NOVI SAD J. MATH. VOL. 54, NO. 2, 2024, 193-211
https://doi.org/10.30755/NSJOM.15679

On the Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_b$

Mirko Lepović[1]

Abstract. We say that a simple graph G is Seidel integral if its Seidel spectrum consists entirely of integers. If $\alpha K_a \cup \beta K_b$ is Seidel integral we show that it belongs to one of the following classes of Seidel integral graphs

$$\left[\frac{k(2t-1)}{\tau}\,x_0 + \frac{m(2t-1)}{\tau}\,z\right]K_a \cup \left[\frac{k(2t-1)}{\tau}\,y_0 + \frac{a}{\tau}\,z\right](2n-1)K_b\,,$$

where (i) $a = (t + 2\ell n - (\ell + n))k + (2\ell - 1)m$ and $b = (2\ell - 1)m$; (ii) $t, k, \ell, m, n \in \mathbb{N}$ such that $(m, 2n-1) = 1$, $(2n-1, 2t-1) = 1$ and $(2\ell - 1, 2t-1) = 1$; (iii) $\tau = (a, m(2t-1))$ such that $\tau \mid k\,(2t-1)$; (iv) (x_0, y_0) is a particular solution of the linear Diophantine equation $ax - m(2t-1)y = \tau$ and (v) $z \geq z_0$ where z_0 is the least integer such that $\left(\frac{k(2t-1)}{\tau}\,x_0 + \frac{m(2t-1)}{\tau}\,z_0\right) \geq 1$ and $\left(\frac{k(2t-1)}{\tau}\,y_0 + \frac{a}{\tau}\,z_0\right) \geq 1\,;$

$$\left[\frac{2kt}{\tau}\,x_0 + \frac{tm}{\tau}\,z\right]K_a \cup \left[\frac{2kt}{\tau}\,y_0 + \frac{a}{\tau}\,z\right]n\,K_b\,,$$

where (i) $a = (t + \ell n)k + \ell m$ and $b = \ell m$; (ii) $t, k, \ell, m, n \in \mathbb{N}$ such that $(m, n) = 1$, $(n, t) = 1$, $(\ell, t) = 1$ and $(t + \ell n, 2) = 1$; (iii) $\tau = (a, tm)$ such that $\tau \mid 2kt$; (iv) (x_0, y_0) is a particular solution of the linear Diophantine equation $ax - (tm)y = \tau$ and (v) $z \geq z_0$ where z_0 is the least integer such that $\left(\frac{2kt}{\tau}\,x_0 + \frac{tm}{\tau}\,z_0\right) \geq 1$ and $\left(\frac{2kt}{\tau}\,y_0 + \frac{a}{\tau}\,z\right) \geq 1$.

AMS Mathematics Subject Classification (2010): 05C50

Key words and phrases: graph; eigenvalue; Diophantine equation

1. Introduction

Let G be a simple graph of order n and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of its (0,1) adjacency matrix of G. The spectrum of G is the set of its eigenvalues and is denoted by $\sigma(G)$. A graph G is said to be integral if its spectrum $\sigma(G)$ consists only of integers [1]. We say that $A^* = [s_{ij}]$ is the Seidel adjacency matrix of the graph G if $s_{ij} = -1$ for any two adjacent vertices i and j, $s_{ij} = 1$ for any two non-adjacent vertices i and j, and $s_{ij} = 0$ if $i = j$. The Seidel spectrum of G is the set of eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of its $(0, -1, 1)$ adjacency matrix $A^* = A^*(G)$ and is denoted by $\sigma^*(G)$. A graph G is said to be Seidel integral if its Seidel spectrum $\sigma^*(G)$ consists only of integers.

[1]Tihomira Vuksanovića 32, 34000 Kragujevac, Serbia. e-mail: lepovic@kg.ac.rs, ORCID iD: orcid.org/0000-0002-2150-1483

We say that an eigenvalue μ is main if and only if $\langle \mathbf{j}, \mathbf{P}\mathbf{j}\rangle = n\cos^2\alpha > 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The quantity $\beta = |\cos\alpha|$ is called the main angle of μ. Similarly, we say that a Seidel eigenvalue μ^* is the Seidel main eigenvalue if and only if $\langle \mathbf{j}, \mathbf{P}^*\mathbf{j}\rangle = n\cos^2\alpha^* > 0$, where $\mathbf{P}^*$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_{A^*}(\mu^*)$. The quantity $\beta^* = |\cos\alpha^*|$ is called the Seidel main angle of μ^*. In [1] it was proved that the graph G and its complement $\overline{G}$ have the same number of main eigenvalues. We also know that $|\mathcal{M}(G)| = |\mathcal{M}^*(G)|$, where $\mathcal{M}(G)$ and $\mathcal{M}^*(G)$ denote the sets of all main and the Seidel main eigenvalues of G, respectively.

Let G be a graph of order n with exactly two main eigenvalues μ_1 and μ_2 and let $n_1 = n\beta_1^2$ and $n_2 = n\beta_2^2$.

Theorem 1.1 (Lepović [3])**.** *Let G be a graph of order n with two main eigenvalues μ_1 and μ_2. Then*

$$\mu^*_{1,2} = \frac{n-2-2\mu_1-2\mu_2}{2} \pm \frac{\sqrt{\left(2\mu_1-2\mu_2+n\right)^2 - 8n_1\left(\mu_1-\mu_2\right)}}{2}. \tag{1.1}$$

Besides, we have[2]

$$n^*_{1,2} = \frac{n}{2} \pm \frac{n^2+2\left(n-2n_1\right)\left(\mu_1-\mu_2\right)}{2\sqrt{\left(2\mu_1-2\mu_2+n\right)^2 - 8n_1\left(\mu_1-\mu_2\right)}}, \tag{1.2}$$

where $n_1^* = n(\beta_1^*)^2$ and $n_2^* = n(\beta_2^*)^2$.

We note that $\alpha K_a \cup \beta K_b$ is an integral graph with two main eigenvalues $\mu_1 = a-1$ and $\mu_2 = b-1$, for any $\alpha, \beta, a, b \in \mathbb{N}$ with $a > b$. Of course, K_n is the complete graph on n vertices while mG denotes the m-fold union of the graph G. As is pointed out in [3], if G is an integral graph then it is Seidel integral if and only if the main Seidel spectrum of G contains integral values. In view of this fact, $\alpha K_a \cup \beta K_b$ is Seidel integral if and only if its largest Seidel main eigenvalue $\mu_1^* \in \mathbb{N}$. We have established in [4] a characterization of integral graphs which belong to the class $\overline{\alpha K_a \cup \beta K_b}$. We now proceed to establish a characterization of Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_b$, as follows.

2. Main results

First, note that $o = \alpha a + \beta b$ is the order of $\alpha K_a \cup \beta K_b$. Then according to (1.1) we get implicitly

$$\mu^*_{1,2} = \frac{\alpha a + \beta b + 2 - 2a - 2b \pm \delta}{2}, \tag{2.1}$$

[2]If G is a graph of order n with k main eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$ and Seidel main eigenvalues $\mu_1^*, \mu_2^*, \ldots, \mu_k^*$ then $n_1 + n_2 + \cdots + n_k = n$ and $n_1^* + n_2^* + \cdots + n_k^* = n$, where $n_i = n\beta_i^2$ and $n_i^* = n(\beta_i^*)^2$ for $i = 1, 2, \ldots, k$. Of course, if G is a graph with exactly two main eigenvalues then according to (1.2) we also have $n_1^* + n_2^* = n$.

where $\delta = \sqrt{\big((\alpha+2)a+(\beta-2)b\big)^2 - 8\alpha a\big(a-b\big)}$. Then $\alpha K_a \cup \beta K_b$ is Seidel integral if and only if $(\alpha, \beta, a, b, \delta)$ represents a positive integral solution of the Diophantine equation

$$\big((\alpha+2)a+(\beta-2)b\big)^2 - 8\alpha a\big(a-b\big) = \delta^2 . \tag{2.2}$$

Therefore, the characterization of Seidel integral graphs which are related to the class $\alpha K_a \cup \beta K_b$ is reduced to the problem of finding the most general positive solution of the equation (2.2).

Next, $\mu_1^*\mu_2^* = 4\mu_1\mu_2 - 2(n_1-1)\mu_2 - 2(n_2-1)\mu_1 - (n-1)$ for any G with two main eigenvalues (see [3]). In the case that $G = \alpha K_a \cup \beta K_b$ this relation is transformed into

$$\big(\,\mu_1^* - 1\big)\big(\,\mu_2^* - 1\big) = 2ab(2-\alpha-\beta) . \tag{2.3}$$

In the sequel (m, n) denotes the highest common divisor of integers $m, n \in \mathbb{N}$ while $m \mid n$ means that m divides n. With this notation, in order to demonstrate a method applied in this paper, we prove first the following two results:

Theorem 2.1. *If $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = 2ab+1$ then it belongs to the class of Seidel integral graphs*

$$tm\, K_{sn-1} \cup (2s-t)n\, K_{sm-1} , \tag{2.4}$$

where $m, n \in \mathbb{N}$ and $n > m$, $sm \geq 2$, $t < 2s$ such that $(s,t) = 1$.

Proof. Assume that $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = 2ab+1$. Using (2.3) we obtain $\mu_2^* = 3-\alpha-\beta$ and $\delta = 2ab+\alpha+\beta-2$. Then Diophantine equation (2.2) is reduced to

$$(b+1)\big(2a-(\alpha+\beta-2)\big) = \alpha(a-b) .$$

Let $b+1 = r\alpha$ where $r = \frac{s}{t}$ such that $(s,t) = 1$. Then from the last relation we obtain $a-b = r\big(2a-(\alpha+\beta-2)\big)$. In view of this, we get

$$\alpha = \frac{t}{s}\,(b+1) \quad \text{and} \quad \beta = \frac{2s-t}{s}\,(a+1) .$$

Since $(s,t) = 1$ it follows that $(2s-t, s) = 1$. Then it must be $s \mid (b+1)$ and $s \mid (a+1)$. Let $b+1 = sm$ and $a+1 = sn$. So we get $\alpha = tm$ and $\beta = (2s-t)n$, where $t < 2s$. □

Theorem 2.2. *If $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = ab+1$ then it belongs to the class of Seidel integral graphs*

$$tm\, K_{sn-2} \cup (s-t)n\, K_{sm-2} , \tag{2.5}$$

where $m, n \in \mathbb{N}$ and $n > m$, $sm \geq 3$, $t < s$ such that $(s,t) = 1$.

PROOF. Assume that $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = ab + 1$. Using (2.3) we obtain $\mu_2^* = 5 - 2\alpha - 2\beta$ and $\delta = ab + 2(\alpha + \beta - 2)$. Then Diophantine equation (2.2) is reduced to

$$(b+2)\big(a - (\alpha + \beta - 2)\big) = \alpha(a - b)\,.$$

Let $b + 2 = r\alpha$ where $r = \frac{s}{t}$ such that $(s,t) = 1$. Then from the last relation we obtain $a - b = r\big(a - (\alpha + \beta - 2)\big)$. In view of this, we get

$$\alpha = \frac{t}{s}\,(b+2) \quad \text{and} \quad \beta = \frac{s-t}{s}\,(a+2)\,.$$

Since $(s,t) = 1$ it follows that $(s-t, s) = 1$. Then it must be $s \mid (b+2)$ and $s \mid (a+2)$. Let $b + 2 = sm$ and $a + 2 = sn$. So we get $\alpha = tm$ and $\beta = (s-t)n$, where $t < s$. $\square$

Remark 2.3. With the condition $a > b$ note that the parameters α, β, a, b determine the graph $\alpha K_a \cup \beta K_b$ up to isomorphism.

In what follows, we show that there exists a one-to-one correspondence between the Seidel integral graphs $\alpha K_a \cup \beta K_b$ with $\mu_1^* = 2ab + 1$ and the parameters m, n, s, t.

Proposition 2.4. *If $\alpha K_a \cup \beta K_b$ is a Seidel integral graph with $\mu_1^* = 2ab + 1$ then it uniquely determines the parameters m, n, s, t.*

PROOF. Let us assume that m_1, n_1, s_1, t_1 and m_2, n_2, s_2, t_2 determine the same Seidel integral graph $\alpha K_a \cup \beta K_b$ with the largest Seidel main eigenvalue $\mu_1^* = 2ab + 1$. Then according to Remark 2.3 and relation (2.4) we have: (i) $t_1 m_1 = t_2 m_2$; (ii) $(2s_1 - t_1)n_1 = (2s_2 - t_2)n_2$; (iii) $s_1 n_1 - 1 = s_2 n_2 - 1$ and (iv) $s_1 m_1 - 1 = s_2 m_2 - 1$. Using (i) and (iv) we get $\frac{t_1}{s_1} = \frac{t_2}{s_2}$. Since $(t_1, s_1) = 1$ and $(t_2, s_2) = 1$ it follows that $t_1 = t_2$ and $s_1 = s_2$. Consequently, using (i) and (ii) we obtain $m_1 = m_2$ and $n_1 = n_2$. $\square$

In a quite analogous manner, using Remark 2.3 and relation (2.5) we can obtain the following result.

Proposition 2.5. *If $\alpha K_a \cup \beta K_b$ is an Seidel integral graph with $\mu_1^* = ab + 1$ then it uniquely determines the parameters m, n, s, t.*

Further, using a procedure similar to the proofs of Theorems 2.1 and 2.2 we proceed to establish a characterization of Seidel integral graphs for the class $\alpha K_a \cup \beta K_b$. The proof is based on the following statement [2].

Theorem 2.6. *The linear Diophantine equation $ax + by = c$ has at least one solution if and only if $d \mid c$ where $d = (a,b)$. In that case the most general solution of this equation is given in the form*

$$x = \frac{c}{d}\,x_0 - \frac{b}{d}\,z \quad \text{and} \quad y = \frac{c}{d}\,y_0 + \frac{a}{d}\,z \qquad (z \in \mathbb{Z})\,,$$

where (x_0, y_0) represents a particular solution[3] *of the equation $ax + by = d$.*

[3] A particular solution of the equation $ax + by = d$ may be obtained by using the EUCLID algorithm. In that case the coefficients a and b uniquely determine x_0 and y_0.

Theorem 2.7. *If $\alpha K_a \cup \beta K_b$ is Seidel integral then it belongs to one of the following classes of Seidel integral graphs*

$$\left[\frac{k(2t-1)}{\tau}\,x_0 + \frac{m(2t-1)}{\tau}\,z\right]K_a \cup \left[\frac{k(2t-1)}{\tau}\,y_0 + \frac{a}{\tau}\,z\right](2n-1)K_b\,, \tag{2.6}$$

where (i) $a = (t+2\ell n-(\ell+n))k+(2\ell-1)m$ and $b = (2\ell-1)m$; (ii) $t,k,\ell,m,n \in \mathbb{N}$ such that $(m,2n-1) = 1$, $(2n-1,2t-1) = 1$ and $(2\ell-1,2t-1) = 1$; (iii) $\tau = (a, m(2t-1))$ such that $\tau \mid k\,(2t-1)$; (iv) (x_0,y_0) is a particular solution of the linear Diophantine equation $ax - m(2t-1)y = \tau$ and (v) $z \geq z_0$ where z_0 is the least integer such that $\left(\frac{k(2t-1)}{\tau}\,x_0 + \frac{m(2t-1)}{\tau}\,z_0\right) \geq 1$ and $\left(\frac{k(2t-1)}{\tau}\,y_0 + \frac{a}{\tau}\,z_0\right) \geq 1$;

$$\left[\frac{2kt}{\tau}\,x_0 + \frac{tm}{\tau}\,z\right]K_a \cup \left[\frac{2kt}{\tau}\,y_0 + \frac{a}{\tau}\,z\right]n\,K_b\,, \tag{2.7}$$

where (i) $a = (t+\ell n)k + \ell m$ and $b = \ell m$; (ii) $t,k,\ell,m,n \in \mathbb{N}$ such that $(m,n) = 1$, $(n,t) = 1$, $(\ell,t) = 1$ and $(t+\ell n, 2) = 1$; (iii) $\tau = (a, tm)$ such that $\tau \mid 2kt$; (iv) (x_0,y_0) is a particular solution of the linear Diophantine equation $ax - (tm)y = \tau$ and (v) $z \geq z_0$ where z_0 is the least integer such that $\left(\frac{2kt}{\tau}\,x_0 + \frac{tm}{\tau}\,z_0\right) \geq 1$ and $\left(\frac{2kt}{\tau}\,y_0 + \frac{a}{\tau}\,z\right) \geq 1$.

Proof. Let us assume that $\mu_1^* \in \mathbb{N}$ and let $\theta = \frac{\rho}{\varphi}$ so that $\mu_1^* - 1 = \theta a$ and $(\rho,\varphi) = 1$. Using (2.1) and (2.3) we obtain

$$\mu_2^* = -\frac{2b\big(\alpha+\beta-2\big)}{\theta} + 1 \quad \text{and} \quad \delta = \theta a + \frac{2b\big(\alpha+\beta-2\big)}{\theta}\,.$$

Then by a straightforward calculation it is not difficult to see that (2.2) may be transformed in the form $\frac{\theta+2}{\theta} = \frac{\alpha(a-b)}{\theta a - b(\alpha+\beta-2)}$. Let c be a constant such that $(\overline{1})$ $\alpha(a-b) = c(\theta+2)$ and $(\overline{2})$ $\theta a - b(\alpha+\beta-2) = c\theta$. Combining $(\overline{1})$ and $(\overline{2})$ we find that $2c = (\alpha-\theta)a + (\beta-2)b$. Observe that $2c$ is an integer because $\theta a = (\mu_1^*-1) \in \mathbb{N}$. Consequently, using $(\overline{1})$ or $(\overline{2})$ we arrive at $2\alpha(a-b) = \big[(\alpha-\theta)a + (\beta-2)b\big](\theta+2)$. Hence,

$$(a-b) = r\big[(\alpha-\theta)a + (\beta-2)b\big] \quad \text{and} \quad (\theta+2) = 2r\alpha\,, \tag{2.8}$$

where $r = \frac{s}{t}$ such that $(s,t) = 1$. Making use of (2.8), by an easy calculation we obtain $(\overline{3})$ $r\beta b = (2r-1)\big[r\alpha a - (a-b)\big]$.

Using now the right-hand side of relation (2.8), note that $2r\alpha a = \mu_1^* + 2a - 1$, which shows that $(2r\alpha a)$ is integral and $2r-1 = \frac{2s-t}{t} > 0$. Since $\beta b = (2-\frac{1}{r})\big[r\alpha a - (a-b)\big]$ (see $(\overline{3})$) it turns out that $r \mid (a-b)$. Let $(\overline{4})$ $(a-b) = \gamma r$ and let $(\overline{5})$ $\gamma = kt$. Then $(\overline{3})$ is reduced to the form:

$$\beta = \frac{\big(2s-t\big)}{b}\,\frac{\big(\alpha a - kt\big)}{t}\,. \tag{2.9}$$

Further, let $(2s-t,b) = \ell$ and let $m,n \in \mathbb{N}$ such that $(\overline{6})$ $(2s-t) = \ell n$ and $(\overline{7})$ $b = \ell m$, where $(m,n) = 1$. We note that $(2s-t,t) = 1$ or $(2s-t,t) = 2$. We shall now consider the following two cases:

Case 1. (t is odd). Let $t \to 2t-1$ where $p \to q$ means that 'p is replaced with q', which provides that $(2s-(2t-1), 2t-1) = 1$. Since $2s-(2t-1)$ is an odd number and $2s-(2t-1) = \ell n$ it follows that ℓ and n are two odd numbers. Setting $\ell \to 2\ell-1$ and $n \to 2n-1$, we find that $s-t = 2\ell n-(\ell+n)$. Then according to $(\overline{6})$ we obtain $(2n-1, 2t-1) = 1$ and $(2\ell-1, 2t-1) = 1$. Consequently, using (2.9) we have $\beta = \frac{(\alpha a - k(2t-1))\,(2n-1)}{m(2t-1)}$. Since $(2n-1, m(2t-1)) = 1$ it follows that $m(2t-1) \mid (\alpha a - k(2t-1))$. Therefore, setting (1.1) $\alpha a - k(2t-1) = \eta(m(2t-1))$ we get (1.2) $\beta = \eta(2n-1)$. We note that (1.1) represents a linear Diophantine equation in variables α and η. Of course, if $(a, m(2t-1)) = \tau$ then (1.1) has at least one solution if and only if $\tau \mid k(2t-1)$. In that case, according to Theorem 2.6 we obtain that

$$\alpha = \frac{k(2t-1)}{\tau}\,x_0 + \frac{m(2t-1)}{\tau}\,z \quad \text{and} \quad \eta = \frac{k(2t-1)}{\tau}\,y_0 + \frac{a}{\tau}\,z\,,$$

where $ax_0 - m(2t-1)y_0 = \tau$. Finally, from $(\overline{4})$ through $(\overline{7})$, and according to (1.2) and the last relation, we get easily that $a = (t+2\ell n-(\ell+n))k + (2\ell-1)m$ and $\beta = \left[\frac{k(2t-1)}{\tau}\,y_0 + \frac{a}{\tau}\,z\right](2n-1)$. So we arrive at the corresponding class of Seidel integral graphs displayed in (2.6).

Case 2. (t is even). Since $(s,t) = 1$ it follows that s is an odd number. Setting $t \to 2t$ and $s \to 2s-1$ we obtain $((2s-1)-t, t) = 1$, which provides that

$$\beta = \frac{\big((2s-1)-t\big)}{b}\,\frac{\big(\alpha a - 2kt\big)}{t}\,. \tag{2.10}$$

Further, let $((2s-1)-t, b) = \ell$ and let $m, n \in \mathbb{N}$ such that (2.1) $(2s-1)-t = \ell n$ and (2.2) $b = \ell m$, where $(m,n) = 1$. Since $2s-1 = t+\ell n$ is an odd number it must be $(t+\ell n, 2) = 1$. Using (2.10) we get $\beta = \frac{(\alpha a-2kt)\,n}{tm}$. Since $((2s-1)-t, t) = 1$ we obtain $(\ell n, t) = 1$, which provides that $(\ell, t) = 1$ and $(n,t) = 1$. In view of this, it follows that $tm \mid (\alpha a - 2kt)$. Therefore, setting (2.3) $\alpha a - 2kt = \eta(tm)$ we get (2.4) $\beta = \eta n$. We note that (2.3) represents a linear Diophantine equation in variables α and η. Of course, if $(a, tm) = \tau$ then (2.3) has at least one solution if and only if $\tau \mid 2kt$. In that case, according to Theorem 2.6 we obtain that

$$\alpha = \frac{2kt}{\tau}\,x_0 + \frac{tm}{\tau}\,z \quad \text{and} \quad \eta = \frac{2kt}{\tau}\,y_0 + \frac{a}{\tau}\,z\,,$$

where $ax_0 - (tm)y_0 = \tau$. Finally, using $(\overline{4})$, $(\overline{5})$ and (2.1), (2.2), (2.3), (2.4), we get easily that $a = (t+\ell n)k + \ell m$ and $\beta = \left[\frac{2kt}{\tau}\,y_0 + \frac{a}{\tau}\,z\right]n$. So we arrive at the corresponding class of Seidel integral graphs displayed in (2.7), which completes the proof. □

Proposition 2.8. *If $\alpha K_a \cup \beta K_b$ is a Seidel integral graph then it uniquely determines the parameters τ, t, k, ℓ, m, n.*

PROOF. Let us assume that $\tau_1, t_1, k_1, \ell_1, m_1, n_1$ and $\tau_2, t_2, k_2, \ell_2, m_2, n_2$ determine the same Seidel integral graph $\alpha K_a \cup \beta K_b$. Since the parameters α, β, a, b determine the graph $\alpha K_a \cup \beta K_b$ up to isomorphism, using the second equality of (2.8) we have $2r\alpha a = \mu_1^* - 1 + 2a$, which shows that $s_1 = s_2$ and $t_1 = t_2$ because $(s,t) = 1$. In view of this, we note that the classes represented by relations (2.6), (2.7) are mutually disjoint. Consequently, without loss of generality, we can assume that the corresponding Seidel integral graph determined by the parameters $\tau_1, t_1, k_1, \ell_1, m_1, n_1$ and $\tau_2, t_2, k_2, \ell_2, m_2, n_2$ belongs to the class of Seidel integral graphs displayed in relation (2.6). Next, using $(\overline{4})$ and $(\overline{5})$ we get $k_1 = k_2$. Since $(2s - (2t-1), b) = 2\ell - 1$ (see Case 1), we also have $\ell_1 = \ell_2$. Since $b = (2\ell - 1)m$ and $s - t = 2\ell n - (\ell + n)$, we find that $m_1 = m_2$ and $n_1 = n_2$. Finally, since $(a, m(2t-1)) = \tau$ it follows that $\tau_1 = \tau_2$. □

Remark 2.9. If (x_0, y_0) is obtained by using the EUCLID algorithm then a fixed Seidel integral graph $\alpha K_a \cup \beta K_b$ also uniquely determines the parameters x_0, y_0, z_0, z.

Proposition 2.10. *If $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = 1$ then it belongs to the class of Seidel integral graphs $K_a \cup K_b$ for any $a, b \in \mathbb{N}$.*

PROOF. Let us assume that $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = 1$. Using (2.3) we obtain $2ab(2-\alpha-\beta) = 0$, which provides that $\alpha = 1$ and $\beta = 1$. Using (2.2) we find that $\delta = a+b$. Since $\delta = \mu_1^* - \mu_2^*$ we obtain $\mu_2^* = -(a+b-1)$. □

Proposition 2.11. *If $\alpha K_a \cup \beta K_b$ is a Seidel integral graph then $\mu_1^* \geq 1$ and $\mu_2^* \leq -1$ for any $\alpha, \beta, a, b \in \mathbb{N}$.*

PROOF. We demonstrate first that $\mu_1^* \geq 1$ for any $\alpha, \beta, a, b \in \mathbb{N}$. On the contrary, assume that $\mu_1^* \leq 0$ for some $\alpha, \beta, a, b \in \mathbb{N}$. Since $\mu_1^* > \mu_2^*$ it follows that $\mu_2^* \leq -1$. Using (2.3) we get

$$(\mu_2^* - 1) \geq (\mu_1^* - 1)(1 - \mu_2^*) = 2ab(\alpha + \beta - 2)\,,$$

from which we obtain $\mu_2^* \geq 2ab(\alpha + \beta - 2) + 1 > \mu_1^*$, a contradiction.

We now demonstrate that $\mu_2^* \leq -1$ for any $\alpha, \beta, a, b \in \mathbb{N}$. On the contrary, assume that $\mu_2^* \geq 0$ for some $\alpha, \beta, a, b \in \mathbb{N}$. Consider the case when $\mu_2^* \geq 2$. Then using (2.3) we obtain

$$(\mu_1^* - 1) \leq (\mu_1^* - 1)(\mu_2^* - 1) = 2ab(2 - \alpha - \beta)\,,$$

which provides that $\alpha = 1$ and $\beta = 1$. Then according to Proposition 2.10, we find that $\mu_1^* = 1 < \mu_2^*$, a contradiction. The case when $\mu_2^* = 1$ is also trivial. Indeed, in this situation we have $2ab(2-\alpha-\beta) = 0$, which provides that $\alpha = 1$ and $\beta = 1$, a contradiction. Finally, consider the case when $\mu_2^* = 0$. Using (2.3) we obtain $\mu_1^* = 2ab(\alpha + \beta - 2) + 1$. Using the right-hand side of relation (2.1) we obtain $\delta = \alpha a + \beta b + 2 - 2a - 2b$. On the other hand, since $\delta = \mu_1^* - \mu_2^*$ we obtain $\delta = 2ab(\alpha + \beta - 2) + 1$. So we arrive at

$$\alpha a(2b - 1) + \beta b(2a - 1) = 4ab - 2a - 2b + 1\,. \tag{2.11}$$

In view of Proposition 2.10, it must be $\alpha \geq 2$ or $\beta \geq 2$. Consider the case when $\alpha \geq 2$. Using relation (2.11) we obtain

$$2a(2b-1) + \beta b(2a-1) \leq 4ab - 2a - 2b + 1\,,$$

from which we obtain $\beta b(2a-1) \leq -2b+1 < 0$, a contradiction. Consider the case when $\beta \geq 2$. In this situation, using (2.11) we obtain $\alpha a(2b-1) \leq -2a+1 < 0$, a contradiction. This completes the proof. □

In order to demonstrate a procedure for obtaining the Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_b$ for a fixed Seidel main eigenvalue μ_1^*, we prove the following two results:

Proposition 2.12. *There exists no Seidel integral graph from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* = 2$ for any $\alpha, \beta, a, b \in \mathbb{N}$.*

PROOF. First, according to the proof of Theorem 2.7, we have $\mu_1^* - 1 = \theta a$. Using that $2s - t > 0$ and using the right-hand side of relation (2.8), we obtain

$$\frac{2a+1}{\alpha a} = \frac{2s}{t} > 1\,,$$

which provides that $\alpha = 1$ or $\alpha = 2$.

CASE 1. ($\alpha = 1$). Since $\frac{2a+1}{2a} = \frac{s}{t}$ and $(2a+1, 2a) = 1$, $(s,t) = 1$, we obtain $s = 2a+1$ and $t = 2a$. Using (2.9) we find that $a(1-2k) < 0$, a contradiction.

CASE 2. ($\alpha = 2$). Since $\frac{2a+1}{4a} = \frac{s}{t}$ and $(2a+1, 4a) = 1$, $(s,t) = 1$, we obtain $s = 2a+1$ and $t = 4a$. Using (2.9) we find that $2a(1-2k) < 0$, a contradiction. □

Proposition 2.13. *If $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = 3$ then it is $2K_3 \cup K_1$ for any $\alpha, \beta, a, b \in \mathbb{N}$.*

PROOF. Using that $\mu_1^* - 1 = \theta a$ and using the right-hand side of relation (2.8), we find that $\frac{2a+2}{\alpha a} = \frac{2s}{t}$, which provides that $\alpha = 1$ or $\alpha = 2$.

CASE 1. ($\alpha = 1$). Since $\frac{a+1}{a} = \frac{s}{t}$ and $(a+1, a) = 1$, $(s,t) = 1$, we obtain $s = a+1$ and $t = a$. Using (2.9) we find that $a(1-k) \leq 0$, a contradiction.

CASE 2. ($\alpha = 2$). We note that $(a+1, 2a) = 1$ or $(a+1, 2a) = 2$. Consider the case when $(a+1, 2a) = 1$. Since $\frac{a+1}{2a} = \frac{s}{t}$ and $(s,t) = 1$, we obtain $s = a+1$ and $t = 2a$. Using (2.9) we find that $2a(1-k) \leq 0$, a contradiction.

Consider the case when $(a+1, 2a) = 2$. In this situation a is an odd number. Let $a = 2\varepsilon + 1$ where $\varepsilon \in \mathbb{N}$. Since $\frac{\varepsilon+1}{2\varepsilon+1} = \frac{s}{t}$ and $(2\varepsilon+1, \varepsilon+1) = 1$, $(s,t) = 1$, we obtain $s = \varepsilon + 1$ and $t = 2\varepsilon + 1$. Then $(\alpha a - kt) = (2-k)(2\varepsilon+1)$, which provides that $k = 1$. Using (2.9) we get

$$\beta = \frac{2(\varepsilon+1) - (2\varepsilon+1)}{b} \cdot \frac{(2-1)(2\varepsilon+1)}{2\varepsilon+1}\,,$$

from which we obtain $b = 1$ and $\beta = 1$. Finally, using $(\overline{4})$ and $(\overline{5})$ we find that $a = (\varepsilon+1)+1$. Since $a = 2\varepsilon+1$ it follows that $\varepsilon = 1$. □

Theorem 2.14. *If $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = 2a+1$ then it belongs to one of the following classes of Seidel integral graphs:* (1^0) $K_{(2\beta+3)m} \cup (\beta+3)K_{3m}$ *or* (2^0) $K_{(2\beta+1)m} \cup 3(\beta+1)K_m$ *or* (3^0) $2K_{(\beta+1)m} \cup (\beta+2)K_m$ *or* (4^0) $3K_{(2\beta+1)m} \cup (\beta+1)K_m$ *for any $\beta, m \in \mathbb{N}$.*

Proof. Let us assume that $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = 2a+1$. Using that $\mu_1^* - 1 = \theta a$ we obtain $\theta = 2$. Using the right-hand side of relation (2.8), we find that $2r\alpha = 4$. Since $(2r) > 1$ it follows that $\alpha = 1$ or $\alpha = 2$ or $\alpha = 3$.

Case 1. ($\alpha = 1$). In this situation $r = 2$, which means that $s = 2$ and $t = 1$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = 2k + b$. Using (2.9) we obtain

$$\beta = \frac{3\big((2k+b) - k\big)}{b}.$$

Consider the case when $3 \mid b$. Setting $b = 3m$ it follows that $m \mid (k + 3m)$. Setting $k = \ell m$ we obtain $\beta = \ell + 3$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (1^0).

Consider the case when $3 \nmid b$. In this situation $b \mid k$. Setting $k = \ell b$, we obtain $\beta = 3(\ell + 1)$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (2^0).

Case 2. ($\alpha = 2$). In this situation $r = 1$, which means that $s = 1$ and $t = 1$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = k+b$. Using (2.9) we obtain $\beta = \frac{k+2b}{b}$. Setting $k = \ell b$, we obtain $\beta = \ell + 2$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (3^0).

Case 3. ($\alpha = 3$). In this situation $s = 2$ and $t = 3$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = 2k + b$. Using (2.9) we obtain $\beta = \frac{k+b}{b}$. Setting $k = \ell b$, we obtain $\beta = \ell + 1$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (4^0). □

Theorem 2.15. *If $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = a+1$ then it belongs to one of the following classes of Seidel integral graphs:* (1^0) $K_{(3\beta+2)m} \cup (\beta+2)K_{2m}$ *or* (2^0) $K_{(3\beta+1)(2m-1)} \cup 2(\beta+1)K_{2m-1}$ *or* (3^0) $2K_{(3\beta+1)m} \cup (\beta+1)K_m$ *for any $\beta, m \in \mathbb{N}$.*

Proof. Let us assume that $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = a+1$. Using that $\mu_1^* - 1 = \theta a$ we obtain $\theta = 1$. Using the right-hand side of relation (2.8), we find that $2r\alpha = 3$. Since $(2r) > 1$ it follows that $\alpha = 1$ or $\alpha = 2$.

Case 1. ($\alpha = 1$). In this situation $s = 3$ and $t = 2$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = 3k + b$. Using (2.9) we obtain

$$\beta = \frac{2\big((3k+b) - 2k\big)}{b}.$$

Consider the case when $2 \mid b$. Setting $b = 2m$ it follows that $m \mid (k + 2m)$. Setting $k = \ell m$, we obtain $\beta = \ell + 2$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (1^0).

Consider the case when $2 \nmid b$. Then b is an odd number. Setting $b = 2m-1$ it follows that $(2m-1) \mid (k + (2m-1))$. Setting $k = \ell(2m-1)$, we obtain $\beta = 2(\ell + 1)$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (2^0).

Case 2. $(\alpha = 2)$. In this situation $s = 3$ and $t = 4$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = 3k + b$. Using (2.9) we obtain $\beta = \frac{k+b}{b}$. Setting $k = \ell b$, we obtain $\beta = \ell + 1$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (3^0). □

Theorem 2.16. *If $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = a$ then it is $2K_3 \cup K_1$ for any $\alpha, \beta, a, b \in \mathbb{N}$.*

Proof. Let us assume that $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = a$. Using that $\mu_1^* - 1 = \theta a$ and using the right-hand side of relation (2.8), we find that $2r\alpha = \frac{3a-1}{a}$. Since $(2r) > 1$ it follows that $\alpha = 1$ or $\alpha = 2$.

Case 1. $(\alpha = 1)$. In this situation we have $\frac{s}{t} = \frac{3a-1}{2a}$. We note that $(3a-1, 2a) = 1$ or $(3a-1, 2a) = 2$. Consider the case when $(3a-1, 2a) = 1$. Then a is an even number. Let $a = 2\varepsilon$ where $\varepsilon \in \mathbb{N}$. Since $(s,t) = 1$ we find that $s = 6\varepsilon - 1$ and $t = 4\varepsilon$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (6\varepsilon - 1)k + b$. So we arrive at

$$2\varepsilon = (6\varepsilon - 1)k + b \geq (5\varepsilon)k + b\,,$$

a contradiction. Consider the case when $(3a-1, 2a) = 2$. Then a is an odd number. Let $a = 2\varepsilon + 1$ where $\varepsilon \in \mathbb{N}$. Since $\frac{s}{t} = \frac{3\varepsilon+1}{2\varepsilon+1}$ and $(s,t) = 1$, $(3\varepsilon + 1, 2\varepsilon + 1) = 1$, we find that $s = 3\varepsilon + 1$ and $t = 2\varepsilon + 1$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (3\varepsilon + 1)k + b$. So we obtain $2\varepsilon + 1 = (3\varepsilon + 1)k + b$, a contradiction.

Case 2. $(\alpha = 2)$. In this situation we have $\frac{s}{t} = \frac{3a-1}{4a}$. Consider the case when a is an even number. Let $a = 2\varepsilon$ where $\varepsilon \in \mathbb{N}$. Since $\frac{s}{t} = \frac{6\varepsilon-1}{8\varepsilon}$ and $(s,t) = 1$, $(6\varepsilon - 1, 8\varepsilon) = 1$, we find that $s = 6\varepsilon - 1$ and $t = 8\varepsilon$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (6\varepsilon - 1)k + b$. So we get $2\varepsilon = (6\varepsilon - 1)k + b$, a contradiction.

Consider the case when a is an odd number. Setting $a = 2\varepsilon + 1$ we obtain $\frac{s}{t} = \frac{3\varepsilon+1}{2(2\varepsilon+1)}$, where $\varepsilon \in \mathbb{N}$. We note that $(3\varepsilon + 1, 2(2\varepsilon + 1)) = 1$ or $(3\varepsilon + 1, 2(2\varepsilon + 1)) = 2$. Consider the case when $(3\varepsilon + 1, 2(2\varepsilon + 1)) = 1$. Then ε is an even number. Let $\varepsilon = 2\varepsilon^\bullet$ where $\varepsilon^\bullet \in \mathbb{N}$. Since $\frac{s}{t} = \frac{6\varepsilon^\bullet+1}{2(4\varepsilon^\bullet+1)}$ and $(s,t) = 1$, $(6\varepsilon^\bullet + 1, 2(4\varepsilon^\bullet + 1)) = 1$, we find that $s = 6\varepsilon^\bullet + 1$ and $t = 2(4\varepsilon^\bullet + 1)$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (6\varepsilon^\bullet + 1)k + b$. So we obtain $4\varepsilon^\bullet + 1 = (6\varepsilon^\bullet + 1)k + b$, a contradiction

Consider the case when $(3\varepsilon + 1, 2(2\varepsilon + 1)) = 2$. Then ε is an odd number. Let $\varepsilon = 2\varepsilon^\bullet - 1$ where $\varepsilon^\bullet \in \mathbb{N}$. Since $\frac{s}{t} = \frac{3\varepsilon^\bullet-1}{4\varepsilon^\bullet-1}$ and $(s,t) = 1$, $(3\varepsilon^\bullet - 1, 4\varepsilon^\bullet - 1) = 1$, we find that $s = 3\varepsilon^\bullet - 1$ and $t = 4\varepsilon^\bullet - 1$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (3\varepsilon^\bullet - 1)k + b$. So we obtain $2(2\varepsilon^\bullet - 1) + 1 = (3\varepsilon^\bullet - 1)k + b$, which provides that $k = 1$. In view of this, we obtain $b = \varepsilon^\bullet$ and $a = 4\varepsilon^\bullet - 1$. Finally, using (2.9) we have

$$\beta = \frac{(2\varepsilon^\bullet - 1)}{\varepsilon^\bullet} \cdot \frac{\big(2(4\varepsilon^\bullet - 1) - (4\varepsilon^\bullet - 1)\big)}{4\varepsilon^\bullet - 1}\,,$$

from which we obtain $\varepsilon^{\bullet} = 1$ and $\beta = 1$. This completes the proof. $\square$

Theorem 2.17. *If $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = 2b+1$ then it belongs to the class of Seidel integral graphs $2K_{3m} \cup K_m$ for any $m \in \mathbb{N}$.*

PROOF. Let us assume that $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = 2b+1$. Using that $\mu_1^* - 1 = \theta a$ we obtain $\theta a = 2b$. Using the right-hand side of relation (2.8), we find that $2r\alpha = \frac{2(a+b)}{a}$. Since $(2r) > 1$ and $a > b$ it follows that $\alpha = 1$ or $\alpha = 2$ or $\alpha = 3$.

CASE 1. ($\alpha = 1$). In this situation $r = \frac{a+b}{a}$. Since $(s,t) = 1$ it follows that there exists $\ell \in \mathbb{N}$ so that $a + b = \ell s$ and $a = \ell t$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = ks + (s-t)\ell$. Using (2.9) we obtain

$$\beta = \frac{(2s-t)}{\ell} \cdot \frac{(k+\ell)}{t}\,.$$

Since $b = (s-t)\ell$ note that $s > t$ and $s \geq 2$. Next, since $a = \ell t$ and $a = ks + (s-t)\ell$ we obtain $2\ell t = (k+\ell)s$. So we arrive at $\beta = \frac{2(2s-t)}{s}$. Since $(2s-t, s) = 1$ we obtain $s \mid 2$, which means that $s = 2$ and $t = 1$. In view of this, we have $2\ell = 2(k+\ell)$, a contradiction because $k \in \mathbb{N}$.

CASE 2. ($\alpha = 2$). In this situation $r = \frac{a+b}{2a}$. Since $(s,t) = 1$ it follows that there exists $\ell \in \mathbb{N}$ so that $a + b = \ell s$ and $2a = \ell t$. Using $(\overline{4})$ and $(\overline{5})$ we find that $2a = 2ks + (2s-t)\ell$. Using (2.9) we obtain

$$\beta = \frac{2(2s-t)}{\ell} \cdot \frac{(k+\ell)}{t}\,.$$

Since $2a = \ell t$ and $2a = 2ks + (2s-t)\ell$ we obtain $\ell t = (k+\ell)s$. So we arrive at $\beta = \frac{2(2s-t)}{s}$. Since $(2s-t, s) = 1$ we obtain $s \mid 2$, which means that $s = 1$ or $s = 2$. Consider the case when $s = 1$. Then $t = 1$ and $\ell = (k+\ell)$, a contradiction.

Consider the case when $s = 2$. Then $t = 1$ or $t = 2$ or $t = 3$. Consider the case when $t = 1$. Then $\ell = 2(k+\ell)$, a contradiction. Consider the case when $t = 2$. Then $2\ell = 2(k+\ell)$, a contradiction. Consider the case when $t = 3$. Then $3\ell = 2(k+\ell)$ from which we obtain $\ell = 2k$. So we obtain that $\beta = 1$, $a = 3k$ and $b = k$. Replacing k with m we obtain the corresponding class of Seidel integral graphs represented in Theorem 2.17.

CASE 3. ($\alpha = 3$). In this situation $r = \frac{a+b}{3a}$. Since $(s,t) = 1$ it follows that there exists $\ell \in \mathbb{N}$ so that $a + b = \ell s$ and $3a = \ell t$. Using $(\overline{4})$ and $(\overline{5})$ we find that $3a = 3ks + (3s-t)\ell$. Using (2.9) we obtain

$$\beta = \frac{3(2s-t)}{\ell} \cdot \frac{(k+\ell)}{t}\,.$$

Since $3a = \ell t$ and $3a = 3ks + (3s-t)\ell$ we obtain $2\ell t = 3(k+\ell)s$. So we arrive at $\beta = \frac{2(2s-t)}{s}$. Since $(2s-t, s) = 1$ we obtain $s \mid 2$, which means that $s = 1$ or $s = 2$. Consider the case when $s = 1$. Then $t = 1$ and $2\ell = 3(k+\ell)$, a contradiction.

Consider the case when $s = 2$. Then $t = 1$ or $t = 2$ or $t = 3$. Consider the case when $t = 1$. Then $\ell = 3(k + \ell)$, a contradiction. Consider the case when $t = 2$. Then $2\ell = 3(k + \ell)$, a contradiction. Consider the case when $t = 3$. Then $\ell = (k + \ell)$, a contradiction. This completes the proof. □

Theorem 2.18. *There exists no Seidel integral graph from the class* $\alpha K_a \cup \beta K_b$ *with* $\mu_1^* = b + 1$ *for any* $\alpha, \beta, a, b \in \mathbb{N}$.

PROOF. Let us assume that $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = b + 1$. Using that $\mu_1^* - 1 = \theta a$ we obtain $\theta a = b$. Using the right-hand side of relation (2.8), we find that $2r\alpha = \frac{2a+b}{a}$. Since $(2r) > 1$ and $a > b$ it follows that $\alpha = 1$ or $\alpha = 2$.

CASE 1. ($\alpha = 1$). In this situation $r = \frac{2a+b}{2a}$. Since $(s, t) = 1$ it follows that there exists $\ell \in \mathbb{N}$ so that $2a + b = \ell s$ and $2a = \ell t$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = ks + (s - t)\ell$. Using (2.9) we obtain

$$\beta = \frac{(2s - t)}{\ell} \cdot \frac{(k + \ell)}{t}.$$

Since $b = (s - t)\ell$ note that $s > t$ and $s \geq 2$. Next, since $2a = \ell t$ and $a = ks + (s - t)\ell$ we obtain $3\ell t = 2(k + \ell)s$. So we arrive at $\beta = \frac{3(2s-t)}{2s}$. Since $(2s - t, s) = 1$ and $(3, 2) = 1$ it follows that $s \mid 3$ and $2 \mid (2s - t)$. In view of this, we find that $s = 3$ and $t = 2$. Then we have $6\ell = 6(k+\ell)$, a contradiction.

CASE 2. ($\alpha = 2$). In this situation $r = \frac{2a+b}{4a}$. Since $(s, t) = 1$ it follows that there exists $\ell \in \mathbb{N}$ so that $2a + b = \ell s$ and $4a = \ell t$. Using $(\overline{4})$ and $(\overline{5})$ we find that $2a = 2ks + (2s - t)\ell$. Using (2.9) we obtain

$$\beta = \frac{2(2s - t)}{\ell} \cdot \frac{(k + \ell)}{t}.$$

Since $4a = \ell t$ and $2a = 2ks + (2s - t)\ell$ we obtain $3\ell t = 4(k + \ell)s$. So we arrive at $\beta = \frac{3(2s-t)}{2s}$. Since $(2s - t, s) = 1$ and $(3, 2) = 1$ it follows that $s \mid 3$ and $2 \mid (2s - t)$. In view of this, we find that $s = 3$ and $t = 2$. Then we have $6\ell = 12(k + \ell)$, a contradiction. This completes the proof. □

Theorem 2.19. *There exists no Seidel integral graph from the class* $\alpha K_a \cup \beta K_b$ *with* $\mu_1^* = b$ *for any* $\alpha, \beta, a, b \in \mathbb{N}$ *and* $a > b \geq 2$.

PROOF. Let us assume that $\alpha K_a \cup \beta K_b$ is Seidel integral with $\mu_1^* = b$. Using that $\mu_1^* - 1 = \theta a$ we obtain $\theta a = b - 1$. Using the right-hand side of relation (2.8), we find that $2r\alpha = \frac{2a+b-1}{a}$. Since $(2r) > 1$ and $a > b$ it follows that $\alpha = 1$ or $\alpha = 2$.

CASE 1. ($\alpha = 1$). In this situation $\frac{s-t}{t} = \frac{b-1}{2a}$. Since $(s - t, t) = 1$ it follows that there exists $\ell \in \mathbb{N}$ so that (i) $b - 1 = \ell(s - t)$ and (ii) $2a = \ell t$. Using (2.3) we obtain[4] (iii) $|\mu_2^*| + 1 = \frac{tb(\beta-1)}{s-t}$. Since $(b, b - 1) = 1$ we obtain $(tb, s - t) = 1$,

[4] If $b = 1$ then we find that $\beta = 1$, which provides the class of Seidel integral graphs $K_a \cup K_1$ with $\mu_1^* = b$ for any $a > 1$ (see Proposition 2.10).

which provides that $(s-t) \mid (\beta-1)$. Let (iv) $\beta-1 = m(s-t)$ for some $m \in \mathbb{N}$. Further, since $\delta = \mu_1^* + |\mu_2^*|$ using (i), (ii), (iii) and (iv), by a straightforward calculation we obtain

$$2\delta = 2\ell mt(s-t) + 2\ell(s-t) + 2mt\,. \tag{2.12}$$

On the other hand, using the left-hand side of relation (2.1) we get $\delta = a + 4b - \beta b - 2$. Then using (i), (ii) and (iv), by a straightforward calculation we obtain

$$2\delta = \ell(6s-5t) - 2\ell m(s-t)^2 - 2ms + 2mt + 2\,. \tag{2.13}$$

Finally, using (2.12) and (2.13) we arrive at (v) $2\ell ms(s-t) = \ell(4s-3t) - 2ms + 2$. We shall now demonstrate that $s = t+1$. On the contrary, assume that $s \geq t+2$. Then using (v) we obtain

$$4\ell ms \leq 2\ell ms(s-t) = 4\ell s - 3\ell t - 2ms + 2\,,$$

a contradiction. Setting $s = t+1$ we can easily see that (v) is reduced to $2m(t+1)(\ell+1) = \ell(t+4)+2$, which provides that $m = 1$. Setting $m = 1$ we now find that $2\ell = \ell t + 2t$, which provides that $t = 1$ and $\ell = 2$. In view of this, we obtain that $a = 1$ and $b = 3$, a contradiction.

Case 2. ($\alpha = 2$). In this situation $\frac{2s-t}{t} = \frac{b-1}{2a}$. We note that $(2s-t, t) = 1$ or $(2s-t, t) = 2$. We shall now consider the following two cases:

Case 2.1 (t is odd). Setting $t \to 2t-1$ we obtain $\frac{2s-(2t-1)}{2t-1} = \frac{b-1}{2a}$. Therefore, since $(2s-(2t-1), 2t-1) = 1$ it follows that there exists $\ell \in \mathbb{N}$ so that (i) $b-1 = \ell(2s-(2t-1))$ and (ii) $2a = \ell(2t-1)$. In view of this, since $b-1 \geq 1$ and $b-1 = \ell(2s-(2t-1))$ note that $s \geq t$. Using (2.3) we obtain (iii) $|\mu_2^*|+1 = \frac{(2t-1)b\beta}{2s-(2t-1)}$. Since $(b, b-1) = 1$ we obtain $((2t-1)b, 2s-(2t-1)) = 1$, which provides that $(2s-(2t-1)) \mid \beta$. Let (iv) $\beta = m(2s-(2t-1))$ for some $m \in \mathbb{N}$. Further, since $\delta = \mu_1^* + |\mu_2^*|$ using (i), (ii), (iii) and (iv), by a straightforward calculation we obtain

$$\delta = 4\ell mt(s-t+1) - \ell m(2s+1) + \ell\big(2s-(2t-1)\big) + m(2t-1)\,. \tag{2.14}$$

On the other hand, using the left-hand side of relation (2.1) we get (v) $\delta = (4-\beta)b - 2$. In view of this, it follows that $\beta = 1$ or $\beta = 2$ or $\beta = 3$. Consider the case when $\beta = 1$. Using (iv) we find that $s = t$ and $m = 1$. Using (i), (v) and (2.14) we arrive at $2\ell t + 2t = 3\ell + 2$, which provides that $t = 1$. So we obtain $2\ell + 2 = 3\ell + 2$, a contradiction. Consider the case when $\beta = 2$. Using (iv) we find that $s = t$ and $m = 2$. Using (i), (v) and (2.14) we arrive at $4\ell t + 2(2t-1) = 3\ell$, a contradiction.

Consider the case when $\beta = 3$. Using (iv) we find that $s = t$ and $m = 3$ or $s = t+1$ and $m = 1$. Consider the case when $s = t$ and $m = 3$. Using (i), (v) and (2.14) we arrive at $6\ell t + 3(2t-1) = 3\ell - 1$, a contradiction. Consider the case when $s = t+1$ and $m = 1$. Using (i), (v) and (2.14) we arrive at $6\ell t + 2t = 3\ell$, a contradiction.

Case 2.2 (t is even). Since $(s,t)=1$ it follows that s is an odd number. Setting $t\to 2t$ and $s\to 2s-1$ we obtain $\frac{(2s-1)-t}{t}=\frac{b-1}{2a}$. Since $((2s-1)-t,t)=1$ it follows that there exists $\ell\in\mathbb{N}$ so that (i) $b-1=\ell((2s-1)-t)$ and (ii) $2a=\ell t$. Using (2.3) we obtain (iii) $|\mu_2^*|+1=\frac{tb\beta}{(2s-1)-t}$. Since $(b,b-1)=1$ we obtain $(tb,(2s-1)-t)=1$, which provides that $((2s-1)-t)\mid\beta$. Let (iv) $\beta=m((2s-1)-t)$ for some $m\in\mathbb{N}$. Further, since $\delta=\mu_1^*+|\mu_2^*|$ using (i), (ii), (iii) and (iv), by a straightforward calculation we obtain

$$\delta=\ell mt\big((2s-1)-t\big)+\ell\big((2s-1)-t\big)+mt\,. \tag{2.15}$$

On the other hand, using the left-hand side of relation (2.1) we get (v) $\delta=(4-\beta)b-2$. In view of this, it follows that $\beta=1$ or $\beta=2$ or $\beta=3$. Consider the case when $\beta=1$. Using (iv) we find that $(2s-1)-t=1$ and $m=1$. Using (i), (v) and (2.15) we arrive at $\ell t+t=2\ell+1$, which provides that $t=2$. So we obtain $2\ell+2=2\ell+1$, a contradiction.

Consider the case when $\beta=2$. Using (iv) we find that $(2s-1)-t=1$ and $m=2$ or $(2s-1)-t=2$ and $m=1$. Consider the case when $(2s-1)-t=1$ and $m=2$. Using (i), (v) and (2.15) we arrive at $2\ell t+2t=\ell$, a contradiction. Consider the case when $(2s-1)-t=2$ and $m=1$. Using (i), (v) and (2.15) we arrive at $2\ell t+t=2\ell$, a contradiction.

Consider the case when $\beta=3$. Using (iv) we find that $(2s-1)-t=1$ and $m=3$ or $(2s-1)-t=3$ and $m=1$. Consider the case when $(2s-1)-t=1$ and $m=3$. Using (i), (v) and (2.15) we arrive at $3\ell t+3t=-1$, a contradiction. Consider the case when $(2s-1)-t=3$ and $m=1$. Using (i), (v) and (2.15) we arrive at $3\ell t+t=-1$, a contradiction. □

Theorem 2.20. *If $(\alpha,\beta,a,b,\delta)$ is a positive integral solution of the Diophantine equation (2.2) then it could be represented by one of the following forms:*

- $a=(t+2\ell n-(\ell+n))k+(2\ell-1)m$ *and* $b=(2\ell-1)m\,;$
- $\alpha=\dfrac{k(2t-1)}{\tau}\,x_0+\dfrac{m(2t-1)}{\tau}\,z\,;$
- $\beta=\left[\dfrac{k(2t-1)}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right](2n-1)\,;$
- $\delta=(2\ell-1)(2n-1)k+\left[\dfrac{k(2t-1)}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right]2\big(t+2\ell n-(\ell+n)\big)m\,,$

with the same conditions (ii)–(v) which are related to (2.6);

- $a=(t+\ell n)k+\ell m$ *and* $b=\ell m\,;$
- $\alpha=\dfrac{2kt}{\tau}\,x_0+\dfrac{tm}{\tau}\,z\,;$
- $\beta=\left[\dfrac{2kt}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right]n\,;$
- $\delta=2k\ell n+\left[\dfrac{2kt}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right]\big(t+\ell n\big)m\,;$

with the same conditions (ii)–(v) which are related to (2.7).

PROOF. According to Theorem 2.7 it suffices to derive the expression for δ. First, from (2.1) we have (i) $\mu_1^* - \mu_2^* = \delta$ and (ii) $\mu_1^* + \mu_2^* = \alpha a + \beta b - 2(a + b - 1)$. Using (i), (ii) and the equality $\mu_1^* = 2(r\alpha - 1)a + 1$ (see (2.8)), by a straightforward calculation we obtain that $\delta = 4r\alpha a - (\alpha a + \beta b) - 2(a - b)$.

CASE 1. (t is odd). Using ($\overline{4}$), ($\overline{5}$), (1.1) and (1.2) we obtain $a - b = ks$, $r\alpha a = ks + \eta ms$ and $\beta = \eta(2n - 1)$. So we find that $\delta = (2\ell - 1)(2n - 1)k + 2(t + 2\ell n - (\ell + n))\eta m$, which provides the statement related to (2.6).

CASE 2. (t is even). Using ($\overline{4}$), ($\overline{5}$), (2.1), (2.2), (2.3) and (2.4) we obtain that $a - b = k(2s - 1)$, $2r\alpha a = 2k(2s - 1) + \eta m(2s - 1)$ and $\beta = \eta n$. So we find that $\delta = 2k\ell n + (t + \ell n)\eta m$, which provides the statement related to (2.7). □

3. Appendix

In this section we present the data given in Tables 1 and 2, which represent the set of all Siedel integral graphs from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* \geq 2$, whose order does not exceed 20. In these tables a Seidel integral graph is described by the parameters α, β, a, b and ones presented in the class of Seidel integral graphs in Theorem 2.7. In Tables 1 and 2 the symbol 'i' is the identification number of an integral graph.

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
1	0	−1	2	7	2	1	3	1	3	2	1	1	1	1	3	−2
2	0	−1	2	7	2	3	2	1	1	1	1	1	1	1	5	−2
3	0	−1	1	9	1	6	3	1	1	1	1	1	1	2	7	−4
4	0	−1	2	10	2	4	3	1	1	1	2	1	1	1	7	−3
5	0	−1	3	11	3	2	3	1	3	2	1	1	1	1	7	−2
6	0	−1	3	11	3	5	2	1	1	1	1	1	1	1	9	−2
7	0	−1	2	13	2	3	5	1	5	3	1	1	1	2	7	−4
8	0	−1	2	13	2	5	4	1	1	1	3	1	1	1	9	−4
9	0	−1	1	13	1	7	6	1	3	2	1	1	1	4	9	−8
10	0	−1	2	14	2	1	6	2	6	2	2	1	2	1	5	−5
11	0	−1	1	14	1	5	9	1	3	2	2	1	1	3	7	−11
12	0	−1	2	14	2	3	4	2	2	1	2	1	2	1	9	−5
13	0	−1	1	14	1	9	5	1	1	1	2	1	1	2	11	−7
14	0	−1	4	15	4	3	3	1	3	2	1	1	1	1	11	−2
15	0	−1	4	15	4	7	2	1	1	1	1	1	1	1	13	−2
16	0	−1	2	16	2	6	5	1	1	1	4	1	1	1	11	−5
17	0	−1	3	16	3	7	3	1	1	1	2	1	1	1	13	−3
18	−1	−1	2	17	3	1	5	2	5	3	1	1	2	1	9	−4
19	0	−1	1	17	1	4	5	3	1	1	1	2	1	1	11	−8
20	1	1	1	17	3	4	3	2	1	1	1	1	2	1	13	−4
21	2	3	−3	18	3	3	5	1	1	2	2	1	1	1	11	−3
22	0	−1	1	18	1	6	6	2	2	1	2	1	2	2	13	−9
23	0	−1	1	19	1	2	9	5	3	2	1	3	1	1	7	−14

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
24	−1	−1	2	19	3	1	6	1	3	5	1	1	1	1	9	−2
25	0	−1	1	19	1	3	10	3	5	3	1	2	1	2	9	−14
26	0	−1	2	19	2	5	7	1	7	4	1	1	1	3	11	−6
27	0	−1	2	19	2	7	6	1	1	1	5	1	1	1	13	−6
28	0	−1	5	19	5	4	3	1	3	2	1	1	1	1	15	−2
29	0	−1	1	19	1	12	7	1	1	1	3	1	1	2	15	−10
30	0	−1	5	19	5	9	2	1	1	1	1	1	1	1	17	−2
31	0	−1	1	19	1	15	4	1	1	1	1	1	1	3	17	−6
32	0	−1	2	20	2	2	9	1	3	2	4	1	1	1	7	−5
33	0	−1	2	20	2	4	6	2	2	1	4	1	2	1	13	−7

Table 1.

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
1	0	−1	1	8	1	4	4	1	1	1	1	1	1	2	5	−5
2	0	−1	1	9	1	3	6	1	2	2	1	1	1	3	4	−7
3	0	−1	2	10	2	2	4	1	2	2	1	1	1	1	5	−3
4	0	−1	1	11	1	3	5	2	1	1	1	2	1	1	6	−7
5	1	1	−1	13	2	1	6	1	2	4	1	1	1	1	4	−3
6	0	−1	1	13	1	2	9	2	3	3	1	2	1	2	4	−11
7	0	−1	1	13	1	6	7	1	1	1	2	1	1	2	8	−9
8	0	−1	1	14	1	2	8	3	2	2	1	3	1	1	5	−11
9	0	−1	2	16	2	4	6	1	3	3	1	1	1	2	9	−5
10	0	−1	1	16	1	4	8	2	1	1	2	2	1	1	9	−11
11	0	−1	3	16	3	4	4	1	2	2	1	1	1	1	11	−3
12	1	3	−3	17	2	3	7	1	1	2	2	1	1	1	8	−5
13	0	−1	1	18	1	3	12	2	4	2	2	1	2	3	7	−15
14	0	−1	1	18	1	8	10	1	1	1	3	1	1	2	11	−13
15	0	−1	1	18	1	10	8	1	2	2	1	1	1	5	13	−11
16	0	−1	1	19	1	4	15	1	3	3	2	1	1	4	6	−17
17	0	−1	1	19	1	7	12	1	4	4	1	1	1	7	10	−15
18	0	−1	2	20	2	2	8	2	4	2	2	1	2	1	9	−7
19	0	−1	2	20	2	12	4	1	1	1	1	1	1	2	17	−5

Table 2.

In this section, the Seidel integral graphs represented in Table 1 are obtained by using relation Theorem 2.7 (2.6), while the Seidel integral graphs represented in Table 2 are obtained by using relation Theorem 2.7 (2.7). In view of this, there exist exactly $33+19=52$ non-isomorphic Seidel integral graphs from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* \geq 2$, whose order does not exceed 20.

Next, graphs represented in Table 1 with identification numbers $i = 2, 3, 4, 5, 8, 13, 16, 20, 21, 27, 29$ are Seidel integral graphs with $\mu_1^* = 2ab + 1$, while there is no graph represented in Table 2 with $\mu_1^* = 2ab + 1$. In view of this, there exist exactly $11 + 0 = 11$ non-isomorphic Seidel integral graphs from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* = 2ab + 1$, whose order does not exceed 20.

Next, (*i*) graphs represented in Table 1 with identification numbers $i = 12, 22, 33$ are Seidel integral graphs with $\mu_1^* = ab+1$ and (*ii*) graphs represented in Table 2 with identification numbers $i = 1, 3, 7, 12, 14$ are Seidel integral graphs with $\mu_1^* = ab + 1$. In view of this, there exist exactly $3 + 5 = 8$ non-isomorphic Seidel integral graphs from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* = ab + 1$, whose order does not exceed 20.

Next, graphs represented in Table 1 with identification numbers $i = 2, 3, 4, 5, 8, 12, 13, 16, 19, 21, 22, 27, 29, 33$ are Seidel integral graphs with $\mu_1^* = 2a+1$, while there is no graph represented in Table 2 with $\mu_1^* = 2a+1$. In view of this, there exist exactly $14+0 = 14$ non-isomorphic Seidel[5] integral graphs from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* = 2a + 1$, whose order does not exceed 20.

Next, there is no graph represented in Table 1 with $\mu_1^* = a+1$, while graphs represented in Table 2 with identification numbers $i = 1, 3, 4, 7, 10, 12, 14, 18$ are Seidel integral graphs with $\mu_1^* = a + 1$. In view of this, there exist exactly $0 + 8 = 8$ non-isomorphic Seidel[6] integral graphs from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* = a + 1$, whose order does not exceed 20.

Next, graphs represented in Table 1 with identification numbers $i = 1, 10$ are Seidel integral graphs with $\mu_1^* = 2b+1$, while there is no graph represented in Table 2 with $\mu_1^* = 2b + 1$. In view of this, there exist exactly $2 + 0 = 2$ non-isomorphic Seidel integral graphs from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* = 2b + 1$, whose order does not exceed 20. This completes my explanation on Tables 1 and 2.

007^{02}	008^{01}	009^{02}	010^{02}	011^{03}	013^{06}	014^{05}	015^{02}	016^{05}
017^{04}	018^{05}	019^{11}	020^{04}	021^{07}	022^{10}	023^{06}	024^{05}	025^{05}
026^{11}	027^{08}	028^{13}	029^{08}	030^{04}	031^{16}	032^{12}	033^{14}	034^{18}
035^{08}	036^{11}	037^{10}	038^{21}	039^{20}	040^{14}	041^{08}	042^{12}	043^{17}
044^{18}	045^{12}	046^{18}	047^{18}	048^{18}	049^{14}	050^{11}	051^{19}	052^{23}
053^{15}	054^{24}	055^{17}	056^{25}	057^{25}	058^{25}	059^{18}	060^{12}	061^{17}
062^{28}	063^{17}	064^{28}	065^{14}	066^{34}	067^{22}	068^{32}	069^{22}	070^{21}

[5] We note (*i*) graph represented in Table 1 with identification number $i = 19$ belong to the class Theorem 2.14 (1^0); (*ii*) graphs represented in Table 1 with identification numbers $i = 3, 13, 22, 29$ belong to the class Theorem 2.14 (2^0); (*iii*) graphs represented in Table 1 with identification numbers $i = 2, 4, 8, 12, 16, 27, 33$ belong to the class Theorem 2.14 (3^0) and (*iv*) graphs represented in Table 1 with identification numbers $i = 5, 21$ belong to the class Theorem 2.14 (4^0).

[6] We note (*i*) graphs represented in Table 2 with identification numbers $i = 4, 10$ belong to the class Theorem 2.15 (1^0); (*ii*) graphs represented in Table 2 with identification numbers $i = 1, 7, 14$ belong to the class Theorem 2.15 (2^0) and (*iii*) graphs represented in Table 2 with identification numbers $i = 3, 12, 18$ belong to the class Theorem 2.15 (3^0).

071^{20}	072^{21}	073^{33}	074^{28}	075^{15}	076^{45}	077^{23}	078^{39}	079^{22}
080^{28}	081^{27}	082^{32}	083^{27}	084^{32}	085^{23}	086^{35}	087^{25}	088^{32}
089^{21}	090^{25}	091^{34}	092^{36}	093^{35}	094^{44}	095^{23}	096^{37}	097^{29}
098^{30}	099^{42}	100^{31}	101^{30}	102^{45}	103^{29}	104^{40}	105^{26}	106^{37}
107^{24}	108^{46}	109^{26}	110^{41}	111^{30}	112^{45}	113^{28}	114^{49}	115^{34}
116^{46}	117^{39}	118^{52}	119^{23}	120^{41}	121^{28}	122^{37}	123^{31}	124^{67}
125^{28}	126^{38}	127^{33}	128^{47}	129^{47}	130^{43}	131^{30}	132^{56}	133^{50}
134^{51}	135^{29}	136^{70}	137^{29}	138^{60}	139^{41}	140^{42}	141^{40}	142^{40}
143^{38}	144^{59}	145^{26}	146^{56}	147^{35}	148^{57}	149^{33}	150^{38}	151^{39}
152^{70}	153^{44}	154^{57}	155^{41}	156^{68}	157^{46}	158^{52}	159^{47}	160^{59}
161^{31}	162^{60}	163^{44}	164^{59}	165^{48}	166^{63}	167^{34}	168^{62}	169^{36}
170^{59}	171^{53}	172^{66}	173^{34}	174^{67}	175^{42}	176^{64}	177^{44}	178^{62}
179^{34}	180^{52}	181^{46}	182^{58}	183^{55}	184^{64}	185^{32}	186^{68}	187^{52}
188^{80}	189^{46}	190^{65}	191^{40}	192^{71}	193^{47}	194^{66}	195^{47}	196^{62}
197^{31}	198^{83}	199^{50}	200^{62}	201^{61}	202^{75}	203^{42}	204^{95}	205^{31}
206^{74}	207^{60}	208^{75}	209^{45}	210^{56}	211^{41}	212^{72}	213^{58}	214^{72}
215^{45}	216^{88}	217^{61}	218^{65}	219^{65}	220^{75}	221^{45}	222^{77}	223^{39}
224^{74}	225^{53}	226^{61}	227^{39}	228^{95}	229^{53}	230^{71}	231^{61}	232^{76}
233^{44}	234^{91}	235^{57}	236^{82}	237^{53}	238^{67}	239^{39}	240^{79}	241^{45}
242^{65}	243^{71}	244^{80}	245^{46}	246^{89}	247^{57}	248^{98}	249^{66}	250^{68}

Table 3.

There exist exactly 9620 non-isomorphic Seidel[7] integral graphs with[8] $\mu_1^* \geq 2$, which belong to the class $\alpha K_a \cup \beta K_b$ whose order does not exceed 250. In particular[9], the total numbers of such integral graphs which belong to the classes Theorem 2.7 (2.6) and (2.7) are 5249 and 4371, respectively. Table 3 contains a distribution of those graphs in respect to their orders. In Table 3 the symbol o^n denotes the number of integral graphs of the corresponding order $o = 1, 2, \ldots, 250$. In this table o^n is omitted if the corresponding number $n = 0$.

Conclusion. Using results and a similar procedure presented in this work, it is possible to investigate any class of Seidel integral graphs which have exactly two main eigenvalues.

[7] In this work the data given in Tables 1, 2 and 3 are obtained in two different ways: (i) they are generated by using relations $\big((2.6)$ and $(2.7)\big)$ and (ii) by varying the parameters α, β, a, b in all possible ways in equation (2.2).

[8] Since $K_a \cup K_b$ is Seidel integral with $\mu_1^* = 1$ for any $a > b$ (in order to reduced the data), we consider in Tables 1, 2 and 3 only Seidel integral graphs from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* \geq 2$.

[9] In particular, there exist exactly 53023 and 49299 non-isomorphic Seidel integral graphs with $\mu_1^* \geq 2$ and order $o \leq 1000$, which belong to the classes Theorem 2.7 (2.6) and (2.7), respectively. In view of this, there exist exactly $53023 + 49299 = 102322$ non-isomorphic Seidel integral graphs from the class $\alpha K_a \cup \beta K_b$ with $\mu_1^* \geq 2$, whose order does not exceed 1000.

Acknowledgement

The author is very grateful to referee for his valuable remarks, comments and suggestions concerning this paper.

References

[1] Dragoš M. Cvetković, Michael Doob, and Horst Sachs. *Spectra of graphs.* Johann Ambrosius Barth, Heidelberg, third edition, 1995. Theory and applications.

[2] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers.* The Clarendon Press, Oxford University Press, New York, fifth edition, 1979.

[3] Mirko Lepović. On the Seidel eigenvectors of a graph. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*, 14:4–10 (2004), 2003. `doi:10.2298/PETF0314004L`.

[4] Mirko Lepović. On integral graphs which belong to the class $\overline{\alpha K_a \cup \beta K_b}$. *J. Appl. Math. Comput.*, 14(1-2):39–49, 2004. `doi:10.1007/BF02936097`.

Received by the editors February 27, 2023
First published online July 18, 2023