Novi Sad J. Math. Vol. 54, No. 2, 2024, 213-229
https://doi.org/10.30755/NSJOM.15867

# $A$-numerical radius inequalities via Dragomir inequalities and related results

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**Abstract.** The purpose of this paper is to establish some new inequalities involving $A$-numerical radius and $A$-operator seminorm of semi-Hilbert space operators. For this aim, we generalize some known Dragomir inequalities for Hilbert space operators. In addition, related inequalities on a semi-Hilbert space are also given.

*AMS Mathematics Subject Classification* (2010): 47B15; 47B20; 47A15

*Key words and phrases:* positive operator; $A$-numerical radius; semi-Hilbert space; inequality

## 1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a non trivial complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$ and associated norm $\|.\|$. For every operator $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\overline{\mathcal{R}(T)}$ stand for, respectively, the null space, the range and the closure of the range of $T$, and the adjoint of $T$ is denoted by $T^*$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for every $x \in \mathcal{H}$.

Let $\mathcal{B}(\mathcal{H})^+$ be the cone of positive (semi-definite) operators i.e.,

$$\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H}) : \langle Ax, x\rangle \geq 0, \forall x \in \mathcal{H}\}.$$

Any positive operator $A \in \mathcal{B}(\mathcal{H})^+$ defines a positive semi-definite sesquilinear form

$$\langle .,. \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad \langle x, y\rangle_A = \langle Ax, y\rangle.$$

Naturally, this semi-inner product induces a semi-norm $\|.\|_A$ defined by

$$\|x\|_A = \sqrt{\langle x, x\rangle_A} = \left\| A^{\frac{1}{2}} x \right\|, \forall x \in \mathcal{H}.$$

Observe that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. Then $\|.\|_A$ is a norm on $\mathcal{H}$ if and only if $A$ is an injective operator and the semi-normed space $(\mathcal{B}(\mathcal{H}), \|.\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed. For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an $A$-adjoint of $T$ if for every $x, y \in \mathcal{H}$

$$\langle Tx, y\rangle_A = \langle x, Sy\rangle_A,$$

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i.e., $AS = T^*A$. Moreover, $T$ is called $A$-selfadjoint if $AT = T^*A$, and it is called $A$-positive if $AT$ is positive, and we write $T \geq_A 0$. The existence of an $A$-adjoint operator is not guaranteed.

Let $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ be the set of all operators admitting $A^{\frac{1}{2}}$-adjoint. By the Douglas theorem, we have

$$\begin{aligned} \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) &= \left\{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}\left(T^*A^{\frac{1}{2}}\right) \subseteq \mathcal{R}\left(A^{\frac{1}{2}}\right)\right\} \\ &= \left\{T \in \mathcal{B}(\mathcal{H}) : \exists\, c > 0; \|Tx\|_A \leq c\,\|x\|_A, \forall x \in \mathcal{H}\right\}. \end{aligned}$$

An operator in $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ is called an $A$-bounded operator. Moreover, it was proved in [2] that if $T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, then

$$\|T\|_A := \sup_{\substack{x \in \overline{\mathcal{R}(A)} \\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{\substack{x \in \mathcal{H} \\ \|x\|_A = 1}} \|Tx\|_A < \infty.$$

In addition, if $T$ is $A$-bounded, then $T\left(\mathcal{N}\left(A\right)\right) \subset \mathcal{N}\left(A\right)$ and

$$\|Tx\|_A \leq \|T\|_A \|x\|_A, \forall x \in \mathcal{H}.$$

Moreover, for $T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ we have

$$\|T\|_A = \sup\left\{\left|\langle Tx, y\rangle_A\right| : x, y \in \mathcal{H} \text{ and } \|x\|_A = \|y\|_A = 1\right\}.$$

The set of all operators which admit $A$-adjoints is denoted by $\mathcal{B}_A(\mathcal{H})$. Again, by the Douglas theorem [6], we get

$$\begin{aligned} \mathcal{B}_A(\mathcal{H}) &= \left\{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}\left(T^*A\right) \subset \mathcal{R}\left(A\right)\right\} \\ &= \left\{T \in \mathcal{B}(\mathcal{H}) : \exists\, c > 0; \|ATx\| \leq c\,\|Ax\|, \forall x \in \mathcal{H}\right\}. \end{aligned}$$

Note that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ are two subalgebras of $\mathcal{B}(\mathcal{H})$ which are neither closed nor dense in $\mathcal{B}(\mathcal{H})$ (see [2, 1]). Moreover, the following inclusions

$$\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}),$$

hold with equality if $A$ is injective and has a closed range.

If $T \in \mathcal{B}_A(\mathcal{H})$, then $T$ admits $A$-adjoint operators. Moreover, there exists a distinguished $A$-adjoint operator of $T$, namely the reduced solution of the equation $AX = T^*A$, i.e., $T^{\sharp_A} = A^\dagger T^*A$, where $A^\dagger$ is the Moore-Penrose inverse of $A$. The $A$-adjoint operator $T^{\sharp_A}$ satisfies

$$AT^{\sharp_A} = T^*A,\ \mathcal{R}\left(T^{\sharp_A}\right) \subset \overline{\mathcal{R}\left(A\right)} \text{ and } \mathcal{N}\left(T^{\sharp_A}\right) = \mathcal{N}\left(T^*A\right).$$

We collect now some properties of $T^{\sharp_A}$ and its relationship with the seminorm $\|.\|_A$ . Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold:

(1) If $AT = TA$, then $T^{\sharp_A} = P_{\overline{\Re(A)}}T^*$.

(2) $T^{\sharp_A} \in \mathcal{B}_A(\mathcal{H})$, $\left(T^{\sharp_A}\right)^{\sharp_A} = P_{\overline{\Re(A)}}TP_{\overline{\Re(A)}}$ and $\left(\left(T^{\sharp_A}\right)^{\sharp_A}\right)^{\sharp_A} = T^{\sharp_A}$.

(3) $T^{\sharp_A}T$ and $TT^{\sharp_A}$ are $A$-selfadjoint and $A$-positive.

(4) If $S \in \mathcal{B}_A(\mathcal{H})$, then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$.

(5) $\|T\|_A = \left\|T^{\sharp_A}\right\|_A = \left\|T^{\sharp_A}T\right\|_A^{\frac{1}{2}} = \left\|TT^{\sharp_A}\right\|_A^{\frac{1}{2}}$.

From now on, to simplify notation, we write $P$ instead of $P_{\overline{\Re(A)}}$.

Recall that an operator $T \in \mathcal{B}_A(\mathcal{H})$ is called $A$-normal if $TT^{\sharp_A} = T^{\sharp_A}T$. It is familiar that every selfadjoint operator is normal. However, an $A$-selfadjoint operator is not necessarily $A$-normal. For example, consider the operators $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$. By a simple computation show that $T$ is $A$-selfadjoint and $TT^{\sharp_A} = \begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = T^{\sharp_A}T$. For more facts about this class of operators, we refer the reader to [18].

Any operator $T \in \mathcal{B}_A(\mathcal{H})$ can be represented as

$$T = \mathrm{Re}_A(T) + i\,\mathrm{Im}_A(T),$$

where

$$\mathrm{Re}_A(T) := \frac{T + T^{\sharp_A}}{2} \text{ and } \mathrm{Im}_A(T) := \frac{T - T^{\sharp_A}}{2i}.$$

The concept of the classical numerical radius was generalized to the $A$-numerical radius as follows

$$\omega_A(T) = \sup\left\{\left|\langle Tx, x\rangle_A\right| : x \in \mathcal{H},\ \|x\|_A = 1\right\}.$$

It follows that

$$\omega_A(T) = \omega_A\left(T^{\sharp_A}\right) \text{ for any } T \in \mathcal{B}_A(\mathcal{H}).$$

It was shown in [20] that for $T \in \mathcal{B}_A(\mathcal{H})$,

$$\omega_A(T) = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}T + e^{-i\theta}T^{\sharp_A}\right\|_A. \tag{1.1}$$

Another formula for $\omega_A(T)$ in terms of the operator semi-norm $\|\cdot\|_A$ is the following useful identity that has been recently defined in [20] as follows

$$\omega_A(T) = \sup_{\theta\in\mathbb{R}}\left\|\mathrm{Re}_A\left(e^{i\theta}T\right)\right\|_A = \sup_{\theta\in\mathbb{R}}\left\|\mathrm{Im}_A\left(e^{i\theta}T\right)\right\|_A. \tag{1.2}$$

A fundamental inequality for the $A$-numerical radius is the power inequality (see [4, 17]) which says that for $T \in \mathcal{B}_A(\mathcal{H})$,

$$\omega_A(T^n) \le \omega_A^n(T),\ \ n \in \mathbb{N}.$$

Further, $A$-numerical radius is a semi-norm on $\mathcal{B}_A(\mathcal{H})$ and that for every $T \in \mathcal{B}_A(\mathcal{H})$,

$$\frac{1}{2}\|T\|_A \le \omega_A(T) \le \|T\|_A. \tag{1.3}$$

Moreover, it is known that if $T$ is $A$-selfadjoint (or $A$-positive), then

$$\omega_A(T) = \|T\|_A . \tag{1.4}$$

Recently, some refinements of the inequalities (1.3) have been proved by many authors (e.g., see [3, 14, 20], and the references therein). Further generalizations and refinements of $A$-numerical radius are discussed in [4, 3, 14, 15, 17, 19], and the references therein.

One major topic of the present article is to study some inequalities of semi-Hilbertian space operators involving $A$-numerical radius and $A$-operator semi-norm, which generalize some classical numerical radius inequalities of complex Hilbert space operators due to Dragomir. The motivation comes from [7, 9, 12, 8, 11, 10].

## 2. Main results

In this section, we present our results. We first state the following lemma.

**Lemma 2.1.** *Let $x, y, z \in \mathcal{H}$. Then, the following two statements are equivalent:*

(i) $\operatorname{Re}(\langle y - x, x - z\rangle_A) \geq 0$.
(ii) $\left\|x - \frac{z+y}{2}\right\|_A \leq \frac{1}{2}\|y - z\|_A$.

*Proof.* Utilizing the fact that in any Hilbert space the following two statements are equivalent:
(i) $\operatorname{Re}(\langle b - a, a - c\rangle) \geq 0$, $a, b, c \in \mathcal{H}$;
(ii) $\left\|a - \frac{c+b}{2}\right\| \leq \frac{1}{2}\|b - c\|$.
Let $x, y, z \in \mathcal{H}$. Putting $x = A^{\frac{1}{2}}a, y = A^{\frac{1}{2}}b$ and $z = A^{\frac{1}{2}}c$, then we deduce the desired result. □

**Theorem 2.1.** *Let $T \in \mathcal{B}_A(\mathcal{H})$ . If $\lambda \in \mathbb{C}\backslash\{0\}$ and $r > 0$ such that*

$$\|T - \lambda I\|_A \leq r,$$

then

$$\|T\|_A \leq \omega_A(T) + \frac{1}{2}\frac{r^2}{|\lambda|}.$$

*Proof.* Let $x \in \mathcal{H}$ be an $A$-unit vector i.e, $\|x\|_A = 1$. Notice first that

$$\|Tx - \lambda x\|_A = \|(T - \lambda I)x\|_A \leq \|(T - \lambda I)\|_A \leq r.$$

It follows that

$$\|Tx - \lambda x\|_A^2 \leq r^2.$$

This implies that

$$\begin{aligned} \|Tx\|_A^2 + |\lambda|^2 &\leq 2\operatorname{Re}\left(\overline{\lambda}\left\langle Tx, x\right\rangle_A\right) + r^2 \\ &\leq 2|\lambda|\left|\left\langle Tx, x\right\rangle_A\right| + r^2. \end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get

$$\|T\|_A^2 + |\lambda|^2 \leq 2|\lambda|\,\omega_A(T) + r^2. \tag{2.1}$$

Since, obviously

$$2\|T\|_A|\lambda| \leq \|T\|_A^2 + |\lambda|^2. \tag{2.2}$$

From the inequalities (2.1) and (2.2), we have

$$\|T\|_A \leq \omega_A(T) + \frac{1}{2}\frac{r^2}{|\lambda|},$$

which proves the inequality of the theorem. □

As a consequence of Theorem 2.1, we have the following two corollaries.

**Corollary 2.1.** *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$ with $\beta \neq -\alpha, \alpha$. If*

$$\operatorname{Re}_A\left((T-\alpha)^{\sharp_A}(\beta - T)\right) \geq_A 0,$$

*then*

$$\|T\|_A \leq \omega_A(T) + \frac{1}{4}\frac{|\beta-\alpha|^2}{|\beta+\alpha|}.$$

*Proof.* If $\operatorname{Re}_A\left((T-\alpha)^{\sharp_A}(\beta - T)\right) \geq_A 0$, then

$$\left\langle (T-\alpha)^{\sharp_A}(\beta - T)x, x\right\rangle_A \geq 0.$$

Hence

$$\left\langle (\beta - T)x, (T-\alpha)x\right\rangle_A \geq 0, \text{ for all } x \in \mathcal{H}. \tag{2.3}$$

By using Lemma 2.1, we conclude that (2.3) is equivalent to

$$\left\| Tx - \frac{\beta+\alpha}{2} Ix \right\|_A \leq \frac{1}{2}|\beta - \alpha|, \text{ for any } x \in \mathcal{H}.$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get

$$\left\| T - \frac{\beta+\alpha}{2} I \right\|_A \leq \frac{1}{2}|\beta - \alpha|.$$

Now, applying Theorem 2.1 for $\lambda = \frac{\beta+\alpha}{2}$ and $r = \frac{1}{2}|\beta - \alpha|$, we deduce the desired result. □

**Corollary 2.2.** *Assume that $T, \lambda, r$ are as in Theorem 2.1. If, in addition, there exists $\rho \geq 0$ such that*

$$\left| |\lambda| - \omega_A(T) \right| \geq \rho,$$

then

$$\|T\|_A^2 - \omega_A^2(T) \leq r^2 - \rho^2.$$

*Proof.* From (2.1) of Theorem 2.1, we have

$$\begin{aligned} \|T\|_A^2 - \omega_A^2(T) &\leq r^2 - \omega_A^2(T) + 2|\lambda|\omega_A(T) - |\lambda|^2 \\ &= r^2 - (|\lambda| - \omega_A(T))^2 \\ &\leq r^2 - \rho^2. \end{aligned}$$

Hence, we get the desired inequality. □

**Remark 2.1.** *In particular, if $\|T - \lambda I\|_A \leq r$ and $|\lambda| = \omega_A(T), \lambda \in \mathbb{C}$, then*

$$\|T\|_A^2 - \omega_A^2(T) \leq r^2.$$

**Theorem 2.2.** *Let $T \in \mathcal{B}_A(\mathcal{H})$. If $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$ such that*

$$\|T - \lambda I\|_A \leq r,$$

then

$$\|T\|_A \leq \frac{|\lambda|}{\sqrt{|\lambda|^2 - r^2}} \omega_A(T). \tag{2.4}$$

*Proof.* From the inequality (2.1), we have

$$\|T\|_A^2 + |\lambda|^2 - r^2 \leq 2|\lambda|\omega_A(T),$$

which implies that

$$\frac{\|T\|_A^2}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \leq \frac{2|\lambda|}{\sqrt{|\lambda|^2 - r^2}} \omega_A(T). \tag{2.5}$$

Since, obviously as (2.2) we see that

$$2\|T\|_A \sqrt{|\lambda|^2 - r^2} \leq \|T\|_A^2 + |\lambda|^2 - r^2,$$

which implies

$$2\|T\|_A \leq \frac{\|T\|_A^2}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2}. \tag{2.6}$$

Now, from the inequalities (2.5) and (2.6), we get the desired inequality. □

**Remark 2.2.** *Note that if*

$$\frac{1}{2} \leq \frac{\sqrt{|\lambda|^2 - r^2}}{|\lambda|},$$

*which is equivalent to* $\frac{r}{|\lambda|} \leq \frac{\sqrt{3}}{2}$*, then (2.4) is a refinement of the first inequality in (1.3).*

The following corollary follows from Theorem 2.2.

**Corollary 2.3.** *Let* $\alpha, \beta \in \mathbb{C}$ *with* $\mathrm{Re}\,(\beta\overline{\alpha}) > 0$*. If* $T \in \mathcal{B}_A(\mathcal{H})$ *such that* $\mathrm{Re}_A\left((T-\alpha)^{\sharp_A}(\beta - T)\right)$ *is an A-positive operator, then*

$$\frac{2\sqrt{\mathrm{Re}\,(\beta\overline{\alpha})}}{|\beta+\alpha|} \leq \frac{\omega_A(T)}{\|T\|_A}, \tag{2.7}$$

and

$$\|T\|_A^2 - \omega_A^2(T) \leq \left|\frac{\beta-\alpha}{\beta+\alpha}\right|^2 \|T\|_A^2 .$$

*Proof.* If we put $\lambda = \frac{\beta+\alpha}{2}$ and $r = \frac{1}{2}|\beta - \alpha|$, then

$$|\lambda|^2 - r^2 = \left|\frac{\beta+\alpha}{2}\right|^2 - \left|\frac{\beta-\alpha}{2}\right|^2 = \mathrm{Re}\,(\beta\overline{\alpha}) > 0.$$

Now, by applying Theorem 2.2, we deduce the desired result. □

**Remark 2.3.** *If* $|\beta-\alpha| \leq \frac{\sqrt{3}}{2}|\beta+\alpha|$ *and* $\mathrm{Re}\,(\beta\overline{\alpha}) > 0$*, then (2.7) is a refinement of the first inequality in (1.3).*

**Theorem 2.3.** *Let* $T \in \mathcal{B}_A(\mathcal{H})$*. Then, for any* $\alpha \in [0,1]$ *and* $t \in \mathbb{R}$*, we have*

$$\|T\|_A^2 \leq \left[(1-\alpha)^2 + \alpha^2\right]\omega_A^2(T) + \alpha\|T - tI\|_A^2 + (1-\alpha)\|T - itI\|_A^2 .$$

*Proof.* We use the following inequality was obtained on a Hilbert space in [7]

$$\|x\|^2\|y\|^2 \leq \left[(1-\alpha)^2 + \alpha^2\right]|\langle x, y\rangle|^2 + \alpha\|ty - x\|^2 + (1-\alpha)\|ity - x\|^2 ,$$

for any $x, y \in \mathcal{H}$, $\alpha \in [0,1]$ and $t \in \mathbb{R}$.

Putting $x = A^{\frac{1}{2}}a, y = A^{\frac{1}{2}}b$ and $z = A^{\frac{1}{2}}c$, we deduce that

$$\|a\|_A^2\|b\|_A^2 \leq \left[(1-\alpha)^2 + \alpha^2\right]|\langle a, b\rangle_A|^2 + \alpha\|tb - a\|_A^2 + (1-\alpha)\|itb - a\|_A^2 .$$

Let $x \in \mathcal{H}$ with $\|x\|_A = 1$, we take $a = Tx$, $b = x$ in the above inequality, we get

$$\|Tx\|_A^2 \leq \left[(1-\alpha)^2 + \alpha^2\right]|\langle Tx, x\rangle_A|^2 + \alpha\|tx - Tx\|_A^2 + (1-\alpha)\|itx - Tx\|_A^2 .$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get

$$\|T\|_A^2 \leq \left[(1-\alpha)^2 + \alpha^2\right] \omega_A^2(T) + \alpha \|T - tI\|_A^2 + (1-\alpha)\|T - itI\|_A^2,$$

as required. □

**Corollary 2.4.** *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then*

$$0 \leq \|T\|_A^2 - \omega_A^2(T) \leq \begin{cases} \inf_{t\in\mathbb{R}} \|T - tI\|_A^2 \\ \inf_{t\in\mathbb{R}} \|T - itI\|_A^2 \end{cases}$$

*and*

$$\|T\|_A^2 \leq \frac{1}{2}\omega_A^2(T) + \frac{1}{2}\inf_{t\in\mathbb{R}}\left(\|T - tI\|_A^2 + \|T - itI\|_A^2\right).$$

**Theorem 2.4.** *Let $T, S \in \mathcal{B}_A(\mathcal{H})$ and let $r > 0$. If*

$$\|T - S\|_A \leq r,$$

*then*

$$\left\|\frac{T^{\sharp_A}T + S^{\sharp_A}S}{2}\right\|_A \leq \omega_A\left(S^{\sharp_A}T\right) + \frac{1}{2}r^2.$$

*Proof.* Since $\|T - S\|_A \leq r$, then for any $x \in \mathcal{H}$ with $\|x\|_A = 1$, it follows that

$$\begin{aligned} \|Tx\|_A^2 + \|Sx\|_A^2 &\leq 2\operatorname{Re}\left(\langle Tx, Sx\rangle_A\right) + r^2 \\ &\leq 2\left|\left\langle S^{\sharp_A}Tx, x\right\rangle_A\right| + r^2. \end{aligned}$$

However

$$\|Tx\|_A^2 + \|Sx\|_A^2 = \left\langle \left(T^{\sharp_A}T + S^{\sharp_A}S\right)x, Sx\right\rangle_A.$$

Therefore, we infer that

$$\left\langle \left(T^{\sharp_A}T + S^{\sharp_A}S\right)x, Sx\right\rangle_A \leq 2\left|\left\langle S^{\sharp_A}Tx, x\right\rangle_A\right| + r^2.$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get

$$\omega_A\left(T^{\sharp_A}T + S^{\sharp_A}S\right) \leq 2\omega_A\left(S^{\sharp_A}T\right) + r^2.$$

Since $T^{\sharp_A}T + S^{\sharp_A}S$ is $A$-positive operator, so by (1.4) we obtain

$$\omega_A\left(T^{\sharp_A}T + S^{\sharp_A}S\right) = \left\|T^{\sharp_A}T + S^{\sharp_A}S\right\|_A.$$

Therefore, we obtain

$$\left\|\frac{T^{\sharp_A}T + S^{\sharp_A}S}{2}\right\|_A \leq \omega_A\left(S^{\sharp_A}T\right) + \frac{1}{2}r^2,$$

as required. □

For $S=\lambda T^{\sharp_A}, \lambda\in\mathbb{C}$ in Theorem 2.4, we get the following result.

**Corollary 2.5.** *Let $T\in\mathcal{B}_A(\mathcal{H})$ and let $r>0$, $\lambda\in\mathbb{C}$. If*

$$\left\|T-\lambda T^{\sharp_A}\right\|_A\leq r,$$

*then*

$$\left\|\frac{T^{\sharp_A}T+|\lambda|^2\,TT^{\sharp_A}}{2}\right\|_A-|\lambda|\,\omega_A\left(T^2\right)\leq\frac{1}{2}r^2.$$

*Proof.* Letting $S=\lambda T^{\sharp_A}$ in Theorem 2.4, we obtain

$$\left\|\frac{T^{\sharp_A}T+|\lambda|^2\,T^{\sharp_A^{\sharp_A}}T^{\sharp_A}}{2}\right\|_A-|\lambda|\,\omega_A\left(T^{\sharp_A^{\sharp_A}}T\right)\leq\frac{1}{2}r^2.$$

Using the facts $\|X\|_A=\left\|X^{\sharp_A}\right\|_A$, $\left(X^{\sharp_A^{\sharp_A}}\right)^{\sharp_A}=X^{\sharp_A}$ and $(XY)^{\sharp_A}=Y^{\sharp_A}X^{\sharp_A}$ for every $X,Y\in\mathcal{B}_A(\mathcal{H})$, we have

$$\begin{aligned}\left\|\frac{T^{\sharp_A}T+|\lambda|^2\,T^{\sharp_A^{\sharp_A}}T^{\sharp_A}}{2}\right\|_A &= \left\|\left(\frac{T^{\sharp_A}T+|\lambda|^2\,T^{\sharp_A^{\sharp_A}}T^{\sharp_A}}{2}\right)^{\sharp_A}\right\|_A\\ &= \left\|\frac{T^{\sharp_A}T^{\sharp_A^{\sharp_A}}+|\lambda|^2\,T^{\sharp_A^{\sharp_A}}\left(T^{\sharp_A^{\sharp_A}}\right)^{\sharp_A}}{2}\right\|_A\\ &= \left\|\frac{T^{\sharp_A}T^{\sharp_A^{\sharp_A}}+|\lambda|^2\,T^{\sharp_A^{\sharp_A}}T^{\sharp_A}}{2}\right\|_A\\ &= \left\|\left(\frac{T^{\sharp_A}T+|\lambda|^2\,TT^{\sharp_A}}{2}\right)^{\sharp_A}\right\|_A\\ &= \left\|\frac{T^{\sharp_A}T+|\lambda|^2\,TT^{\sharp_A}}{2}\right\|_A.\end{aligned}$$

Also, by using the facts $\omega_A\left(X\right)=\omega_A\left(X^{\sharp_A}\right)$ and $\left(X^{\sharp_A^{\sharp_A}}\right)^{\sharp_A}=X^{\sharp_A}$ for every $X\in\mathcal{B}_A(\mathcal{H})$, we can observe that

$$\omega_A\left(T^{\sharp_A^{\sharp_A}}T\right)=\omega_A\left(T^{\sharp_A}\left(T^{\sharp_A^{\sharp_A}}\right)^{\sharp_A}\right)=\omega_A\left(\left(T^{\sharp_A}\right)^2\right)=\omega_A\left(T^2\right).$$

Therefore, we get the required result. □

**Corollary 2.6.** *Let $T\in\mathcal{B}_A(\mathcal{H})$ and let $r>0$, $\lambda\in\mathbb{C}$. If*

$$\|T-\lambda I\|_A\leq r,$$

*then*

$$\left\|\frac{T^{\sharp_A}T+|\lambda|^2\,I}{2}\right\|_A-\lambda\omega_A\left(T\right)\leq\frac{1}{2}r^2.$$

**Theorem 2.5.** *Let $T, S \in \mathcal{B}_A(\mathcal{H})$ and let $r \geq 1$. If $\|Sx\|_A \leq \|Tx\|_A$ for any $x \in \mathcal{H}$, then*

$$\left\|\frac{T^{\sharp_A}T + S^{\sharp_A}S}{2}\right\|_A^r \leq \left(\|T\|_A \|S\|_A\right)^{r-1} \omega_A\left(S^{\sharp_A}T\right) + \frac{1}{2}r^2 \|T\|_A^{2r-2} \|T-S\|_A^2 .$$

*Proof.* The following inequality obtained on a Hilbert space in [13]

$$\|x\|^{2r} + \|y\|^{2r} \leq 2\|x\|^{r-1}\|y\|^{r-1} \operatorname{Re}\left(\langle x, y\rangle\right) + r^2 \|x\|^{2r-2}\|x-y\|_A^2 ,$$

for all $x, y \in \mathcal{H}$ with $\|y\| \leq \|x\|$.

Putting $x = A^{\frac{1}{2}}a, y = A^{\frac{1}{2}}b$ then, we deduce that

$$\|a\|_A^{2r} + \|b\|_A^{2r} \leq 2\|a\|_A^{r-1}\|b\|_A^{r-1} \operatorname{Re}\left(\langle a, b\rangle_A\right) + r^2 \|a\|_A^{2r-2}\|a-b\|_A^2 ,$$

for all $a, b \in \mathcal{H}$ with $\|b\|_A \leq \|a\|_A$.

Let $x \in \mathcal{H}$, if we take $a = Tx$, $b = Sx$ in the above inequality, then we get

(2.8)
$$\|Tx\|_A^{2r} + \|Sx\|_A^{2r} \leq 2\|Tx\|_A^{r-1}\|Sx\|_A^{r-1} \operatorname{Re}\left(\langle Tx, Sx\rangle_A\right) + r^2 \|Tx\|_A^{2r-2}\|Tx-Sx\|_A^2 ,$$

such that $\|Sx\|_A \leq \|Tx\|_A$.

Now, by using the elementary inequality $\frac{\alpha^q+\beta^q}{2} \geq \left(\frac{\alpha+\beta}{2}\right)^q$ for $\alpha, \beta \geq 0$ and $q \geq 1$, we can see that

$$\begin{aligned} \frac{\|Tx\|_A^{2r} + \|Sx\|_A^{2r}}{2} &\geq \left(\frac{\|Tx\|_A^2 + \|Sx\|_A^2}{2}\right)^r \\ &= \left(\left\langle\left(\frac{T^{\sharp_A}T + S^{\sharp_A}S}{2}\right)x, x\right\rangle_A\right)^r . \end{aligned}$$

Thus

(2.9)
$$\left(\left\langle\left(\frac{T^{\sharp_A}T + S^{\sharp_A}S}{2}\right)x, x\right\rangle_A\right)^r \leq \frac{\|Tx\|_A^{2r} + \|Sx\|_A^{2r}}{2}.$$

Therefore, from (2.8) and (2.9), we deduce that

$$\begin{aligned} &\left(\left\langle\left(\frac{T^{\sharp_A}T + S^{\sharp_A}S}{2}\right)x, x\right\rangle_A\right)^r \\ \leq\ & \|Tx\|_A^{r-1}\|Sx\|_A^{r-1} \operatorname{Re}\left(\langle Tx, Sx\rangle_A\right) + \frac{1}{2}r^2 \|Tx\|_A^{2r-2}\|Tx-Sx\|_A^2 \\ \leq\ & \|Tx\|_A^{r-1}\|Sx\|_A^{r-1} \left|\left\langle S^{\sharp_A}Tx, x\right\rangle_A\right| + \frac{1}{2}r^2 \|Tx\|_A^{2r-2}\|Tx-Sx\|_A^2 . \end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality, we get

$$\omega_A^r\left(\frac{T^{\sharp_A}T + S^{\sharp_A}S}{2}\right) \leq \left(\|T\|_A \|S\|_A\right)^{r-1} \omega_A\left(S^{\sharp_A}T\right) + \frac{1}{2}r^2 \|T\|_A^{2r-2} \|T-S\|_A^2 .$$

So, by (1.4) we the deduce desired result. ∎

**Corollary 2.7.** *If $T \in \mathcal{B}_A(\mathcal{H})$ is an A-hyponormal operator, then*

$$\left\| \frac{T^{\sharp_A}T + TT^{\sharp_A}}{2} \right\|_A^r \le \|T\|_A^{2r-2} \left( \|T\|_A^2 + \frac{1}{2} r^2 \left\| T - T^{\sharp_A} \right\|_A^2 \right),$$

for any $r \ge 1$.

*Proof.* We recall that an operator $T \in \mathcal{B}_A(\mathcal{H})$ is $A$-hyponormal if $\left\|T^{\sharp_A}x\right\|_A \le \|Tx\|_A$ for all $x \in \mathcal{H}$ (see [16]). Now, if we choose in Theorem 2.4, $S = T^{\sharp_A}$ then, we obtain

$$\begin{aligned}
&\left\| \frac{T^{\sharp_A}T + T^{\sharp_A^{\sharp_A}}T^{\sharp_A}}{2} \right\|_A^r \\
\le\ & \left( \|T\|_A \left\|T^{\sharp_A}\right\|_A \right)^{r-1} \omega_A\left( T^{\sharp_A^{\sharp_A}} T \right) + \frac{1}{2} r^2 \|T\|_A^{2r-2} \left\| T - T^{\sharp_A} \right\|_A^2 \\
\le\ & \|T\|_A^{2r-2} \left\| T^{\sharp_A^{\sharp_A}} T \right\|_A + \frac{1}{2} r^2 \|T\|_A^{2r-2} \left\| T - T^{\sharp_A} \right\|_A^2 \\
&\text{(using the facts } \|X\|_A = \left\|X^{\sharp_A}\right\|_A \text{ and } \omega_A(X) \le \|X\|_A \text{ for any } X \in \mathcal{B}_A(\mathcal{H})) \\
\le\ & \|T\|_A^{2r-2} \left\| T^{\sharp_A^{\sharp_A}} \right\|_A \|T\|_A + \frac{1}{2} r^2 \|T\|_A^{2r-2} \left\| T - T^{\sharp_A} \right\|_A^2 \\
&\text{(using the fact } \|XY\|_A \le \|X\|_A \|Y\|_A \text{ for any } X, Y \in \mathcal{B}_A(\mathcal{H})) \\
=\ & \|T\|_A^{2r-2} \left( \|T\|_A^2 + \frac{1}{2} r^2 \left\| T - T^{\sharp_A} \right\|_A^2 \right). \\
&\text{(since } \left\| T^{\sharp_A^{\sharp_A}} \right\|_A = \|T\|_A )
\end{aligned}$$

Therefore, we have

$$\left\| \frac{T^{\sharp_A}T + T^{\sharp_A^{\sharp_A}}T^{\sharp_A}}{2} \right\|_A^r \le \|T\|_A^{2r-2} \left( \|T\|_A^2 + \frac{1}{2} r^2 \left\| T - T^{\sharp_A} \right\|_A^2 \right).$$

Also, by using the fact $\|X\|_A = \left\|X^{\sharp_A}\right\|_A$ for any $X \in \mathcal{B}_A(\mathcal{H})$, we can observe that

$$\begin{aligned}
\left\| T^{\sharp_A}T + T^{\sharp_A^{\sharp_A}}T^{\sharp_A} \right\|_A &= \left\| \left( T^{\sharp_A}T + T^{\sharp_A^{\sharp_A}}T^{\sharp_A} \right)^{\sharp_A} \right\|_A \\
&= \left\| T^{\sharp_A}T^{\sharp_A^{\sharp_A}} + T^{\sharp_A^{\sharp_A}} \left( T^{\sharp_A^{\sharp_A}} \right)^{\sharp_A} \right\|_A \\
&= \left\| T^{\sharp_A}T^{\sharp_A^{\sharp_A}} + T^{\sharp_A^{\sharp_A}}T^{\sharp_A} \right\|_A \\
\text{(since } \left( T^{\sharp_A^{\sharp_A}} \right)^{\sharp_A} &= T^{\sharp_A} ) \\
&= \left\| \left( T^{\sharp_A}T + TT^{\sharp_A} \right)^{\sharp_A} \right\|_A \\
&= \left\| T^{\sharp_A}T + TT^{\sharp_A} \right\|_A .
\end{aligned}$$

Therefore, we infer that

$$\left\| \frac{T^{\sharp_A}T + TT^{\sharp_A}}{2} \right\|_A^r \leq \|T\|_A^{2r-2} \left( \|T\|_A^2 + \frac{1}{2} r^2 \left\| T - T^{\sharp_A} \right\|_A^2 \right),$$

as required. □

The following lemma plays a crucial role in our next proof, which can be found in [5].

**Lemma 2.2.** *Let $T \in \mathcal{B}(\mathcal{H})$ be an $A$-positive operator and let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Then*

$$\langle Tx, x \rangle_A^n \leq \langle T^n x, x \rangle_A \text{ for all } n \in \mathbb{N}^*.$$

**Theorem 2.6.** *Let $T, S, W, R \in \mathcal{B}_A(\mathcal{H})$ and $n, m \in \mathbb{N}^*$. Then*

$$\left\| \frac{S^{\sharp_A}T + R^{\sharp_A}W}{2} \right\|_A^2 \leq \left\| \frac{\left(T^{\sharp_A}T\right)^n + \left(W^{\sharp_A}W\right)^n}{2} \right\|_A^{\frac{1}{n}} \left\| \frac{\left(S^{\sharp_A}S\right)^m + \left(R^{\sharp_A}R\right)^m}{2} \right\|_A^{\frac{1}{m}}.$$

*Proof.* Let $x, y \in \mathcal{H}$, we have

$$\begin{aligned} & \left| \left\langle \left(S^{\sharp_A}T + R^{\sharp_A}W\right)x, y \right\rangle_A \right|^2 \\ = & \left| \left\langle \left(S^{\sharp_A}T\right)x, y \right\rangle_A + \left\langle \left(R^{\sharp_A}W\right)x, y \right\rangle_A \right|^2 \\ \leq & \left( \left| \left\langle \left(S^{\sharp_A}T\right)x, y \right\rangle_A \right| + \left| \left\langle \left(R^{\sharp_A}W\right)x, y \right\rangle_A \right| \right)^2 \\ \leq & \left( \left\langle T^{\sharp_A}Tx, x \right\rangle_A^{\frac{1}{2}} \left\langle S^{\sharp_A}Sy, y \right\rangle_A^{\frac{1}{2}} + \left\langle W^{\sharp_A}Wx, x \right\rangle_A^{\frac{1}{2}} \left\langle R^{\sharp_A}Ry, y \right\rangle_A^{\frac{1}{2}} \right)^2. \\ & \text{(by the Cauchy-Schwarz inequality)} \end{aligned}$$

Now, on utilizing the elementary inequality:

$$(ab + cd)^2 \leq \left(a^2 + c^2\right)\left(b^2 + d^2\right) \text{ for } a, b, c, d \in \mathbb{R},$$

it follows that

$$\begin{aligned} & \left( \left\langle T^{\sharp_A}Tx, x \right\rangle_A^{\frac{1}{2}} \left\langle S^{\sharp_A}Sy, y \right\rangle_A^{\frac{1}{2}} + \left\langle W^{\sharp_A}Wx, x \right\rangle_A^{\frac{1}{2}} \left\langle R^{\sharp_A}Ry, y \right\rangle_A^{\frac{1}{2}} \right)^2 \\ \leq & \left( \left\langle T^{\sharp_A}Tx, x \right\rangle_A + \left\langle W^{\sharp_A}Wx, x \right\rangle_A \right) \left( \left\langle S^{\sharp_A}Sy, y \right\rangle_A + \left\langle R^{\sharp_A}Ry, y \right\rangle_A \right), \end{aligned}$$

for all $x, y \in \mathcal{H}$.

Therefore, we have

$$
\begin{aligned}
&\left|\left\langle\left(\frac{S^{\sharp_A}T+R^{\sharp_A}W}{2}\right)x,y\right\rangle_A\right|^2 \\
&\le \left(\frac{\langle T^{\sharp_A}Tx,x\rangle_A+\langle W^{\sharp_A}Wx,x\rangle_A}{2}\right)\left(\frac{\langle S^{\sharp_A}Sy,y\rangle_A+\langle R^{\sharp_A}Ry,y\rangle_A}{2}\right) \\
&= \left[\left(\frac{\langle T^{\sharp_A}Tx,x\rangle_A+\langle W^{\sharp_A}Wx,x\rangle_A}{2}\right)^n\right]^{\frac{1}{n}} \\
&\quad\times\left[\left(\frac{\langle S^{\sharp_A}Sy,y\rangle_A+\langle R^{\sharp_A}Ry,y\rangle_A}{2}\right)^m\right]^{\frac{1}{m}} \\
&\le \left(\frac{\langle T^{\sharp_A}Tx,x\rangle_A^n+\langle W^{\sharp_A}Wx,x\rangle_A^n}{2}\right)^{\frac{1}{n}} \\
&\quad\times\left(\frac{\langle S^{\sharp_A}Sy,y\rangle_A^m+\langle R^{\sharp_A}Ry,y\rangle_A^m}{2}\right)^{\frac{1}{m}}
\end{aligned}
$$

(as $\left(\frac{\alpha+\beta}{2}\right)^q \le \frac{\alpha^q+\beta^q}{2}$ for $\alpha,\beta\ge 0$ and $q\ge 1$)

$$
\begin{aligned}
&\le \left(\frac{\langle (T^{\sharp_A}T)^n x,x\rangle_A+\langle (W^{\sharp_A}W)^n x,x\rangle_A}{2}\right)^{\frac{1}{n}} \\
&\quad\times\left(\frac{\langle (S^{\sharp_A}S)^m y,y\rangle_A+\langle (R^{\sharp_A}R)^m y,y\rangle_A}{2}\right)^{\frac{1}{m}} \quad \text{(by Lemma 2.2)} \\
&= \left(\frac{\langle [(T^{\sharp_A}T)^n+(W^{\sharp_A}W)^n] x,x\rangle_A}{2}\right)^{\frac{1}{n}} \\
&\quad\times\left(\frac{\langle [(S^{\sharp_A}S)^m+(R^{\sharp_A}R)^m] y,y\rangle_A}{2}\right)^{\frac{1}{m}}
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left|\left\langle\left(S^{\sharp_A}T+R^{\sharp_A}W\right)x,y\right\rangle_A\right|^2 \\
&\quad\le \left(\frac{\langle [(T^{\sharp_A}T)^n+(W^{\sharp_A}W)^n] x,x\rangle_A}{2}\right)^{\frac{1}{n}} \\
&\qquad\times\left(\frac{\langle [(S^{\sharp_A}S)^m+(R^{\sharp_A}R)^m] y,y\rangle_A}{2}\right)^{\frac{1}{m}}.
\end{aligned}
$$

Taking the supremum over $x,y\in\mathcal{H}$ with $\|x\|_A=\|y\|_A=1$ in the above inequality, we get

$$\left\|\frac{S^{\sharp_A}T+R^{\sharp_A}W}{2}\right\|_A^2 \leq \left\|\frac{\left(T^{\sharp_A}T\right)^n+\left(W^{\sharp_A}W\right)^n}{2}\right\|_A^{\frac{1}{n}}\left\|\frac{\left(S^{\sharp_A}S\right)^m+\left(R^{\sharp_A}R\right)^m}{2}\right\|_A^{\frac{1}{m}},$$

as required. □

**Remark 2.4.** *If $n=m$, then we have*

$$\left\|\frac{S^{\sharp_A}T+R^{\sharp_A}W}{2}\right\|_A^{2n} \leq \left\|\frac{\left(T^{\sharp_A}T\right)^n+\left(W^{\sharp_A}W\right)^n}{2}\right\|_A\left\|\frac{\left(S^{\sharp_A}S\right)^n+\left(R^{\sharp_A}R\right)^n}{2}\right\|_A.$$

Theorem 2.6 includes several $A$-operator seminorm inequalities. Some of these inequalities are demonstrated in the following corollaries.

**Corollary 2.8.** *If $S,R\in\mathcal{B}_A(\mathcal{H})$ and $n\in\mathbb{N}^*$, then*

$$\left\|\frac{S+R}{2}\right\|_A^{2n} \leq \left\|\frac{\left(S^{\sharp_A}S\right)^n+\left(R^{\sharp_A}R\right)^n}{2}\right\|_A.$$

*Proof.* If we take $T=W=I$, then we get

$$\left\|\frac{S^{\sharp_A}+R^{\sharp_A}}{2}\right\|_A^{2n} \leq \left\|\frac{\left(S^{\sharp_A}S\right)^n+\left(R^{\sharp_A}R\right)^n}{2}\right\|_A.$$

Therefore, by using the fact $\|X\|_A=\left\|X^{\sharp_A}\right\|_A$ for every $X\in\mathcal{B}_A(\mathcal{H})$, we deduce the desired result. □

Putting $R=T$ and $W=S$ in Theorem 2.6, we get the following corollary.

**Corollary 2.9.** *Let $T,S\in\mathcal{B}_A(\mathcal{H})$ and $n,m\in\mathbb{N}^*$. Then*

$$\left\|\frac{S^{\sharp_A}T+T^{\sharp_A}S}{2}\right\|_A^2 \leq \left\|\frac{\left(T^{\sharp_A}T\right)^n+\left(S^{\sharp_A}S\right)^n}{2}\right\|_A^{\frac{1}{n}}\left\|\frac{\left(T^{\sharp_A}T\right)^m+\left(S^{\sharp_A}S\right)^m}{2}\right\|_A^{\frac{1}{m}}.$$

*In particular, we have*

$$\left\|\frac{S^{\sharp_A}T+T^{\sharp_A}S}{2}\right\|_A^{n} \leq \left\|\frac{\left(T^{\sharp_A}T\right)^n+\left(S^{\sharp_A}S\right)^n}{2}\right\|_A.$$

**Corollary 2.10.** *If $T\in\mathcal{B}_A(\mathcal{H})$ and $T=\operatorname{Re}_A(T)+i\operatorname{Im}_A(T)$, then*

$$\|T\|_A^{2n}\leq 2^{2n-1}\left\|\left(\operatorname{Re}_A(T)\right)^{2n}+\left(\operatorname{Im}_A(T)\right)^{2n}\right\|_A, \textit{ for all } n\in\mathbb{N}^*.$$

*Proof.* If we take $S = \text{Re}_A(T)$ and $R = \text{Im}_A(T)$ in Corollary 2.8 then we see that

$$\left\|\frac{T}{2}\right\|_A^{2n} \leq \left\|\frac{\left(\text{Re}_A(T)^{\sharp_A}\,\text{Re}_A(T)\right)^n + \left(\text{Im}_A(T)^{\sharp_A}\,\text{Im}_A(T)\right)^n}{2}\right\|_A.$$

This implies that

$$\|T\|_A^{2n} \leq 2^{2n-1}\left\|\left(\text{Re}_A(T)^{\sharp_A}\,\text{Re}_A(T)\right)^n + \left(\text{Im}_A(T)^{\sharp_A}\,\text{Im}_A(T)\right)^n\right\|_A.$$

One can easily check that

$$\text{Re}_A(T)^{\sharp_A} = \text{Re}_A(T)^{\sharp_A^{\sharp_A}} \text{ and } \text{Im}_A(T)^{\sharp_A} = \text{Im}_A(T)^{\sharp_A^{\sharp_A}}.$$

Moreover, by using the fact $\|X\|_A = \left\|X^{\sharp_A}\right\|_A$ for every $X \in \mathcal{B}_A(\mathcal{H})$, we can observe that

$$\begin{aligned}
&\left\|\left(\text{Re}_A(T)^{\sharp_A}\,\text{Re}_A(T)\right)^n + \left(\text{Im}_A(T)^{\sharp_A}\,\text{Im}_A(T)\right)^n\right\|_A \\
&= \left\|\left(\text{Re}_A(T)^{\sharp_A}\,\text{Re}_A(T)^{\sharp_A^{\sharp_A}}\right)^n + \left(\text{Im}_A(T)^{\sharp_A}\,\text{Im}_A(T)^{\sharp_A^{\sharp_A}}\right)^n\right\|_A \\
&= \left\|\left(\text{Re}_A(T)^{\sharp_A}\right)^{2n} + \left(\text{Im}_A(T)^{\sharp_A}\right)^{2n}\right\|_A \\
&\text{(since } \text{Re}_A(T)^{\sharp_A} = \text{Re}_A(T)^{\sharp_A^{\sharp_A}} \text{ and } \text{Im}_A(T)^{\sharp_A} = \text{Im}_A(T)^{\sharp_A^{\sharp_A}}\text{)} \\
&= \left\|\left((\text{Re}_A(T))^{2n} + (\text{Im}_A(T))^{2n}\right)^{\sharp_A}\right\|_A \\
&= \left\|(\text{Re}_A(T))^{2n} + (\text{Im}_A(T))^{2n}\right\|_A.
\end{aligned}$$

Therefore, we infer that

$$\|T\|_A^{2n} \leq 2^{2n-1}\left\|(\text{Re}_A(T))^{2n} + (\text{Im}_A(T))^{2n}\right\|_A.$$

Hence, the proof is complete. □

**Remark 2.5.** *Using similar arguments as used in Corollary 2.10, we have*

$$\|\text{Re}_A(T)\|_A^{2n} \leq \left\|\frac{\left(T^{\sharp_A}T\right)^n + \left(TT^{\sharp_A}\right)^n}{2}\right\|_A,$$

*and*

$$\|\text{Im}_A(T)\|_A^{2n} \leq \left\|\frac{\left(T^{\sharp_A}T\right)^n + \left(TT^{\sharp_A}\right)^n}{2}\right\|_A,$$

*for any* $n \in \mathbb{N}^*$.

## Acknowledgment

The author would like to thank the referee for careful reading and kind suggestions.

## References

[1] M. Laura Arias, Gustavo Corach, and M. Celeste Gonzalez. Metric properties of projections in semi-Hilbertian spaces. *Integral Equations Operator Theory*, 62(1):11–28, 2008. doi:10.1007/s00020-008-1613-6.

[2] M. Laura Arias, Gustavo Corach, and M. Celeste Gonzalez. Partial isometries in semi-Hilbertian spaces. *Linear Algebra Appl.*, 428(7):1460–1475, 2008. doi:10.1016/j.laa.2007.09.031.

[3] Pintu Bhunia, Raj Kumar Nayak, and Kallol Paul. Refinements of A-numerical radius inequalities and their applications. *Adv. Oper. Theory*, 5(4):1498–1511, 2020. doi:10.1007/s43036-020-00056-8.

[4] Pintu Bhunia, Kallol Paul, and Raj Kumar Nayak. On inequalities for $A$-numerical radius of operators. *Electron. J. Linear Algebra*, 36:143–157, 2020.

[5] C. Conde and K. Feki. On some inequalities for the generalized joint numerical radius of semi-hilbert space operators. *Ricerche Mat.*, 73(2):661–679, 2024. doi:10.1007/s11587-021-00629-6.

[6] R. G. Douglas. On majorization, factorization, and range inclusion of operators on Hilbert space. *Proc. Amer. Math. Soc.*, 17:413–415, 1966. doi:10.2307/2035178.

[7] S. S. Dragomir. A potpourri of Schwarz related inequalities in inner product spaces. II. *JIPAM. J. Inequal. Pure Appl. Math.*, 7(1):Article 14, 11, 2006.

[8] S. S. Dragomir. Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. *Tamkang J. Math.*, 39(1):1–7, 2008.

[9] S. S. Dragomir. Power inequalities for the numerical radius of a product of two operators in Hilbert spaces. *Sarajevo J. Math.*, 5(18)(2):269–278, 2009.

[10] Sever S. Dragomir. *Advances in inequalities of the Schwarz, triangle and Heisenberg type in inner product spaces.* Nova Science Publishers, Inc., New York, 2007.

[11] Sever S. Dragomir. A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces. *Banach J. Math. Anal.*, 1(2):154–175, 2007. doi:10.15352/bjma/1240336213.

[12] Silvestru Sever Dragomir. *Inequalities for the numerical radius of linear operators in Hilbert spaces.* SpringerBriefs in Mathematics. Springer, Cham, 2013. doi:10.1007/978-3-319-01448-7.

[13] A. A. Goldstein, J. V. Ryff, and L. E. Clarke. Problems and Solutions: Solutions of Advanced Problems: 5473. *Amer. Math. Monthly*, 75(3):309–310, 1968. doi:10.2307/2314992.

[14] Messaoud Guesba. Some generalizations of $A$-numerical radius inequalities for semi-Hilbert space operators. *Boll. Unione Mat. Ital.*, 14(4):681–692, 2021. doi:10.1007/s40574-021-00307-3.

[15] Messaoud Guesba, Pintu Bhunia, and Kallol Paul. $A$-numerical radius inequalities and $A$-translatable radii of semi-Hilbert space operators. *Filomat*, 37(11):3443–3456, 2023. doi:10.2298/fil2306741s.

[16] Ould Ahmed Mahmoud Sid Ahmed and Abdelkader Benali. Hyponormal and $k$-quasi-hyponormal operators on semi-Hilbertian spaces. *Aust. J. Math. Anal. Appl.*, 13(1):Art. 7, 22, 2016.

[17] M. S. Moslehian, Q. Xu, and A. Zamani. Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces. *Linear Algebra Appl.*, 591:299–321, 2020. doi:10.1016/j.laa.2020.01.015.

[18] Adel Saddi. $A$-normal operators in semi Hilbertian spaces. *Aust. J. Math. Anal. Appl.*, 9(1):Art. 5, 12, 2012.

[19] Qingxiang Xu, Zhongming Ye, and Ali Zamani. Some upper bounds for the $\mathbb{A}$-numerical radius of $2 \times 2$ block matrices. *Adv. Oper. Theory*, 6(1):Paper No. 1, 13, 2021. doi:10.1007/s43036-020-00102-5.

[20] Ali Zamani. $A$-numerical radius inequalities for semi-Hilbertian space operators. *Linear Algebra Appl.*, 578:159–183, 2019. doi:10.1016/j.laa.2019.05.012.

*Received by the editors April 15, 2023*
*First published online August 25, 2023*