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Basic Chen inequalities for statistical submanifolds of cosymplectic statistical space forms

Zamrooda Jabeen[1], Mehraj Ahmad Lone[2][3] and Mohamd Saleem Lone[4]

Abstract. In this paper, we obtain the basic Chen inequalities for the statistical submanifolds of cosymplectic statistical space form. Moreover, we discuss some special cases and equality cases.

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1. Introduction

The notion of statistical manifolds was introduced by Amari [2] in 1985. Statistical manifolds act as a differential geometric analogue of the statistical model in the sense that the sample space of density functions is replaced by a Riemannian manifold and a Fisher information matrix is replaced by Riemannian metric. A statistical manifold arises with a pair of dual torsion-free affine connections, an analogue to conjugate connections in affine geometry (see [10]).

A general issue with the statistical manifolds is that the dual connections are not metric. So using the standard notions of Riemannian geometry, geometers were not able to define precisely sectional curvature with respect to the dual connection involved. However, in 2016, Opozda [11] discovered a way to define the sectional curvature on a statistical manifold.

One of the basic interests in Riemannian geometry as well as in physics is the study of curvature invariants. The sectional curvature and the Ricci curvature are the most such curvature invariants studied from the ages of Gauss and Riemann. While studying the submanifold theory, one of the important question asked is: what are the optimal relationships between the intrinsic and the extrinsic invariants? For example, with respect to a surface, we have the famous Euler inequality: $G \leq |H|^2$, where H is the mean curvature and G is the Gaussian curvature, with the equality at umbilical points. Chen [4] extended this inequality for submanifolds of general real space forms. However, the known curvature invariants (Ricci and scalar) didn't happen to explore the sufficient

[1]Department of Mathematics, National Institute of Technology Srinagar, J& K, India
e-mail: zjabeen19@gmail.com, ORCID iD: orcid.org/0000-0003-2548-9079
[2]Department of Mathematics, National Institute of Technology Srinagar, J& K, India
e-mail: mehraj.jmi@gmail.com, ORCID iD: orcid.org/0000-0002-4764-922479
[3]Corresponding author
[4]Department of Mathematics, University of Kashmir, Srinagar, J& K, India
e-mail: saleemraja2008@gmail.com, ORCID iD: orcid.org/0000-0001-5833-3594

relationships between the intrinsic and extrinsic invariants of submanifolds. In this connection, Chen discovered new type of curvature invariants known as δ-invariants (cf. [5] for more details). With this notion, Chen was able to define a new family of curvature invariants. For example, the first Chen invariant is given by $\delta_M = \tau - \inf K$, where τ and K are the scalar and sectional curvatures of a submanifold, respectively.

In recent years, the statistical submanifolds of statistical manifolds are being examined by various geometers very actively and some interesting results have been obtained ([1, 3, 7, 9, 12, 13, 14]).

In [3], Aydin *et al.* obtained certain inequalities for the scalar and Ricci curvature of statistical submanifolds of constant curvature. Recently, Chen *et al.* [6] derived a Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature. In [8], Malek and Akbari obtained a couple of inequalities with Casorati curvature for statistical submanifolds of cosymplectic statistical space form with constant ϕ-sectional curvature.

The above studies serve as the main motivation for our study. The structure of this paper is as follows. In the second section, we collect all definitions and lemmas which are helpful to prove the main results of the paper and in the last two sections, we discuss our main results.

2. Preliminaries

Let $(\hat{M}, \hat{g})$ be a Riemannian manifold with a pair of torsion-free affine connections $\hat{\nabla}$ and $\hat{\nabla}^*$. Then $(\hat{\nabla}, \hat{g})$ is called statistical structure on $(\hat{M}, \hat{g})$ if

$$(\hat{\nabla}_Z \hat{g})(X, Y) - (\hat{\nabla}_Z \hat{g})(Z, Y) = 0,$$

for $X, Y, Z \in T\hat{M}$. A Riemannian manifold $(\hat{M}, \hat{g})$ with statistical structure satisfying the compatibility condition

$$Z\hat{g}(X, Y) = \hat{g}(\hat{\nabla}_Z X, Y) + \hat{g}(X, \hat{\nabla}^*_Z Y)$$

is said to be a statistical manifold and is denoted as $(\hat{M}, \hat{g}, \hat{\nabla}, \hat{\nabla}^*)$. Any torsion-free connection $\hat{\nabla}$ has a dual connection $\hat{\nabla}^*$ satisfying

$$\hat{\nabla}^\circ = \frac{\hat{\nabla} + \hat{\nabla}^*}{2},$$

where $\hat{\nabla}^\circ$ is the Levi-Civita connection on $\hat{M}$.

The curvature tensor fields with respect to dual connections $\hat{\nabla}$ and $\hat{\nabla}^*$ are denoted by $\hat{R}$ and $\hat{R}^*$. The curvature tensor field $\hat{R}^\circ$ associated with the $\hat{\nabla}^\circ$ is called Riemannian curvature tensor.

In case of curvature tensor, a notable difference here is that while defining the curvature tensor, there is a correction term in the form of R° contributed by the Levi-Civita connection ∇°.

Let $\hat{M}$ be a statistical manifold then the sectional curvature is defined as [11]

$$K(\pi) = \frac{1}{2}\Big[\hat{R}(X, Y)Z + \hat{R}^*(X, Y)Z - 2\hat{R}^\circ(X, Y)Z\Big]. \tag{2.1}$$

The curvature tensors $\hat{R}$ and $\hat{R}^*$ satisfy the following property

$$\hat{g}(\hat{R}(X,Y)Y,W) = -\hat{g}(\hat{R}^*(X,Y)W,Z).$$

Let $(\hat{M},\hat{g},\hat{\nabla})$ be a statistical manifold, then the difference $(1,2)$ -tensor of the torsion free connection $\hat{\nabla}$ and the Levi-Civita connection ∇° is defined as [7]:

$$\mathcal{K}_X Y = \mathcal{K}(X,Y) = \hat{\nabla}_X Y - \hat{\nabla}^\circ_X Y.$$

$\mathcal{K}$ is a symmetric tensor field on $\hat{M}$, i.e., $\mathcal{K}_X Y = \mathcal{K}_Y X$ and

$$\hat{g}(\mathcal{K}_X Y, Z) = \hat{g}(Y, \mathcal{K}_X Z).$$

Next, we define the compatibility condition for a statistical cosymplectic manifold.

Definition 2.1. Let $(\hat{M},\hat{g},\hat{\nabla})$ be a $(2n+1)$-dimensional statistical manifold, the structure $(\hat{\nabla},\hat{g},\phi,\xi,\eta)$ is called a cosymplectic structure on $\hat{M}$, if:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(X) = 1, \quad \phi\xi = 0, \quad \hat{g}(\phi X, Y) = -\hat{g}(X,\phi Y),$$
$$\hat{\nabla}^0_X \phi = 0,$$

that is $(\hat{g},\phi,\xi,\eta)$ is a cosymplectic structure on $\hat{M}$ and $\mathcal{K}_X\phi Y + \phi\mathcal{K}_X Y = 0$.

Definition 2.2. A Riemannian metric $\hat{g}$ is said to be compatible with the structure (ϕ,ξ,η) if $\hat{g}(\phi X,\phi Y) = \hat{g}(X,Y) - \eta(X)\eta(Y)$, or equivalently $\hat{g}(X,\phi Y) = -\hat{g}(\phi Y, X)$ and $\hat{g}(X,\xi) = \eta(X)$, for all $X,Y \in \Gamma(T\hat{M})$.

Let M be a $(m+1)$-dimensional submanifold of a $(2n+1)$-dimensional cosymplectic manfiold $\hat{M}$, we can write:

$$\phi X = PX + FX,$$

where PX is the tangent part of ϕX and FX is the normal part of ϕX. P is an endomorphism of the tangent bundle and F is a normal bundle valued 1-form of the tangent bundle of M, satisfying

$$g(X,PY) = -g(PX,Y)$$

for all $X,Y \in \Gamma(TM)$. Also, for a plane section $\pi \in T_pM$ at a point $p \in M$,

$$a(\pi) = g^2(e_1, Pe_2), \quad b(\pi) = (\eta(e_1))^2 + (\eta(e_2))^2 \tag{2.2}$$

independent of the orthonormal basis $\{e_1,e_2\}$ of the plane section π.

Definition 2.3. Let $(\hat{M},\hat{\nabla},\hat{g},\phi,\xi)$ be a cosymplectic statistical manifold and c be a real constant, then cosymplectic statistical structure is said to be of constant ϕ-sectional curvature if

$$\begin{aligned}\hat{R}(X,Y)Z \;=\; & \frac{c}{4}\big[\hat{g}(Y,Z)X - \hat{g}(X,Z)Y + \hat{g}(X,\phi Z)\phi Y \\ & -\hat{g}(Y,\phi Z)\phi X + 2\hat{g}(X,\phi Y)\phi Z + \eta(X)\eta(Z)Y \\ & -\eta(Y)\eta(Z)X + \hat{g}(X,Z)\eta(Y)\xi - \hat{g}(Y,Z)\eta(X)\xi\big]\end{aligned} \tag{2.3}$$

for all $X,Y,Z \in \Gamma(T\hat{M})$.

Let (M, g, ∇, ∇^*) be statistical submanifold of $(\hat{M}, \hat{g}, \hat{\nabla}, \hat{\nabla}^*)$. The Gauss and Weingarten formulae are given as

$$\hat{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \hat{\nabla}_X x = -A_x X + \nabla^{\perp}_X x$$
$$\hat{\nabla}^*_X Y = \nabla^*_X Y + \sigma^*(X,Y), \quad \hat{\nabla}^*_X x = -A^*_x X + \nabla^{*\perp}_X x$$

for all $X, Y \in TM$ and $x \in T^{\perp}M$ respectively. Moreover, we have the following equations

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$$
$$\hat{g}(\sigma(X,Y), x) = g(A^*_x X, Y), \quad \hat{g}(\sigma^*(X,Y), x) = g(A_x X, Y)$$
$$X\hat{g}(x,y) = \hat{g}(\nabla^{\perp}_X x, y) + \hat{g}(x, \nabla^{*\perp}_X y),$$

where $x, y \in T^{\perp}M$. The mean curvature vector fields for orthonormal tangent and normal frames $\{e_1, e_2, \dots, e_{m+1}\}$ and $\{e_{m+2}, e_{m+3}, \dots, e_{2n+1}\}$, respectively, are defined as

$$H = \frac{1}{m+1}\sum_{i=1}^{m+1}\sigma(e_i,e_i) = \frac{1}{m+1}\sum_{\gamma=1}^{2n+1}\left(\sum_{i=1}^{m+1}\sigma^{\gamma}_{ii}\right)\xi_\gamma, \ \sigma^{\gamma}_{ij} = g(\sigma(e_i,e_j), e_\gamma)$$

and

$$H^* = \frac{1}{m+1}\sum_{i=1}^{m+1}\sigma^*(e_i,e_i) = \frac{1}{m+1}\sum_{\gamma=1}^{2n+1}\left(\sum_{i=1}^{2n+1}\sigma^{*\gamma}_{ii}\right)\xi_\gamma, \ \sigma^{*\gamma}_{ij} = g(\sigma^*(e_i,e_j), e_\gamma)$$

for $1 \leq i, j \leq m+1$ and $1 \leq \gamma \leq 2n+1$.

Now, we state the following fundamental results on statistical manifolds.

Proposition 2.4. *Let (M, g, ∇, ∇^*) be statistical submanifold of $(\hat{M}, \hat{g}, \hat{\nabla}, \hat{\nabla}^*)$. Let $\hat{R}$ and $\hat{R}^*$ be the Riemannian curvature tensors on $\hat{M}$ for $\hat{\nabla}$ and $\hat{\nabla}^*$, respectively. Then the Gauss and Ricci equations are given by the following result.*

$$\hat{g}(\hat{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + \hat{g}(\sigma(X,Z),\sigma^*(Y,W)) - \hat{g}(\sigma^*(X,W),\sigma(Y,Z)),$$

$$\hat{g}(\hat{R}^*(X,Y)Z,W) = g(R^*(X,Y)Z,W) + \hat{g}(\sigma^*(X,Z),\sigma(Y,W)) - \hat{g}(\sigma(X,W),\sigma^*(Y,Z)),$$

$$\hat{g}(R^{\perp}(X,Y)x,y) = \hat{g}(\hat{R}(X,Y)x,y) + g([A^*_x, A_y]X, Y),$$

$$\hat{g}(R^{*\perp}(X,Y)x,y) = \hat{g}(\hat{R}^*(X,Y)x,y) + g([A_x, A^*_y]X, Y),$$

where $[A_x, A^*_y] = A_x A^*_y - A^*_y A_x$ and $[A^*_x, A_y] = A^*_x A_y - A_y A^*_x$, for $X, Y, Z, W \in TM$ and $x, y \in T^{\perp}M$.

Now, we state two important lemmas which we use to prove the main results in the upcoming sections.

Lemma 2.5. *[6] Let $n \geq 3$ be an integer and $a_1, a_2, \ldots, a_n$ are n real numbers. Then, we have*

$$\sum_{1\leq i<j\leq n}^{n} a_i a_j - a_1 a_2 \leq \frac{n-2}{2(n-1)}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Moreover, the equality holds if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Lemma 2.6. *[9] Let $n > 4$ be an integer and $a_1, a_2, \ldots, a_n$ are n real numbers. Then, we have*

$$\sum_{1\leq i<j\leq n}^{n} a_i a_j - a_1 a_2 - a_3 a_4 \leq \frac{n-3}{2(n-2)}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Moreover, the equality holds if and only if $a_1 + a_2 = a_3 + a_4 = a_5 = \cdots = a_n$.

3. Main result

In the present section, we obtain first Chen inequality for the submanifolds of cosympletic statistical manifolds.

Theorem 3.1. *Let M^{m+1} be statistical submanifold of a cosymplectic statistical manifold $\hat{M}^{2n+1}(c)$ of constant ϕ-sectional curvature c, such that the structure vector field ξ is tangent to M. Then for any point $p \in M^{n+1}$ and any plane section π at p, we have*

$$\begin{aligned}\tau_0 - K_0(\pi) \quad &\leq \quad \tau - K(\pi) + \frac{(m+1)^2(m-1)}{2m}(\|H\|^2 + \|H^*\|^2) \\ &\quad -\frac{c}{8}[m^2 - m + 3\|P\|^2 + 2(b(\pi) - 3a(\pi) - 1)] \\ &\quad -2\hat{K}_0(\pi) + 2\hat{\tau}_0(\pi).\end{aligned}$$

Moreover, the equalities hold for any $\gamma \in \{m+2, \ldots, 2n+1\}$ if and only if

$$\sigma^{\gamma}_{11} + \sigma^{\gamma}_{22} = \sigma^{\gamma}_{33} = \cdots = \sigma^{\gamma}_{(m+1)(m+1)}$$

$$\sigma^{*\gamma}_{11} + \sigma^{*\gamma}_{22} = \sigma^{*\gamma}_{33} = \cdots = \sigma^{*\gamma}_{(m+1)(m+1)}$$

$$\sigma^{\gamma}_{ij} = \sigma^{*\gamma}_{ij} = 0 \qquad \forall 1 \leq i \neq j \leq m+1.$$

Proof. Let $p \in M$ and $\{e_1, e_2, \cdots, e_{m+1}\}$ and $\{e_{m+2}, \cdots, e_{2n+1}\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$, respectively. The scalar curvature corresponding to the sectional K-curvature is

$$\tau = \frac{1}{2}\sum_{1\leq i<j\leq m+1}\Big[g(R(e_i, e_j)e_j), e_i) + g(R^*(e_i, e_j)e_j), e_i) - 2g(R^{\circ}(e_i, e_j)e_j), e_i)\Big].$$

Using (2.3) and Gauss equation for R and R^*, we obtain
(3.1)
$$\tau = \frac{c}{8}\Big[m(m-1)+3\|P\|^2\Big]-\tau_0+\frac{1}{2}\sum_{\gamma=m+2}^{2n+1}\sum_{1\le i<j\le m+1}\Big[\sigma_{ii}^{*\gamma}\sigma_{jj}^{\gamma}+\sigma_{ii}^{\gamma}\sigma_{jj}^{*\gamma}-2\sigma_{ij}^{*\gamma}\sigma_{ij}^{\gamma}\Big].$$

The above equality can also be written as

$$\begin{aligned}\tau &= \frac{c}{8}\Big[m(m-1)+3\|P\|^2\Big]+2\sum_{\gamma=m+2}^{2n+1}\sum_{1\le i<j\le m+1}\Big[\sigma_{ii}^{0\gamma}\sigma_{jj}^{0\gamma}-(\sigma_{ij}^{0\gamma})^2\Big]\\ &\quad -\frac{1}{2}\sum_{\gamma=m+2}^{2n+1}\sum_{1\le i<j\le m+1}\Big[\sigma_{ii}^{\gamma}\sigma_{jj}^{\gamma}-(\sigma_{ij}^{\gamma})^2\Big]\\ &\quad -\frac{1}{2}\sum_{\gamma=m+2}^{2n+1}\sum_{1\le i<j\le m+1}\Big[\sigma_{ii}^{*\gamma}\sigma_{jj}^{*\gamma}-(\sigma_{ij}^{*\gamma})^2\Big]-\tau_0.\end{aligned}$$

Also using the Gauss equation for Levi-Civita connection, we have

$$\begin{aligned}\tau &= \tau_0+\frac{c}{8}\Big[m(m-1)+3\|P\|^2\Big]-\frac{1}{2}\sum_{\gamma=m+2}^{2n+1}\sum_{1\le i<j\le m+1}\Big[\sigma_{ii}^{\gamma}\sigma_{jj}^{\gamma}-(\sigma_{ij}^{\gamma})^2\Big]\\ &\quad -\frac{1}{2}\sum_{\gamma=m+2}^{2n+1}\sum_{1\le i<j\le m+1}\Big[\sigma_{ii}^{*\gamma}\sigma_{jj}^{*\gamma}-(\sigma_{ij}^{*\gamma})^2\Big]-2\hat{\tau}_0. \qquad (3.2)\end{aligned}$$

Next, we calculate the sectional curvature $K(\pi)$ of the plane section π as

$$K(\pi)=\frac{1}{2}\Big[g(R(e_1,e_2)e_2,e_1)+g(R^*(e_1,e_2)e_2,e_1)-2g(R^\circ(e_1,e_2)e_2,e_1)\Big].$$

Using (2.3) and Gauss equation, we obtain

$$\begin{aligned}K(\pi) &= \frac{c}{4}[1+3a(\pi)-b(\pi)]+\frac{1}{2}[g(\sigma^*(e_1,e_1),\sigma(e_2,e_2))\\ &\quad +g(\sigma(e_1,e_1),\sigma^*(e_2,e_2))-2g(\sigma(e_1,e_2),\sigma^*(e_1,e_2))]-K_0(\pi)\\ &= \frac{c}{4}[1+3a(\pi)-b(\pi)]+\frac{1}{2}\sum_{\gamma=m+2}^{2n+1}\Big[\sigma_{11}^{*\gamma}\sigma_{22}^{\gamma}+\sigma_{11}^{\gamma}\sigma_{22}^{*\gamma}-2\sigma_{12}^{*\gamma}\sigma_{12}^{\gamma}\Big]\\ &\quad -K_0(\pi).\\ &= \frac{c}{4}[1+3a(\pi)-b(\pi)]+\sum_{\gamma=m+2}^{2n+1}\Big[2\{\sigma_{11}^{0\gamma}\sigma_{22}^{0\gamma}-(\sigma_{12}^{0\gamma})^2\}\\ &\quad -\frac{1}{2}\{\sigma_{11}^{\gamma}\sigma_{22}^{\gamma}-(\sigma_{12}^{\gamma})^2\}-\frac{1}{2}\{\sigma_{11}^{*\gamma}\sigma_{22}^{*\gamma}-(\sigma_{12}^{*\gamma})^2\}\Big]-K_0(\pi).\end{aligned}$$

Using the Gauss equation for the Levi-Civita connection, we get

$$K(\pi) = K_0(\pi) + \frac{c}{4}[1 + 3a(\pi) - b(\pi)] - \sum_{\gamma=m+2}^{2n+1} \Big[\frac{1}{2}\{\sigma_{11}^{\gamma}\sigma_{22}^{\gamma} - (\sigma_{12}^{\gamma})^2\}$$

$$-\frac{1}{2}\{\sigma_{11}^{*\gamma}\sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2\}\Big] - 2\hat{K}_0(\pi). \tag{3.3}$$

Subtracting (3.3) from (3.2), we get

$$(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) = \frac{c}{8}[m^2 - m + 3\|P\|^2 + 2(b(\pi) - 3a(\pi) - 1)]$$

$$-\frac{1}{2}\sum_{\gamma=m+2}^{2n+1}\sum_{1\le i<j\le m+1}\Big\{\Big[\sigma_{ii}^{\gamma}\sigma_{jj}^{\gamma} - (\sigma_{ij}^{\gamma})^2\Big] - \frac{1}{2}\Big[\sigma_{ii}^{*\gamma}\sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\Big]\Big\}$$

$$+\sum_{\gamma=m+2}^{2n+1}\Big[\frac{1}{2}\{\sigma_{11}^{\gamma}\sigma_{22}^{\gamma} - (\sigma_{12}^{\gamma})^2\} + \frac{1}{2}\{\sigma_{11}^{*\gamma}\sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2\}\Big]$$

$$+2\hat{K}_0(\pi) - 2\hat{\tau}_0(\pi). \tag{3.4}$$

Using Lemma2.5, we get

$$\begin{aligned}(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) \ \ge\ & \frac{c}{8}[m^2 - m + 3\|P\|^2 + 2(b(\pi) - 3a(\pi) - 1)] \\ & -\frac{(m+1)^2(m-1)}{2m}(\|H\|^2 + \|H^*\|^2) \\ & +2\hat{K}_0(\pi) - 2\hat{\tau}_0(\pi).\end{aligned}$$

This proves our claim. □

Corollary 3.2. *Let M^{m+1} be statistical submanifold of a cosymplectic statistical manifold $\hat{M}^{2n+1}(c)$ of constant ϕ-sectional curvature c, such that the structure vector field ξ is tangent to M. If a point $p \in M^{n+1}$ and a plane section π at p satisfy that*

$$\begin{aligned}(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) \ <\ & \frac{c}{8}[m^2 - m + 3\|P\|^2 + 2(b(\pi) - 3a(\pi) - 1)] \\ & +2\hat{K}_0(\pi) - 2\hat{\tau}_0(\pi),\end{aligned}$$

then M^{m+1} is non-minimal in $\hat{M}^{2n+1}(c)$.

4. $\delta(2,2)$ Inequality

Let $\pi_1, \pi_2 \subset T_pM$ be two mutually orthogonal planes spanned by $\{e_1, e_2\}$ and $\{e_3, e_4\}$, respectively. Also let $\{e_1, e_2, \cdots, e_{m+1}\}$ and $\{e_{m+2}, e_{m+3}, \cdots, e_{2n+1}\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$, respectively. Using the Gauss equation for the Levi-Civita connection, the sectional curvatures $K(\pi_1)$ and $K(\pi_2)$ of the plane sections π_1 and π_2 are given by

$$\begin{aligned} K(\pi_1) &= K_0(\pi_1) + \frac{c}{4}[1 + 3a(\pi_1) - b(\pi_1)] - \sum_{\gamma=m+2}^{2n+1} \Big[\frac{1}{2}\{\sigma_{11}^{\gamma}\sigma_{22}^{\gamma} - (\sigma_{12}^{\gamma})^2\} \\ & -\frac{1}{2}\{\sigma_{11}^{*\gamma}\sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2\}\Big] - 2\hat{K}_0(\pi_1), \end{aligned}$$

$$\begin{aligned} K(\pi_2) &= K_0(\pi_2) + \frac{c}{4}[1 + 3a(\pi_2) - b(\pi_2)] - \sum_{\gamma=m+2}^{2n+1} \Big[\frac{1}{2}\{\sigma_{33}^{\gamma}\sigma_{44}^{\gamma} - (\sigma_{34}^{\gamma})^2\} \\ & -\frac{1}{2}\{\sigma_{33}^{*\gamma}\sigma_{44}^{*\gamma} - (\sigma_{34}^{*\gamma})^2\}\Big] - 2\hat{K}_0(\pi_2). \end{aligned}$$

Using the above two relations and by doing simple calculations and using Lemma 2.6, we obtain the following inequality

$$\begin{aligned} &(\tau - K(\pi_1) - K(\pi_2)) - (\tau_0 - K_0(\pi_1) - K_0(\pi_2)) \geq \frac{c}{8}[m^2 - m - 4 + 3\|P\|^2] \\ &+\frac{c}{4}[b(\pi_1) + b(\pi_2) - 3(a(\pi_1) + a(\pi_2))] - \frac{(m+1)^2(m-2)}{2(m-1)}(\|H\|^2 + \|H^*\|^2) \\ &+2\hat{K}_0(\pi_1) + 2\hat{K}_0(\pi_2) - 2\hat{\tau}_0. \end{aligned}$$

Theorem 4.1. *Let M^{m+1} be statistical submanifold of a cosymplectic statistical manifold $\hat{M}^{2n+1}(c)$ of constant ϕ-sectional curvature c, such that the structure vector field ξ is tangent to M. Then for any point $p \in M^{n+1}$ and any two plane sections π_1 and π_2 at p, we have*

$$\begin{aligned} &(\tau_0 - K_0(\pi_1) - K_0(\pi_2)) \leq (\tau - K(\pi_1) - K(\pi_2)) - \frac{c}{8}[m^2 - m - 4 + 3\|P\|^2] \\ &-\frac{c}{4}[b(\pi_1) + b(\pi_2) - 3(a(\pi_1) + a(\pi_2))] + \frac{(m+1)^2(m-2)}{2(m-1)}(\|H\|^2 + \|H^*\|^2) \\ &-2\hat{K}_0(\pi_1) - 2\hat{K}_0(\pi_2) + 2\hat{\tau}_0. \end{aligned}$$

Moreover, the equalities holds for any $\gamma \in \{m+2, n+3, \ldots, 2n+1\}$ if and only if

$$\begin{aligned} &\sigma_{11}^{\gamma} + \sigma_{22}^{\gamma} = \sigma_{33}^{\gamma} + \sigma_{44}^{\gamma} = \sigma_{55}^{\gamma} \cdots = \sigma_{(m+1)(m+1)}^{\gamma} \\ &\sigma_{11}^{*\gamma} + \sigma_{22}^{*\gamma} = \sigma_{33}^{*\gamma} + \sigma_{44}^{*\gamma} = \sigma_{55}^{*\gamma} \cdots = \sigma_{(m+1)(m+1)}^{*\gamma} \\ &\sigma_{ij}^{\gamma} = \sigma_{ij}^{*\gamma} = 0 \ \forall\ 1 \leq i \neq j \leq m+1. \end{aligned}$$

Corollary 4.2. *Let M^{m+1} be statistical submanifold of a cosymplectic statistical manifold $\hat{M}^{2n+1}(c)$ of constant ϕ-sectional curvature c, such that the structure vector field ξ is tangent to M. If a point $p \in M^{n+1}$ and two plane sections π_1 and π_2 at p satisfy that*

$$\begin{aligned} (\tau - K(\pi_1) - K(\pi_2)) &< (\tau_0 - K_0(\pi_1) - K_0(\pi_2)) + \frac{c}{8}[m^2 - m - 4 + 3\|P\|^2] \\ &+\frac{c}{4}[b(\pi_1) + b(\pi_2) - 3(a(\pi_1) + a(\pi_2))] \\ &+2\hat{K}_0(\pi_1) + 2\hat{K}_0(\pi_2) - 2\hat{\tau}_0(\pi), \end{aligned}$$

then M^{m+1} *is non-minimal in* $\hat{M}^{2n+1}(c)$.

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