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Some weighted integral inequalities via the extended Chebyshev functional

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Abstract. In this work by applying the extended Chebyshev functional for monotone and quasi- monotone functions, we obtain some new weighted integral inequalities for weighted Hardy operators. Moreover some known integrals are generalized.

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1. Introduction

Inequalities play a very important role in different fields of mathematics and present an active and attractive area of research. As examples, we cite the field of integration which is dominated by inequalities involving operators and integrals (see [10], [11]). Among the inequalities, we recall two famous integral inequalities, first, the Chebyshev inequality (see [2], [12]), second the Hardy integral inequality. G.H. Hardy stated in 1920 (see [5]) and proved in 1925 (see [4]) his famous inequality if $f(x)$ is non-negative and measurable on $(0,\infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dx\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,\ p>1. \tag{1.1}$$

The well known Chebyshev functional is defined as follows

$$T(h_1,h_2) = \frac{1}{b-a}\int_0^b h_1(x)h_2(x)dx - \frac{1}{b-a}\int_0^b h_1(x)dx\frac{1}{b-a}\int_0^b h_2(x)dx. \tag{1.2}$$

If the functions h_1 and h_2 are synchronous on $[a,b]$ (i.e. $(h_1(x)-h_2(y))(h_2(x)-h_2(y))\ge 0$, for any $x, y_i n[a,b]$), then

$$T(h_i,h_2)\ge 0. \tag{1.3}$$

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If h_1 and h_2 are asynchronous (i.e. $(h_1(x) - h_2(y))(h_2(x) - h_2(y)) \leq 0$, for any $x, y_in[a, b]$), then inequality (1.3) holds in the reversed direction.

The functional (**??**) has received a great deal of attention from mathematicians and a number of inequalities have appeared in the literature, (see [3, 5, 6, 7, 8, 9, 10]). The aim of this paper is to derive some weighted integral inequalities for monotone and quasi-monotone functions involving the weighted Hardy operators and its conjugates, using weighted Chebyshev functional. Some known and new results are deduced.

2. Main results

We first give some results for monotone functions.

The following lemma was established in [10].

Lemma 2.1. *Let h_1 and h_2 are two synchronous functions on (a, b) for each $t_1, t_2 \in (a, b)$, then*

$$\int_a^b p(t)h_1(t)dt \int_a^b p(t)h_2(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)h_1(t)h_2(t)dt, \tag{2.1}$$

where p is weight function defined on (a, b). If the functions h_1 and h_2 are asynchronous on (a, b) for each $t_1, t_2 \in (a, b)$, then

$$\int_a^b p(t)h_1(t)dt \int_a^b p(t)h_2(t)dt \geq \int_a^b p(t)dt \int_a^b p(t)h_1(t)h_2(t)dt. \tag{2.2}$$

Let

$$H_{1,p,q}(x) = \int_0^x h_1(t)p(t)q(t)dt, \quad H_{2,p,q}(x) = \int_0^x h_2(t)p(t)q(t)dt$$

and

$$H^*_{1,p,q}(x) = \int_x^\infty h_1(t)p(t)q(t)dt, \quad H^*_{2,p,q}(x) = \int_x^\infty h_2(t)p(t)q(t)dt,$$

where $t \in (a, b), 0 < a < b \leq \infty$, h_1 and h_2 are non-negative Lebesgue measurable functions on (a, b) and p, q are positive weight functions defined on (a, b).

In this work, we assume that ψ is a non-negative function defined on (a, b).

Theorem 2.2. *Let h_1, h_2, q and ψ be non-decreasing functions on (a, b) and p be a positive weight function on (a, b). If the function*

$$\psi\left[h_1(x)h_2(x)\left(\frac{q(x)}{x}\int_0^x p(t)dt\right)^2\right]$$

is integrable on (a, b), then

$$\int_a^b \psi\left(\frac{H_{1,p,q}(x)H_{2,p,q}(x)}{x^2}\right)dx \leq \int_a^b \psi\left[h_1(x)h_2(x)\left(\frac{q(x)}{x}\int_0^x p(t)dt\right)^2\right]dx. \tag{2.3}$$

Proof. By the weighted Chebyshev integral inequality (2.1) and assumptions on functions h_1, h_2, p, q, ψ for all $x \in (a,b)$, we get

$$\begin{aligned}
&\int_a^b \psi\left(\frac{H_{1,p,q}(x)H_{2,p,q}(x)}{x^2}\right)dx \\
&= \int_a^b \psi\left[\frac{1}{x^2}\left(\int_0^x p(t)q(t)h_1(t)dt\right)\left(\int_0^x p(t)q(t)h_2(t)dt\right)\right]dx \\
&\leq \int_a^b \psi\left[\frac{q^2(x)}{x^2}\left(\int_0^x p(t)h_1(t)dt\right)\left(\int_0^x p(t)h_2(t)dt\right)\right]dx \\
&\leq \int_a^b \psi\left[\frac{q^2(x)}{x^2}\left(\int_0^x p(t)dt\right)\left(\int_0^x p(t)h_1(t)h_2(t)dt\right)\right]dx \\
&\leq \int_a^b \psi\left[\frac{q^2(x)h_1(x)h_2(x)}{x^2}\left(\int_0^x p(t)dt\right)^2\right]dx \\
&= \int_a^b \psi\left[h_1(x)h_2(x)\left(\frac{q(x)}{x}\int_0^x p(t)dt\right)^2\right]dx.
\end{aligned}$$

□

Corollary 2.3. *If the function ψ is non-increasing in Theorem 2.2, inequality* (2.3) *holds in the reversed direction*
(2.4)

$$\int_a^b \psi\left(\frac{H_{1,p,q}(x)H_{2,p,q}(x)}{x^2}\right)dx \geq \int_a^b \psi\left[h_1(x)h_2(x)\left(\frac{q(x)}{x}\int_0^x p(t)dt\right)^2\right]dx.$$

Remark 2.4. In particular, if we set $\psi(x) = x^\mu$ in (2.3), where $\mu \geq 1$, we get

$$\int_a^b \left(\frac{H_{1,p,q}(x)H_{2,p,q}(x)}{x^2}\right)^\mu dx \leq \int_a^b \left[h_1(x)h_2(x)\left(\frac{q(x)}{x}\int_0^x p(t)dt\right)^2\right]^\mu dx.$$

Remark 2.5. If $q(x) = p(x) = 1$ in inequality (2.3) and (2.4), we get Theorem [9, Th. 6] and Theorem [9, Th. 7]. In addition if we set $\psi(x) = x^\mu$, $\mu \geq 1$, and $h_2(x) = 1$, in inequalities (2.3) and (2.4) we get a Hardy-type inequality

$$\int_a^b \left(\frac{H_1(x)}{x}\right)^\mu dx \leq \int_a^b h_1^\mu(x)dx,$$

and its converse, respectively.

Theorem 2.6. *Let h_1, h_2, q be non-increasing functions on (a,b), ψ be a non-decreasing function on (a,b) and p be a positive weight function on (a,b). If the function*

$$\psi\left[h_1(x)h_2(x)\left(\frac{q(x)}{x}\int_x^\infty p(t)dt\right)^2\right]$$

is integrable on (a,b), then
(2.5)

$$\int_a^b \psi\left(\frac{H^*_{1,p,q}(x)H^*_{2,p,q}(x)}{x^2}\right)dx \leq \int_a^b \psi\left[h_1(x)h_2(x)\left(\frac{q(x)}{x}\int_x^\infty p(t)dt\right)^2\right]dx.$$

By using the Chebyshev integral inequality (2.1), the proof is similar to that of Theorem 2.2.

Remark 2.7. In particular if we set $\psi(x) = x^{\mu}$ where $\mu \geq 1$, we get

$$\int_a^b \left(\frac{H^*_{1,p,q}(x)H^*_{2,p,q}(x)}{x^2}\right)^{\mu} dx \leq \int_a^b \left[h_1(x)h_2(x)\left(\frac{q(x)}{x}\int_x^{\infty} p(t)dt\right)^2\right]^{\mu} dx.$$

Corollary 2.8. *If $q(x) = 1$ in inequality* (2.5), *we have*

$$\int_a^b \psi\left(\frac{H^*_{1,p}(x)H^*_{2,p}(x)}{x^2}\right)dx \leq \int_a^b \psi\left[h_1(x)h_2(x)\left(\frac{1}{x}\int_x^{\infty} p(t)dt\right)^2\right]dx.$$

Corollary 2.9. *If the function ψ is non-increasing in Theorem 2.6, inequality* (2.5) *holds in the reversed direction.*

Theorem 2.10. *Let h_1 and h_2 be asynchronous functions on (a, b). We suppose h_1 and q are non-increasing functions on (a, b) and ψ is a non-decreasing function on (a, b) and p is a positive weight function on (a, b), then*

$$\int_a^b \psi\left(\frac{H_{1,p,q}(x)H_{2,p,q}(x)}{x^2}\right)dx \geq \int_a^b \psi\left[h_1(x)\left(\frac{q^2(x)}{x^2}\int_0^x p(t)h_2(t)dt\int_0^x p(t)dt\right)\right]dx. \tag{2.6}$$

Proof. By the Chebyshev integral inequality (2.2) and conditions on functions h_1, h_2, p, q, ψ for all $x \in (a, b)$, we obtain

$$\begin{aligned}
&\int_a^b \psi\left(\frac{H_{1,p,q}(x)H_{2,p,q}(x)}{x^2}\right)dx \\
&= \int_a^b \psi\left[\frac{1}{x^2}\left(\int_0^x p(t)q(t)h_1(t)dt\right)\left(\int_0^x p(t)q(t)h_2(t)dt\right)\right]dx \\
&\geq \int_a^b \psi\left[\frac{q^2(x)}{x^2}\left(\int_0^x p(t)h_1(t)dt\right)\left(\int_0^x p(t)h_2(t)dt\right)\right]dx \\
&\geq \int_a^b \psi\left[\frac{q^2(x)}{x^2}\left(\int_0^x p(t)dt\right)\left(\int_0^x p(t)h_1(t)h_2(t)dt\right)\right]dx \\
&\geq \int_a^b \psi\left[\frac{q^2(x)h_1(x)}{x^2}\int_0^x p(t)g(t)dt\int_0^x p(t)dt\right]dx \\
&= \int_a^b \psi\left[h_1(x)\left(\frac{q^2(x)}{x^2}\int_0^x p(t)h_2(t)dt\int_0^x p(t)dt\right)\right]dx.
\end{aligned}$$

□

Corollary 2.11. *If ψ is a non-increasing function in Theorem 2.10, inequality* (2.6) *holds in the reversed direction.*

Corollary 2.12. *If $p(x) = q(x) = 1$ in inequality (2.6), we have*

$$\int_a^b \psi\left(\frac{H_1(x)H_2(x)}{x^2}\right)dx \geq \int_a^b \psi\left[h_1(x)\left(\frac{1}{x}\int_0^x h_2(t)dt\right)\right]dx.$$

Theorem 2.13. *Let h_1 and h_2 be asynchronous functions on (a, b). We suppose h_1, q and ψ are non-decreasing functions on (a, b) and p is a positive weight function on (a, b), then*

(2.7)
$$\int_a^b \psi\left(\frac{H^*_{1,p,q}(x)H^*_{2,p,q}(x)}{x^2}\right)dx \geq \int_a^b \psi\left[h_1(x)\left(\frac{q^2(x)}{x^2}\int_x^\infty p(t)h_2(t)dt\int_x^\infty p(t)dt\right)\right]dx.$$

The proof is obtained by using arguments similar to those in the proof of Theorem 2.10.

Corollary 2.14. *If ψ is a non-increasing function in Theorem 2.13, inequality (2.7) holds in the reversed direction.*

Corollary 2.15. *If $q(x) = 1$ in (2.7), we obtain*

$$\int_a^b \psi\left(\frac{H^*_{1,p}(x)H^*_{2,p}(x)}{x^2}\right)dx \geq \int_a^b \psi\left[h_1(x)\left(\frac{1}{x^2}\int_x^\infty h_2(t)dt\int_x^\infty p(t)dt\right)\right]dx.$$

Theorem 2.16. *Let h_1, h_2, q be non-increasing functions on (a, b), ψ be a non-decreasing function on (a, b) and p be a positive weight function defined on (x, ∞) where*

$$0 < \left(\int_x^\infty p(t)dt\right) < \infty.$$

If the function $\psi\big(h_1(x)h_2(x)q^2(x)\big)$ is integrable on (a, b), then

$$\int_a^b \psi\left(\frac{H^*_{1,p,q}(x)H^*_{2,p,q}(x)}{\left(\int_x^\infty p(t)dt\right)^2}\right)dx \leq \int_a^b \psi\big(h_1(x)h_2(x)q^2(x)\big)dx. \tag{2.8}$$

Proof. By the Chebyshev integral inequality (2.1) and assumptions on functions h_1, h_2, p, q, ψ for all $x \in (a, b)$, we get

$$\int_a^b \psi\left(\frac{H^*_{1,p,q}(x)H^*_{2,p,q}(x)}{\left(\int_x^\infty p(t)dt\right)^2}\right)dx$$

$$= \int_a^b \psi\left[\frac{1}{\left(\int_x^\infty p(t)dt\right)^2}\left(\int_x^\infty p(t)q(t)h_1(t)dt\right)\left(\int_x^\infty p(t)q(t)h_2(t)dt\right)\right]dx$$

$$\leq \int_a^b \psi\left[\frac{q^2(x)}{\left(\int_x^\infty p(t)dt\right)^2}\left(\int_x^\infty p(t)h_1(t)dt\right)\left(\int_x^\infty p(t)h_2(t)dt\right)\right]dx$$

$$\leq \int_a^b \psi\left[\frac{q^2(x)}{\left(\int_x^\infty p(t)dt\right)^2}\left(\int_x^\infty p(t)dt\right)\left(\int_x^\infty p(t)h_1(t)h_2(t)dt\right)\right]dx$$

$$\leq \int_a^b \psi\Big[\frac{q^2(x)h_1(x)h_2(x)}{\big(\int_x^\infty p(t)dt\big)^2}\Big(\int_x^\infty p(t)dt\Big)^2\Big]dx$$

$$= \int_a^b \psi\big(h_1(x)h_2(x)q^2(x)\big)dx.$$

□

Remark 2.17. In particular if we set $\psi(x) = x^\mu$ where $\mu \geq 1$, we get

$$\int_a^b \left(\frac{H^*_{1,p,q}(x)H^*_{2,p,q}(x)}{\big(\int_x^\infty p(t)dt\big)^2}\right)^\mu dx \leq \int_a^b \big(h_1(x)h_2(x)q^2(x)\big)^\mu dx.$$

Corollary 2.18. *If the function $q(x) = 1$ in (2.8), we obtain*

$$\int_a^b \psi\left(\frac{H^*_{1,p}(x)H^*_{2,p}(x)}{\big(\int_x^\infty p(t)dt\big)^2}\right)dx \leq \int_a^b \psi\big(h_1(x)h_2(x)\big)dx.$$

Corollary 2.19. *If $q(x) = h_2(x) = 1$ and $\psi(x) = x$, we have*

$$\int_a^b \frac{\int_x^\infty h_1(t)p(t)dt}{\int_x^\infty p(t)dt}dx \leq \int_a^b h_1(x)dx.$$

Now we turn our attention to the case of quasi-monotone functions.
The following definition was introduced in [1].

Definition 2.20. We say that a non-negative function h_1 is quasi-monotone on $]0,\infty[$ if for some real number α, $x^\alpha h_1$ is a decreasing or an increasing function of x. More precisely, given $\beta \in \mathbb{R}$ we say that $h_1 \in Q_\beta$ if $x^{-\beta}h_1(x)$ is non-increasing and $h_1 \in Q^\beta$ if $x^{-\beta}h_1(x)$ is non-decreasing.

Let $h_{1,\beta}(x) = x^{-\beta}h_1(x), h_{2,\beta}(x) = x^{-\beta}h_2(x)$ on (a,b), where $0 < a < b \leq \infty$.

Lemma 2.21. *If $h_{1,\beta}$ and $h_{2,\beta}$ are two synchronous functions on (a,b), i.e $h_1, h_2 \in Q_\beta$ or $h_1, h_2 \in Q^\beta$ and p is a positive weight function on (a,b), then*
(2.9)
$$\int_a^b p(x)x^{-\beta}h_1(x)dx \int_a^b p(x)x^{-\beta}h_2(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)x^{-2\beta}h_1(x)h_2(x)dx.$$

If $h_{1,\beta}$ and $h_{2,\beta}$ are asynchronous functions on (a,b), then
(2.10)
$$\int_a^b p(x)x^{-\beta}h_1(x)dx \int_a^b p(x)x^{-\beta}h_2(x)dx \geq \int_a^b p(x)dx \int_a^b p(x)x^{-2\beta}h_1(x)h_2(x)dx.$$

Proof. We replace h_1 and h_2 in Lemma 2.1 by $h_{1,\beta}$ and $h_{2,\beta}$, respectively, then we get inequality (2.9). In the same way, we have inequality (2.10). □

Let

$$H_{1,\beta,p,q}(x)=\int_0^x t^{-\beta}h_1(t)p(t)q(t)dt,\quad H_{2,\beta,p,q}(x)=\int_0^x t^{-\beta}h_2(t)p(t)q(t)dt$$

and

$$H^*_{1,\beta,p,q}(x)=\int_x^\infty t^{-\beta}h_1(t)p(t)q(t)dt,\quad H^*_{2,\beta,p,q}(x)=\int_x^\infty t^{-\beta}h_2(t)p(t)q(t)dt,$$

where $t\in(a,b), 0<a<b\le\infty$, $\beta\in\mathbb{R}$, h_1 and h_2 are non-negative Lebesgue measurable functions on (a,b) and p,q are positive weight functions on (a,b).

Theorem 2.22. *Let $h_1,h_2\in Q^\beta$, q and ψ be non-decreasing functions on (a,b) and p be a positive weight function defined on (a,b). If the function*

$$\psi\left[h_1(x)h_2(x)\left(x^{-(1+\beta)}q(x)\int_0^x p(t)dt\right)^2\right]$$

is integrable on (a,b), then

(2.11)

$$\int_a^b\psi\left(\frac{H_{1,\beta,p,q}(x)H_{2,\beta,p,q}(x)}{x^2}\right)dx\le\int_a^b\psi\left[h_1(x)h_2(x)\left(x^{-(1+\beta)}q(x)\int_0^x p(t)dt\right)^2\right]dx.$$

Proof. We apply integral inequality (2.9) and conditions on functions h_1 h_2, p, q, ψ for all $x\in(a,b)$, we get

$$\begin{aligned}
&\int_a^b\psi\left(\frac{H_{1,\beta,p,q}(x)H_{2,\beta,p,q}(x)}{x^2}\right)dx\\
&=\int_a^b\psi\left[\frac{1}{x^2}\left(\int_0^x p(t)q(t)t^{-\beta}h_1(t)dt\right)\left(\int_0^x p(t)q(t)t^{-\beta}h_2(t)dt\right)\right]dx\\
&\le\int_a^b\psi\left[\frac{q^2(x)}{x^2}\left(\int_0^x p(t)t^{-\beta}h_1(t)dt\right)\left(\int_0^x p(t)t^{-\beta}h_2(t)dt\right)\right]dx\\
&\le\int_a^b\psi\left[\frac{q^2(x)}{x^2}\left(\int_0^x p(t)dt\right)\left(\int_0^x p(t)t^{-2\beta}h_1(t)h_2(t)dt\right)\right]dx\\
&\le\int_a^b\psi\left[\frac{q^2(x)x^{-2\beta}h_1(x)h_2(x)}{x^2}\left(\int_0^x p(t)dt\right)^2\right]dx\\
&=\int_a^b\psi\left[h_1(x)h_2(x)\left(x^{-(1+\beta)}q(x)\int_0^x p(t)dt\right)^2\right]dx.
\end{aligned}$$

□

Remark 2.23. In particular if we set $\psi(x)=x^\mu$ where $\mu\ge1$, we get

$$\int_a^b\left(\frac{H_{1,\beta,p,q}(x)H_{2,\beta,p,q}(x)}{x^2}\right)^\mu dx\le\int_a^b\left[h_1(x)h_2(x)\left(x^{-(1+\beta)}q(x)\int_0^x p(t)dt\right)^2\right]^\mu dx.$$

Corollary 2.24. *If $q(x) = p(x) = 1$ in (2.11), we get*

$$\int_a^b \psi\Big(\frac{H_{1,\beta}(x)H_{2,\beta}(x)}{x^2}\Big)dx \le \int_a^b \psi\big(h_{1,\beta}(x)h_{2,\beta}(x)\big)dx.$$

Corollary 2.25. *If the function ψ is non-increasing in Theorem 2.22, inequality (2.11) holds in the reversed direction.*

Remark 2.26. If $\beta = 0$ in (2.11), we obtain inequality (2.3) of Theorem 2.2.

Theorem 2.27. *Let $h_1, h_2 \in Q_\beta$, ψ be a non-decreasing function on (a, b), q be a non-increasing function on (a, b) and p be a positive weight function on (a, b). If the function $\psi\Big[h_1(x)h_2(x)\Big(x^{-(1+\beta)}q(x)\int_x^\infty p(t)dt\Big)^2\Big]$ is integrable on (a, b), then*

(2.12)
$$\int_a^b \psi\Big(\frac{H^*_{1,\beta,p,q}(x)H^*_{2,\beta,p,q}(x)}{x^2}\Big)dx \le \int_a^b \psi\Big[h_1(x)h_2(x)\Big(x^{-(1+\beta)}q(x)\int_x^\infty p(t)dt\Big)^2\Big]dx.$$

The proof is similar to that of Theorem 2.22.

Remark 2.28. In particular if we set $\psi(x) = x^\mu$ where $\mu \ge 1$, we get

$$\int_a^b \Big(\frac{H^*_{1,\beta,p,q}(x)H^*_{2,\beta,p,q}(x)}{x^2}\Big)^\mu dx \le \int_a^b \Big[h_1(x)h_2(x)\Big(x^{-(1+\beta)}q(x)\int_x^\infty p(t)dt\Big)^2\Big]^\mu dx.$$

Corollary 2.29. *If $q(x) = 1$ in (2.12), we get*

$$\int_a^b \psi\Big(\frac{H^*_{1,\beta,p}(x)H^*_{2,\beta,p}(x)}{x^2}\Big)dx \le \int_a^b \psi\Big[h_1(x)h_2(x)\Big(x^{-(\beta+1)}\int_x^\infty p(t)dt\Big)^2\Big]dx.$$

Corollary 2.30. *If the function ψ is non-increasing in Theorem 2.27, inequality (2.12) holds in the reversed direction.*

Remark 2.31. If $\beta = 0$ in Theorem 2.27, we obtain inequality (2.5) of Theorem 2.6.

Theorem 2.32. *Let $h_1 \in Q_\beta$ and $h_2 \in Q^\beta$, ψ be a non-decreasing function on (a, b), p be a positive weight function on (a, b) and q be a non-increasing function on (a, b), then*

$$\int_a^b \psi\Big(\frac{H_{1,\beta,p,q}(x)H_{2,\beta,p,q}(x)}{x^2}\Big)dx$$

$$\ge \int_a^b \psi\Big[x^{-(2+\beta)}q^2(x)h_1(x)\int_0^x p(t)t^{-\beta}h_2(t)dt\int_0^x p(t)dt\Big]dx. \tag{2.13}$$

Proof. We apply integral inequality (2.10) with the functions h_1, h_2, p, q, ψ for all $x \in (a,b)$, therefore

$$\int_a^b \psi\Big(\frac{H_{1,\beta,p,q}(x)H_{2,\beta,p,q}(x)}{x^2}\Big)dx$$
$$= \int_a^b \psi\Big[\frac{1}{x^2}\Big(\int_0^x p(t)q(t)t^{-\beta}h_1(t)dt\Big)\Big(\int_0^x p(t)q(t)t^{-\beta}h_2(t)dt\Big)\Big]dx$$
$$\geq \int_a^b \psi\Big[\frac{q^2(x)}{x^2}\Big(\int_0^x p(t)t^{-\beta}h_1(t)dt\Big)\Big(\int_0^x p(t)t^{-\beta}h_2(t)dt\Big)\Big]dx$$
$$\geq \int_a^b \psi\Big[\frac{q^2(x)}{x^2}\Big(\int_0^x p(t)dt\Big)\Big(\int_0^x p(t)t^{-2\beta}h_1(t)h_2(t)dt\Big)\Big]dx$$
$$\geq \int_a^b \psi\Big[\frac{q^2(x)x^{-\beta}h_1(x)}{x^2}\int_0^x p(t)t^{-\beta}h_2(t)dt\int_0^x p(t)dt\Big]dx$$
$$= \int_a^b \psi\Big[x^{-(2+\beta)}q^2(x)h_1(x)\int_0^x p(t)t^{-\beta}h_2(t)dt\int_0^x p(t)dt\Big]dx.$$

□

Corollary 2.33. *If ψ is a non-increasing function in Theorem 2.32, inequality* (2.13) *holds in the reversed direction .*

Corollary 2.34. *If $p(x) = q(x) = 1$ in inequality* (2.13)*, we have*

$$\int_a^b \psi\Big(\frac{H_1(x)H_2(x)}{x^2}\Big)dx \geq \int_a^b \psi\Big[x^{-\beta}h_1(x)\Big(\frac{1}{x}\int_0^x t^{-\beta}h_2(t)dt\Big)\Big]dx.$$

Remark 2.35. If $\beta = 0$ in Theorem 2.32, we get inequality (2.4) of Corollary 2.3.

Theorem 2.36. *Let $h_1 \in Q^\beta$ and $h_2 \in Q_\beta$, q and ψ be non-decreasing functions on (a,b) and p be a positive weight function on (a,b), then*

$$\int_a^b \psi\Big(\frac{H^*_{1,\beta,p,q}(x)H^*_{2,\beta,p,q}(x)}{x^2}\Big)dx$$

$$\geq \int_a^b \psi\Big[x^{-(2+\beta)}q^2(x)h_1(x)\int_x^\infty p(t)t^{-\beta}h_2(t)dt\int_x^\infty p(t)dt\Big]dx. \tag{2.14}$$

The proof is similar to that of Theorem 2.32. Hence, it is omitted.

Corollary 2.37. *If ψ is a non-increasing function in Theorem 2.36, inequality* (2.14) *holds in the reversed direction.*

Remark 2.38. If $\beta = 0$ in Theorem 2.36, we obtain inequality (2.7) of Theorem 2.13.

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