(\delta_1, \delta_2)\delta_2, \delta_1) + g(R(\delta_1, \delta_3)\delta_3, \delta_1) \\ &= -2 \\ &= -2g(\delta_1, \delta_1). \end{aligned} \tag{6.2}$$

Similarly, we have

$$S(\delta_2, \delta_2) = -2g(\delta_2, \delta_2) \quad \text{and} \quad S(\delta_3, \delta_3) = -2g(\delta_3, \delta_3).$$

From the expressions of the Ricci tensor we find that $\mathcal{N}$ is an Einstein manifold. Let us assume that $f : \mathcal{N}^3 \to \mathbb{R}$ is a smooth function such that $f = u$. Then we can easily get

$$Df = \frac{\partial}{\partial u} = \delta_1.$$

Using (6.1) we infer

$$Hessf(\delta_1, \delta_1) = 0.$$

Thus gradient ρ-Einstein soliton equation satisfies

$$Hessf(\delta_1, \delta_1) + S(\delta_1, \delta_1) + 2g(\delta_1, \delta_1) = 0.$$

Hence $\mathcal{N}^3$ satisfies

$$Hessf(U, V) + S(U, V) + 2g(U, V) = 0.$$

Thus g is a gradient ρ-Einstein soliton with $f = u$ and $\beta = -2$. Hence **Corollary 5.3** is verified.

7. Acknowledgement

We would like to thank the referees and editor for reviewing the paper carefully and their valuable comments to improve the quality of the paper.

References

[1] A. M. Blaga. Eta-Ricci solitons on para-Kenmotsu manifolds. *Balkan J. Geom. Appl.*, 20(1):1–13, 2015.

[2] G. Calvaruso. Homogeneous paracontact metric three-manifolds. *Illinois J. Math.*, 55(2):697–718 (2012), 2011. URL: `http://projecteuclid.org/euclid.ijm/1359762409`, `doi:10.1215/ijm/1359762409`.

[3] Giovanni Catino. Generalized quasi-Einstein manifolds with harmonic Weyl tensor. *Math. Z.*, 271(3-4):751–756, 2012. `doi:10.1007/s00209-011-0888-5`.

[4] Giovanni Catino and Lorenzo Mazzieri. Gradient Einstein solitons. *Nonlinear Anal.*, 132:66–94, 2016. `doi:10.1016/j.na.2015.10.021`.

[5] Piotr Dacko. On almost para-cosymplectic manifolds. *Tsukuba J. Math.*, 28(1):193–213, 2004. `doi:10.21099/tkbjm/1496164721`.

[6] K. De. A note on gradient solitons on para-Sasakian manifolds. In *Proceedings of The International Conference "Differential Geometry, Dynamical Systems" (DGDS-2020)*, volume 28 of *BSG Proc.*, pages 20–29. Geom. Balkan Press, Bucharest, 2021.

[7] Krishnendu De and Uday Chand De. A note on almost Ricci soliton and gradient almost Ricci soliton on para-Sasakian manifolds. *Korean J. Math.*, 28(4):739–751, 2020.

[8] Krishnendu De and Uday Chand De. Almost quasi-Yamabe solitons and gradient almost quasi-Yamabe solitons in paracontact geometry. *Quaest. Math.*, 44(11):1429–1440, 2021. `doi:10.2989/16073606.2020.1799882`.

[9] Krishnendu De and Uday Chand De. δ-almost Yamabe solitons in paracontact metric manifolds. *Mediterr. J. Math.*, 18(5):Paper No. 218, 11, 2021. `doi:10.1007/s00009-021-01856-9`.

[10] Krishnendu De and Uday Chand De. A note on gradient solitons on para-Kenmotsu manifolds. *Int. J. Geom. Methods Mod. Phys.*, 18(1):Paper No. 2150007, 11, 2021. `doi:10.1142/S0219887821500079`.

[11] Amalendu Ghosh. (m, ρ)-quasi-Einstein metrics in the frame-work of K-contact manifolds. *Math. Phys. Anal. Geom.*, 17(3-4):369–376, 2014. doi:10.1007/s11040-014-9161-6.

[12] G. Huang and Y. Wei. The classification of (m, ρ)-quasi-Einstein manifolds. *Ann. Global Anal. Geom.*, 44:269–282, 2013. doi:10.1007/s10455-013-9366-0.

[13] Soji Kaneyuki and Floyd L. Williams. Almost paracontact and parahodge structures on manifolds. *Nagoya Math. J.*, 99:173–187, 1985. doi:10.1017/S0027763000021565.

[14] I. Küpeli Erken, P. Dacko, and C. Murathan. Almost α-paracosymplectic manifolds. *J. Geom. Phys.*, 88:30–51, 2015. doi:10.1016/j.geomphys.2014.09.011.

[15] Irem Küpeli Erken. Yamabe solitons on three-dimensional normal almost paracontact metric manifolds. *Period. Math. Hungar.*, 80(2):172–184, 2020. doi:10.1007/s10998-019-00303-3.

[16] İrem Küpeli Erken and Cengizhan Murathan. A study of three-dimensional paracontact $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-spaces. *Int. J. Geom. Methods Mod. Phys.*, 14(7):1750106, 35, 2017. doi:10.1142/S0219887817501067.

[17] Isuke Satō. On a structure similar to the almost contact structure. *Tensor (N.S.)*, 30(3):219–224, 1976.

[18] V. Venkatesha and H. Aruna Kumara. Gradient ρ-Einstein soliton on almost Kenmotsu manifolds. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 65(2):375–388, 2019. doi:10.1007/s11565-019-00323-4.

[19] Guofang Wei and Will Wylie. Comparison geometry for the Bakry-Emery Ricci tensor. *J. Differential Geom.*, 83(2):377–405, 2009. doi:10.4310/jdg/1261495336.

Received by the editors June 22, 2021
First published online September 19, 2022