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Tangent bundles of Lorentzian α-Sasakian manifolds

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Abstract. The goal of the present study is to investigate the complete lifts from Lorentzian α-Sasakian manifolds to tangent bundles with the conformally flat and the quasi-conformally flat curvature tensor. It is demonstrated that the manifold is locally isometric with a sphere $S^{2n+1}(c)$ on the tangent bundle in both situations.

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1. Introduction

Tanno categorized connected almost contact metric manifolds in [25] whose automorphism groups have the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He demonstrated that they can be classified into three groups:
(1) homogeneous normal contact Riemannian manifolds with $c > 0$,
(2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and
(3) a warped product space $R \times_f \mathbb{C}$ if $c < 0$.

Kim et. al. [19] studied generalized Ricci-reccurent trans Sasakian manifolds and discussed Sasakian, α-Sasakian, Kenmotsu or β-Kenmotsu manifolds. Later, De and Tripathi [8] established the expressions for the Ricci operator, the Ricci tensor and the curvature tensor on 3-dimensional trans Sasakian manifolds. Furthermore, Yildiz and Murathan [31] studied Lorentzian α-Sasakian manifolds with the conformally flat and the quasi conformally flat curvature tensor. Recently, Yoldas dealt with Ricci solitons on Lorentizian α-Sasakian manifolds in [33]. For the further studies onthe class of almost contactmetric manifolds we refer to ([5], [6], [12], [28], [23], [27], [34], [32]) and many more.

The Weyl conformal curvature tensor $\mathcal{C}$ and the concircular curvature tensor $\tilde{\mathcal{C}}$ on a Riemannian manifold M of dimension n have been studied by Adati and Matsumoto [1], Chaki and Gupta [6] and Yano and Sawaki [30].

On the other hand, Yano and ishihara [29] proposed the notion of the liftings of tensor fields and connections to its tangent bundle and established basic properties of curvature tensors. Dida and Hathout ([10], [9], [11]) investigated

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Ricci soliton structures on tangent bundles of Riemannian manifolds. Numerous researchers ([2], [3], [4], [13], [18], [17], [14], [15], [16]) have discussed various connections and geometric structures on the tangent bundle and determined their basic geometric properties.

Motivated by these studies, we study the complete lifts from Lorentzian α-Sasakian Manifolds with the conformally flat and the quasi conformally flat curvature tensor to tangent bundles.

The main findings of the paper are given as:

- The complete lifts from Lorentzian α-Sasakian manifolds to tangent bundles with the Weyl conformal curvature tensor

$$\mathcal{C} = 0 \tag{1.1}$$

on TM are the subject of our investigation.

- A conformally flat Lorentzian α-Sasakian manifolds on TM is locally isometric to a sphere $S^{2n+1}(c)$ such that $c = \alpha^2$ is shown.

- The complete lifts of Lorentzian α-Sasakian manifolds to tangent bundles with the quasi conformal curvature tensor

$$\tilde{\mathcal{C}} = 0 \tag{1.2}$$

on TM are the subject of our investigation.

- A quasi conformally flat Lorentzian α-Sasakian manifolds on TM is locally isometric to a sphere $S^{2n+1}(c)$ such that $c = \alpha^2$ is shown.

Notations: Throughout the article following notations are used: $\Im_r^s(M)$ and $\Im_r^s(TM)$ denote the set of all tensor fields of type (r, s), that is of contravariant degree r and covariant degree s, in M and TM, respectively.

2. Preliminaries

A differentiable manifold dim $(= 2n + 1)$ is said to be a Lorentzian α-Sasakian manifold if it there exists a (1, 1)-tensor field ϕ, a vector field ξ, a vector field η and a Lorentzian metric g which fulfill ([6], [21], [20], [22], [26])

$$\eta(\xi) = -1, \tag{2.1}$$
$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$
$$g(X, \xi) = \eta(X), \tag{2.4}$$
$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.5}$$

$\forall X, Y \in \Im_0^1(M)$.

In addition, a Lorentzian α-Sasakian manifold M holds the relations [8]

$$\nabla_X \xi = -\alpha\phi X, \tag{2.6}$$
$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y), \tag{2.7}$$

where ∇ represents the covariant differentation operator with respect to the Lorentzian metric g.

A Lorentzian α-Sasakian manifold M is called η-Einstein if its Ricci tensor S is given by

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.8}$$

$\forall X, Y \in \Im_0^1(M)$, where a, b are functions on M. Further, on a Lorentzian α-Sasakian manifold M satisfies ([19], [26])

$$R(\xi, X)Y = \alpha^2(g(X,Y)\xi - \eta(Y)X), \tag{2.9}$$
$$R(X,Y)\xi = \alpha^2(\eta(Y)X - \eta(X)Y), \tag{2.10}$$
$$R(\xi, X)\xi = \alpha^2(\eta(X)\xi + X), \tag{2.11}$$
$$S(X,\xi) = 2n\alpha^2\eta(X), \tag{2.12}$$
$$Q\xi = 2n\alpha^2\xi, \tag{2.13}$$
$$S(\xi,\xi) = -2n\alpha^2, \tag{2.14}$$

where S is the Ricci tensor and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

3. The complete lift

In an n-dimensional differentiable manifold M, $T_p(M)$ is the tangent space at a point p of M i.e. the set of all tangent vectors of M at p. Then the set $TM = \bigcup_{p\in M} T_p(M)$ is the tangent bundle over the manifold M. Let $(x^i), i = 1, ..., n$ be a local co-ordinate system on M and $(x^i, y^i), i = 1, ..., n$ is a induced local co-ordinate system on TM. If $X = X^i \frac{\partial}{\partial x^i}$ is a local vector field on M, then its vertical, complete and horizontal lifts in the terms of partial differential equations are

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{3.1}$$
$$X^C = X^i \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}. \tag{3.2}$$

Let the function, a 1-form, a vector field and a tensor field of type (1,1) be symbolized as f, η, X and ϕ, respectively. The complete and vertical lifts of f, η, X and ϕ are symbolized as f^C, η^C, X^C, ϕ^C and f^V, η^V, X^V, ϕ^V, respectively. The following formulas on f, η, X and ϕ are given by ([7], [24], [29])

$$(fX)^V = f^V X^V, (fX)^C = f^C X^V + f^V X^C, \tag{3.3}$$
$$X^V f^V = 0, X^V f^C = X^C f^V = (Xf)^V, X^C f^C = (Xf)^C, \tag{3.4}$$
$$\eta^V(f^V) = 0, \eta^V(X^C) = \eta^C(X^V) = \eta(X)^V, \eta^C(X^C) = \eta(X)^C, \tag{3.5}$$
$$\phi^V X^C = (\phi X)^V, \phi^C X^C = (\phi X)^C, \tag{3.6}$$
$$[X,Y]^V = [X^C, Y^V] = [X^V, Y^C], [X,Y]^C = [X^C, Y^C], \tag{3.7}$$

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C, \qquad \nabla^C_{X^C} Y^V = (\nabla_X Y)^V, \tag{3.8}$$

Taking the complete lift on (2.1)-(2.8), we infer

$$\eta^C(\xi^C) = \eta^V(\xi^V) = 0, \quad \eta^C(\xi^V) = \eta^V(\xi^C) = -1, \tag{3.9}$$

$$(\phi^2)^C = I + \eta^C \otimes \xi^V + \eta^V \otimes \xi^C, \tag{3.10}$$

$$g^C((\phi X)^C, (\phi Y)^C) = g^C(X^C, Y^C) + \eta^C(X^C)\eta^V(Y^C) \tag{3.11}$$

$$+ \eta^C(X^C)\eta^V(Y^C), \tag{3.12}$$

$$g^C(X^C, \xi^C) = \eta^C(X^C), \tag{3.13}$$

$$\phi^C \xi^C = \phi^V \xi^V = \phi^C \xi^V = \phi^V \xi^C = 0, \tag{3.14}$$

$$\eta^C(\phi X)^C = \eta^V(\phi X)^V = \eta^C(\phi X)^V = \eta^V(\phi X)^C = 0, \tag{3.15}$$

$\forall X^C, Y^C \in \Im^1_0(TM)$ and we also have

$$\nabla^C_{X^C} \xi^C = -\alpha(\phi X)^C, \tag{3.16}$$

$$(\nabla^C_{X^C} \eta^C) Y^C = -\alpha g^C((\phi X)^C, Y^C), \tag{3.17}$$

where ∇^C is the complete lift of ∇ on TM.
A Lorentzian α-Sasakian manifold M on TM is said to be η^C-Einstein if its Ricci tensor S^C is of the form

$$S^C(X^C, Y^C) = a g^C(X^C, Y^C) + b\{\eta^C(X^C)\eta^V(Y^C) + \eta^V(X^C)\eta^C(Y^C)\}, \tag{3.18}$$

$\forall X^C, Y^C \in \Im^1_0(TM)$.

Taking the complete lift on (2.9)-(2.14), we infer

$$\begin{aligned} R^C(\xi^C, X^C)Y^C &= \alpha^2\{g^C(X^C, Y^C)\xi^V + g^C(X^V, Y^C)\xi^V \\ &- \eta^C(Y^C)X^V + \eta^V(Y^C)X^C\}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} R^C(X^C, Y^C)\xi^C &= \alpha^2\{\eta^C(Y^C)X^V + \eta^V(Y^C)X^C \\ &- \eta^C(X^C)Y^V - \eta^V(X^C)Y^C\}, \end{aligned} \tag{3.20}$$

$$R^C(\xi^C, X^C)\xi^C = \alpha^2\{\eta^C(X^C)\xi^V + \eta^V(X^C)\xi^C + X^C\}, \tag{3.21}$$

$$S^C(X^C, \xi^C) = 2n\alpha^2 \eta^C(X^C), \tag{3.22}$$

$$(Q\xi)^C = 2n\alpha^2 \xi^C, \tag{3.23}$$

$$S^C(\xi^C, \xi^C) = -2n\alpha^2, \tag{3.24}$$

where S^C and Q^C are the complete lifts of S and Q, respectively given by $S^C(X^C, Y^C) = g^C((QX)^C, Y^C)$.

4. Lorentzian α-Sasakian manifolds with $\mathcal{C} = 0$

The conformal curvature tensor $\mathcal{C}$ on M^{2n+1} is defined as ([1], [6])

$$\begin{aligned} \mathcal{C}(X,Y)Z &= R(X,Y)Z + \frac{1}{2n-1}[S(X,Z)Y - S(Y,Z)X \\ &+ g(X,Z)QY - g(Y,Z)QX] \\ &+ \frac{r}{2n(2n-1)}[g(X,Z)Y - g(Y,Z)X], \end{aligned} \tag{4.1}$$

where

$$S(X,Y) = g(QX,Y).$$

Taking the complete lift in (4.1), we acquire

$$\begin{aligned}
\mathcal{C}^C(X^C,Y^C)Z^C &= R^C(X^C,Y^C)Z^C \\
&+ \frac{1}{2n-1}[S^C(X^C,Z^C)Y^V + S^C(X^V,Z^C)Y^V \\
&- S^C(X^C,Z^C)Y^V - S^C(X^V,Z^C)Y^C \\
&+ g^C(X^C,Z^C)(QY)^V + g^C(X^V,Z^C)(QY)^C \\
&- g^C(Y^C,Z^C)(QX)^V - g^C(Y^V,Z^C)(QX)^C] \\
&- \frac{r}{2n(2n-1)}[g^C(X^C,Z^C)Y^V + g^C(X^V,Z^C)Y^C \\
&- g^C(Y^C,Z^C)X^V - g^C(Y^V,Z^C)X^C].
\end{aligned} \tag{4.2}$$

Using (1.1) we get from (4.2)

$$\begin{aligned}
R^C(X^C,Y^C)Z^C &= \frac{1}{2n-1}[S^C(X^C,Z^C)Y^V + S^C(X^V,Z^C)Y^C \\
&- S^C(X^C,Z^C)Y^V - S^C(X^V,Z^C)Y^V \\
&- g^C(X^C,Z^C)(QY)^V - g^C(X^V,Z^C)(QY)^C \\
&+ g^C(Y^C,Z^C)(QX)^V + g^C(Y^V,Z^C)(QX)^C] \\
&+ \frac{r}{2n(2n-1)}[g^C(X^C,Z^C)Y^V + g^C(X^V,Z^C)Y^C \\
&- g^C(Y^C,Z^C)X^V - g^C(Y^V,Z^C)X^C].
\end{aligned} \tag{4.3}$$

Taking $Z = \xi$ in (4.3) and using (3.13), (3.20) and (3.22), we find

$$\begin{aligned}
&\left(\alpha^2 + \frac{r}{2n(2n-1)} - \frac{2n\alpha^2}{2n-1}\right)(\eta^C(Y^C)X^V + \eta^V(Y^C)X^C \\
-\ &\eta^C(X^C)Y^V - \eta^V(X^C)Y^C) \\
=\ &\frac{1}{2n-1}(\eta^C(Y^C)(QX)^V + \eta^V(Y^C)(QX)^C \\
-\ &\eta^C(X^C)(QY)^V - \eta^V(X^C)(QY)^C).
\end{aligned}$$

Taking $Y = \xi$ and using (3.9) we get

$$(QX)^C = \left(\frac{\tau}{2n} - \alpha^2\right)X^C + \left(\frac{\tau}{2n} - \alpha^2 - 2n\alpha^2\right)\{\eta^C(X^C)\xi^V + \eta^V(X^C)\xi^C\}. \tag{4.4}$$

Thus the manifold is η^C-Einstein. Contracting (4.4) we get

$$\tau = 2n(2n+1)\alpha^2. \tag{4.5}$$

Using (4.5) in (4.4) we infer

$$(QX)^C = 2n\alpha^2 X^C. \tag{4.6}$$

Substituting (4.6) in (4.3) we infer

$$\begin{aligned} R^C(X^C, Y^C)Z^C &= \alpha^2\{g^C(Y^C, Z^C)X^V + g^C(Y^V, Z^C)X^C \\ &- g^C(X^C, Z^C)Y^V - g^C(X^V, Z^C)Y^C, Z)Y\}. \end{aligned} \tag{4.7}$$

This implies that a conformally flat Lorentzian α-Sasakian manifold on the tangent bundle TM is of constant curvature α^2. Thus we conclude that

Theorem 4.1. *Let TM be the tangent bundle of a Lorentzian α-Sasakian manifold. Then a conformally flat Lorentzian α-Sasakian manifold on TM is locally isometric to a sphere $S^{2n+1}(c)$, where $c = \alpha^2$.*

5. Lorentzian α-Sasakian manifolds with $\tilde{C} = 0$

The quasi conformal curvature tensor $\tilde{C}$ on M^{2n+1} is defined as ([1], [6])

$$\tilde{C}(X,Y)Z = aR(X,Y)Z + b\{S(Y,Z)X - S(X,Z)Y \tag{5.1}$$

$$+ g(Y,Z)QX - g(X,Z)QY\} \tag{5.2}$$

$$- \frac{r}{2n+1}(\frac{a}{2n} + 2b)\{g(Y,Z)X - g(X,Z)Y\} \tag{5.3}$$

where a, b are constants such that $ab \neq 0$ and

$$S(Y,Z) = g(QY,Z)$$

Taking the complete lift in (5.1), we arrive at

$$\begin{aligned} &\tilde{C}^C(X^C, Y^C)Z^C \\ &= aR^C(X^C, Y^C)Z^C + b\{S^C(Y^C, Z^C)X^V + S^C(Y^V, Z^C)X^C \\ &- S^C(X^C, Z^C)Y^V - S^C(X^V, Z^C)Y^C \\ &+ g^C(Y^C, Z^C)(QX)^V + g^C(Y^V, Z^C)(QX)^C \\ &- g^C(X^C, Z^C)(QY)^V - g^C(X^V, Z^C)(QY)^C\} \\ &- \frac{r}{2n+1}(\frac{a}{2n} + 2b)\{g^C(Y^C, Z^C)X^V + g^C(Y^V, Z^C)X^C \\ &- g^C(X^C, Z^C)Y^V - g^C(X^V, Z^C)Y^C\} \end{aligned} \tag{5.4}$$

Making use of (1.2), we find from (5.4)

$$\begin{aligned} &R^C(X^C, Y^C)Z^C \\ &= -\frac{b}{a}\{S^C(Y^C, Z^C)X^V + S^C(Y^V, Z^C)X^C \\ &- S^C(X^C, Z^C)Y^V - S^C(X^V, Z^C)Y^C \\ &+ g^C(Y^C, Z^C)(QX)^V + g^C(Y^V, Z^C)(QX)^C \\ &- g^C(X^C, Z^C)(QY)^V - g^C(X^V, Z^C)(QY)^C\} \\ &+ \frac{r}{2n+1}(\frac{a}{2n} + 2b)\{g^C(Y^C, Z^C)X^V + g^C(Y^V, Z^C)X^C \\ &- g^C(X^C, Z^C)Y^V - g^C(X^V, Z^C)Y^C\} \end{aligned} \tag{5.5}$$

Putting $Z = \xi$ in (5.5) and using (3.13), (3.20) and (3.22), we get

$$\begin{aligned}\alpha^2\{\eta^C(Y^C)X^V &+ \eta^V(Y^C)X^C - \eta^C(X^C)Y^V - \eta^V(X^C)Y^C\} \\ &= -\frac{b}{a}\{\eta^C(Y^C)(QX)^V + \eta^V(Y^C)(QX)^C \\ &- \eta^C(X^C)(QY)^V - \eta^V(X^C)(QY)^C\} \\ &+ \{\frac{\tau}{(2n+1)a}(\frac{a}{2n} + 2b) - \frac{2n\alpha^2 b}{a}\} \\ &\quad \{\eta^C(Y^C)X^V + \eta^V(Y^C)X^C \\ &- \eta^C(X^C)Y^V - \eta^V(X^C)Y^C\}. \end{aligned} \tag{5.6}$$

Taking $Y = \xi$ and applying (2.1) we have

$$\begin{aligned}(QX)^C &= \{\frac{\tau}{(2n+1)b}(\frac{a}{2n} + 2b) - 2n\alpha^2 - \frac{a}{b}\alpha^2\}X^C \\ &+ \{\frac{\tau}{(2n+1)b}(\frac{a}{2n} + 2b) - 4n\alpha^2 - \frac{a}{b}\alpha^2\} \\ &\quad \{\eta^C(X^C)\xi^V + \eta^V(X^C)\xi^C\}. \end{aligned} \tag{5.7}$$

Contracting (5.7), we get after a few steps

$$\tau = 2n(2n+1)\alpha^2. \tag{5.8}$$

From (5.8) in (5.7), one immediately has

$$(QX)^C = 2n\alpha^2 X^C. \tag{5.9}$$

Finally, using (5.9), we find from (5.5)

$$\begin{aligned}R^C(X^C, Y^C)Z^C &= \alpha^2\{g^C(Y^C, Z^C)X^V + g^C(Y^V, Z^C)X^C \\ &- g^C(X^C, Z^C)Y^V - g^C(X^V, Z^C)Y^C\}. \end{aligned} \tag{5.10}$$

Thus we conclude that

Theorem 5.1. *Let TM be the tangent bundle of a Lorentzian α-Sasakian manifold. Then a quasi conformally flat Lorentzian α-Sasakian manifold on TM is locally isometric to a sphere $S^{2n+1}(c)$, where $c = \alpha^2$.*

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References

[1] Tyuzi Adati and Kōzi Matsumoto. On conformally recurrent and conformally symmetric P-Sasakian manifolds. *TRU Math.*, 13(1):25–32, 1977.

[2] Murat Altunbas, Lokman Bilen, and Aydin Gezer. Remarks about the Kaluza-Klein metric on tangent bundle. *Int. J. Geom. Methods Mod. Phys.*, 16(3):1950040, 13, 2019. `doi:10.1142/S0219887819500403`.

[3] Shahroud Azami. General natural metallic structure on tangent bundle. *Iran. J. Sci. Technol. Trans. A Sci.*, 42(1):81–88, 2018. `doi:10.1007/s40995-018-0488-x`.

[4] Lokman Bilen, Sibel Turanli, and Aydin Gezer. On Kähler-Norden-Codazzi golden structures on pseudo-Riemannian manifolds. *Int. J. Geom. Methods Mod. Phys.*, 15(5):1850080, 10, 2018. `doi:10.1142/S0219887818500809`.

[5] D. E. Blair and J. A. Oubiña. Conformal and related changes of metric on the product of two almost contact metric manifolds. *Publ. Mat.*, 34(1):199–207, 1990. URL: `https://doi.org/10.5565/PUBLMAT_34190_15`, `doi:10.5565/PUBLMAT\_34190\_15`.

[6] M. C. Chaki and Bandana Gupta. On conformally symmetric spaces. *Indian J. Math.*, 5:113–122, 1963.

[7] Lovejoy S. Das and Mohd. Nazul Islam Khan. Almost r-contact structures on the tangent bundle. *Differ. Geom. Dyn. Syst.*, 7:34–41, 2005.

[8] U. C. De and Mukut Mani Tripathi. Ricci tensor in 3-dimensional trans-Sasakian manifolds. *Kyungpook Math. J.*, 43(2):247–255, 2003.

[9] H. M. Dida and A. Ikemakhen. A class of metrics on tangent bundles of pseudo-Riemannian manifolds. *Arch. Math. (Brno)*, 47(4):293–308, 2011.

[10] H. Mohammed Dida and Fouzi Hathout. Ricci soliton on the tangent bundle with semi-symmetric metric connection. *Bull. Transilv. Univ. Braşov Ser. III. Math. Comput. Sci.*, 1(63)(2):37–52, 2021. `doi:10.31926/but.mif`.

[11] Mohammed Hamou Dida, Fouzi Hathout, and Mustapha Djaa. On the geometry of the second order tangent bundle with the diagonal lift metric. *Int. J. Math. Anal. (Ruse)*, 3(9-12):443–456, 2009.

[12] Alfred Gray and Luis M. Hervella. The sixteen classes of almost Hermitian manifolds and their linear invariants. *Ann. Mat. Pura Appl. (4)*, 123:35–58, 1980. `doi:10.1007/BF01796539`.

[13] Ahmet Kazan and H. Bayram Karadağ. Locally decomposable golden Riemannian tangent bundles with Cheeger-Gromoll metric. *Miskolc Math. Notes*, 17(1):399–411, 2016. `doi:10.18514/MMN.2016.1534`.

[14] Mohammad Nazrul Islam Khan. Lifts of hypersurfaces with quarter-symmetric semi-metric connection to tangent bundles. *Afr. Mat.*, 25(2):475–482, 2014. `doi:10.1007/s13370-013-0150-x`.

[15] Mohammad Nazrul Islam Khan. Novel theorems for the frame bundle endowed with metallic structures on an almost contact metric manifold. *Chaos Solitons Fractals*, 146:Paper No. 110872, 8, 2021. `doi:10.1016/j.chaos.2021.110872`.

[16] Mohammad Nazrul Islam Khan. Tangent bundles endowed with semi-symmetric non-metric connection on a Riemannian manifold. *Facta Univ. Ser. Math. Inform.*, 36(4):855–878, 2021.

[17] Mohammad Nazrul Islam Khan. Complete lifts of a semi-symmetric metric P-connection on a Riemannian manifold to its tangent bundle. *Int. J. Math. Comput. Sci.*, 17(2):909–916, 2022.

[18] Mohammad Nazrul Islam Khan. Novel theorems for metallic structures on the frame bundle of the second order. *Filomat*, 36(13):4471–4482, 2022.

[19] Jeong-Sik Kim, Rajendra Prasad, and Mukut Mani Tripathi. On generalized Ricci-recurrent trans-Sasakian manifolds. *J. Korean Math. Soc.*, 39(6):953–961, 2002. doi:10.4134/JKMS.2002.39.6.953.

[20] Koji Matsumoto. On Lorentzian paracontact manifolds. *Bull. Yamagata Univ. Natur. Sci.*, 12(2):151–156, 1989.

[21] Koji Matsumoto and Ion Mihai. On a certain transformation in a Lorentzian para-Sasakian manifold. *Tensor (N.S.)*, 47(2):189–197, 1988.

[22] Ion Mihai and Radu Roşca. On Lorentzian P-Sasakian manifolds. In *Classical analysis (Kazimierz Dolny, 1991)*, pages 155–169. World Sci. Publ., River Edge, NJ, 1992.

[23] José A. Oubiña. New classes of almost contact metric structures. *Publ. Math. Debrecen*, 32(3-4):187–193, 1985. doi:10.5486/pmd.1985.32.3-4.07.

[24] E. Peyghan, F. Firuzi, and U. C. De. Golden Riemannian structures on the tangent bundle with g-natural metrics. *Filomat*, 33(8):2543–2554, 2019. doi:10.2298/fil1908543p.

[25] Shûkichi Tanno. The automorphism groups of almost contact Riemannian manifolds. *Tohoku Math. J. (2)*, 21:21–38, 1969. doi:10.2748/tmj/1178243031.

[26] M. Tarafdar and A. Bhattacharyya. On Lorentzian para-Sasakian manifolds. In *Steps in differential geometry (Debrecen, 2000)*, pages 343–348. Inst. Math. Inform., Debrecen, 2001.

[27] Mukut Mani Tripathi. Trans-Sasakian manifolds are generalised quasi-Sasakian. *Nepali Math. Sci. Rep.*, 18(1-2):11–14, 1999/00.

[28] Lieven Vanhecke and Dirk Janssens. Almost contact structures and curvature tensors. *Kodai Math. J.*, 4(1):1–27, 1981. URL: http://projecteuclid.org/euclid.kmj/1138036310.

[29] Kentaro Yano and Shigeru Ishihara. *Tangent and cotangent bundles: differential geometry.* Pure and Applied Mathematics, No. 16. Marcel Dekker, Inc., New York, 1973.

[30] Kentaro Yano and Sumio Sawaki. Riemannian manifolds admitting a conformal transformation group. *J. Differential Geometry*, 2:161–184, 1968. URL: http://projecteuclid.org/euclid.jdg/1214428253.

[31] Ahmet Yildiz and Cengizhan Murathan. On Lorentzian α-Sasakian manifolds. *Kyungpook Math. J.*, 45(1):95–103, 2005.

[32] Halil İbrahim Yoldaş. Some results on α-cosympletic manifolds. *Bull. Transilv. Univ. Braşov Ser. III. Math. Comput. Sci.*, 1(63)(2):115–128, 2021. doi:10.31926/but.mif.

[33] Halil İbrahim Yoldaş. Some classes of Ricci solitons on Lorentzian α-Sasakian manifolds. *Differ. Geom. Dyn. Syst.*, 24:232–244, 2022. doi:10.3390/e24020244.

[34] Halil İbrahim Yoldaş and Erol Yaşar. Some notes on Kenmotsu manifold. *Facta Univ. Ser. Math. Inform.*, 35(4):949–961, 2020. doi:10.22190/fumi2004949y.

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