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On the Minkowski fractional integral inequality using k-Hilfer the fractional derivative

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Abstract. In this paper, we first introduce the k-Hilfer fractional derivative operator. Based on these, we obtain the reverse Minkowski fractional integral and some other fractional inequalities. Moreover, we discuss a fractional integral inequality that is connected to the Minkowski inequality by employing k-Hilfer fractional derivative operator. These studies may motivate further research in a variety of disciplines of pure and applied science.

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1. Introduction

The study of fractional order integral and derivative operators is called fractional calculus. Due to its many applications in the fields of science, finance, and biotechnology, fractional calculus is immensely significant. This topic is just as crucial as calculus itself, and it has been very influential in recent years, see [2, 4, 5, 10, 15, 17, 18, 20, 27]. The ability to solve fractional partial differential equations, fractional boundary value issues, and fractional ordinary differential equations heavily depends on the use of fractional integral inequalities. Additionally, they provide upper and lower bounds for solutions to fractional boundary value problems. Using the Riemann-Liouville fractional integral, Dahmani examined the reverse Minkowski fractional integral inequality in [9]. Nale et al.[22] obtained some Minkowski type fractional integral inequalities by considering a generalized proportional Hadamard fractional integral operator. The fractional integral inequalities presented by Ahmed Anber et al. using the Riemann-Liouville fractional integral are comparable to those found in the Minkowski fractional integral inequality [3]. In [6, 8, 24, 25, 26] the authors obtained the Minkowski inequality and some other fractional inequalities for convex functions by employing Saigo and fractional proportional

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integral operators. In [7], Chinchane studied the reverse Minkowski fractional integral inequality by considering generalized k-fractional integral operator in terms of the Gauss hypergeometric function. Mubeen et al. [21] have introduced the Minkowski inequality involving generalized k-fractional conformable integrals. Moreover, the several kind of fractional integral and derivative operators have been introduced. Some differential inequalities, such as those of the Opial type involving the fractional derivatives of two functions and those involving the Caputo K-fractional derivative, have been demonstrated by a number of researchers in the field of fractional differential inequalities. [13]. Recently, Iqbal et al. [14] used the k-Hilfer fractional derivative operator to achieve the Grüss inequality. Our work is strongly inspired by the references [1, 14, 16, 19, 22, 23]. Our main aim of this study is to prove a new the Minkowski inequality using the k-Hilfer fractional derivative and Young's inequality.

2. Preliminaries

In [11], the definition of the gamma k-function was given by Diaz et al. and is as follows:

Definition 2.1. The Γ_k function is the generalization of the classical Γ function and is defined as follows:

$$\Gamma_k(\varkappa) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{\varkappa}{k}-1}}{(\varkappa)_{n,k}}, k > 0, \Re(\varkappa) > 0,$$

where $(\varkappa)_{n,k} = \varkappa(\varkappa+k)(\varkappa+2k)..., (\varkappa+(n-1)k), n \geq 1$, is called Pochhammer k symbol. The integral representation is provided by

$$\Gamma_k(\varkappa) = \int_0^\infty t^{\varkappa-1}e\frac{-t^k}{k}dt, \Re(\varkappa) > 0. \tag{2.1}$$

Specially for $k = 1$, $\Gamma_1(\varkappa) = \Gamma(\varkappa)$.

The set of complex-valued Lebesgue measurable functions f such that $|f|^2$ is integrable on a finite or infinite interval of the real number set $\mathbb{R}$ is denoted by the symbol $L^p[a,b]$. We denote by $AC^n[a,b]$ the space of complex-valued functions f which have continuous derivatives up to order $(n-1)$ on $[a,b]$ such that $f^{(n-1)}$ belongs to $AC[a,b]$.
The following definition is provided in the citation [12].

Definition 2.2. Let $f \in L^1[a,b]$, $k > 0$, $f * K_{(1-\eta)(1-\xi)} \in AC^1[a,b]$. The k-fractional derivative operator ${}^k\mathcal{D}^{\xi,\eta}_{a+}f$ of order $0 < \xi < 1$ and type $0 < \eta \leq 1$ with respect to $\varkappa \in [a,b]$ is defined by

$$({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa) := I^{\eta(1-\xi)}_{a+,k}\frac{d}{d\varkappa}(I^{(1-\eta)(1-\xi)}_{a+,k}f(\varkappa)), \tag{2.2}$$

whenever the right hand side exists. The derivative (2.2) is usually called k-Hilfer fractional derivative. A more general representation as in equation (2.2) reads like this:

Definition 2.3. Let $f \in L^1[a,b]$, $f * K_{(1-\eta)(n-\xi)} \in AC^n[a,b]$, $n-1<\xi<n$, $0<\eta\leq 1$, $n \in \mathbb{N}$ then the following equation holds true:

$$({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa) = (I^{\eta(\eta-\xi)}_{a+,k}\frac{d^n}{d\varkappa^n}(I^{(1-\eta)(n-\xi)}_{a+,k}f(\varkappa))). \tag{2.3}$$

The relation (2.3) can be expressed in the following way using the Riemann-Liouvile fractional integral properties:

$$\begin{aligned}({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa) &= (I^{\eta(n-\xi)}_{a+,k}(D^{n-(1-\eta)(n-\xi)}_{a+,k}f(\varkappa))) \\ &= \frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-y)^{\frac{\eta(n-\xi)}{k}-1}((D^{\xi+\eta(n-\xi)}_{a+,k}f)(y))dy.\end{aligned} \tag{2.4}$$

We generate many classical fractional derivatives from the derivative (2.3) as special cases by setting
(i) $k=1$, we get Hilfer fractional derivative presented in [13],
(ii)$k=1, \eta=0, D^{\xi,0}_{a+}f = D^{\xi}_{a+}f$, we arrive at Riemann-Liouvile fractional derivative of order ξ given in [27],
(iii)$k=1, \eta=1, n=1$ is a Caputo fractional derivative of order ξ provided in [19].

3. A Reverse Minkowski fractional integral inequality

Here, we prove the k-Hilfer fractional derivative-based reverse Minkowski fractional integral inequality.

Theorem 3.1. *Let $k>0$, $p\geq 1$ and $({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)$ denote the k-Hilfer fractional derivative of order ξ, $0<\xi<1$, and type $0<\eta\leq 1$. Suppose that $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f)(\varkappa)$, $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g)(\varkappa)$ are two integrable functions on $[0,\infty)$, such that $({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)[f^p(\varkappa)]<\infty$, $({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)[g^p(\varkappa)]<\infty$. If $0<\mathfrak{m}\leq \frac{{}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau)}{{}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\tau)} \leq \mathcal{M}$, $\tau \in (a,\varkappa)$, we have*

$$\begin{aligned}&\left[{}^k\mathcal{D}^{\xi,\eta}_{a+}(\varkappa)[f^p(\varkappa)]\right]^{\frac{1}{p}} + \left[{}^k\mathcal{D}^{\xi,\eta}_{a+}(\varkappa)[g^q(\varkappa)]\right]^{\frac{1}{p}} \\ &\leq \frac{1+\mathcal{M}(\mathfrak{m}+2)}{(\mathfrak{m}+1)(\mathcal{M}+1)}\left[{}^k\mathcal{D}^{\xi,\eta}_{a+}(x)[(f+g)^p(\varkappa)]\right]^{\frac{1}{p}}.\end{aligned} \tag{3.1}$$

Proof. Using the condition $\frac{{}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau)}{{}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\tau)} \leq \mathcal{M}$, $\tau\in(a,\varkappa)$, we can write

$$(\mathcal{M}+1)^p\left({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f\right)(\tau) \leq \mathcal{M}^p\left({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}(f+g)^p\right)(\tau). \tag{3.2}$$

Multiplying both sides of (3.2) by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating the resulting identity with respect to τ from a to $\varkappa$, we get

$$\begin{aligned}&\frac{(\mathcal{M}+1)^p}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f)(\tau)d\tau \\ &\leq \frac{\mathcal{M}^p}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}\left({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}(f+g)^p\right)(\tau)d\tau,\end{aligned} \tag{3.3}$$

which is equivalent to

$$ {}^{k}\mathcal{D}_{a+}^{\xi,\eta} f^{p}(\varkappa) \leq \frac{\mathcal{M}^{p}}{(\mathcal{M}+1)^{p}} \left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f+g)^{p}(\varkappa)] \right], \tag{3.4} $$

as a result, we may write

$$ \left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta} f^{p}(\varkappa) \right]^{\frac{1}{p}} \leq \frac{\mathcal{M}}{(\mathcal{M}+1)} \left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f+g)^{p}(\varkappa)] \right]^{\frac{1}{p}}. \tag{3.5} $$

On other hand, using condition $\mathfrak{m} \leq \frac{({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} f)(\tau)}{({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} g)(\tau)}$, we obtain

$$ \left(1+\frac{1}{\mathfrak{m}}\right)({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} g)(\tau) \leq \frac{1}{\mathfrak{m}}\Big(({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} f)(\tau) + ({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} g)(\tau)\Big), \tag{3.6} $$

therefore,

$$ \left(1+\frac{1}{\mathfrak{m}}\right)^{p}({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} g)^{p}(\tau) \leq \left(\frac{1}{\mathfrak{m}}\right)^{p}\Big(({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} f)(\tau) + ({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} g)(\tau)\Big)^{p}. \tag{3.7} $$

Now, multiplying both sides of (3.7) by $\frac{1}{k\Gamma_{k}(\eta(n-\xi))}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating resulting the identity with respect to τ from a to $\varkappa$, we have

$$ \left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[g^{p}(\varkappa)] \right]^{\frac{1}{p}} \leq \left(\frac{1}{\mathfrak{m}+1}\right) \left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f+g)^{p}(\varkappa)] \right]^{\frac{1}{p}}. \tag{3.8} $$

The inequalities (3.1) follows by adding the inequalities (3.5) and (3.8). □

The following is our second finding.

Theorem 3.2. *Let $k>0$, $p \geq 1$ and suppose that $({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} f)$, $({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} g)$ are two positive functions on $[0,\infty)$, such that $[({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} f)^{p}(\varkappa)] < \infty$, $[({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} f)^{p}(\varkappa)] < \infty$. If $0 < \mathfrak{m} \leq \frac{({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} f)(\tau)}{({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} g)(\tau)} \leq \mathcal{M}$, then*

$$ \begin{aligned} &\left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta} f^{p}(\varkappa) \right]^{\frac{2}{p}} + \left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta} g^{q}(\varkappa) \right]^{\frac{2}{p}} \geq \left(\frac{(\mathcal{M}+1)(\mathfrak{m}+1)}{\mathcal{M}}-2\right)\times \\ &\qquad \left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta} f^{p}(\varkappa) \right]^{\frac{1}{p}} + \left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta} g^{q}(\varkappa) \right]^{\frac{1}{p}}, \end{aligned} \tag{3.9} $$

Proof. Multiplying the inequalities (3.5)and (3.8), we obtain

$$ \begin{aligned} &\left(\frac{(\mathcal{M}+1)(\mathfrak{m}+1)}{\mathcal{M}}\right) \left[({}^{k}\mathcal{D}_{a+}^{\xi,\eta} f)^{p}(\varkappa)\right]^{\frac{1}{p}} \left[[({}^{k}\mathcal{D}_{a+}^{\xi,\eta} g)^{p}(\varkappa)]\right]^{\frac{1}{p}} \\ &\leq \left[[({}^{k}\mathcal{D}_{a+}^{\xi,\eta} f)(\varkappa) + ({}^{k}\mathcal{D}_{a+}^{\xi,\eta} g)(\varkappa)^{p}]^{\frac{1}{p}}\right]^{2}. \end{aligned} \tag{3.10} $$

We applied the Minkowski inequality to the right hand side of (3.10) and the result is

$$\begin{aligned}&\left(\left[[({}^{k}\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)+({}^{k}\mathcal{D}^{\xi,\eta}_{a+}g)(\varkappa)]^{p}\right]^{\frac{1}{p}}\right)^{2}\\ &\leq\left(\left[[({}^{k}\mathcal{D}^{\xi,\eta}_{a+}f)^{p}(\varkappa)]\right]^{\frac{1}{p}}+\left[[({}^{k}\mathcal{D}^{\xi,\eta}_{a+}f)^{p}(\varkappa)]\right]^{\frac{1}{p}}\right)^{2},\end{aligned}\tag{3.11}$$

which implies that

$$\begin{aligned}\left[{}^{k}\mathcal{D}^{\xi,\eta}_{a+}(f(\varkappa)+g(\varkappa))^{p}\right]^{\frac{2}{p}}&\leq\left[{}^{k}\mathcal{D}^{\xi,\eta}_{a+}f^{p}(\varkappa)\right]^{\frac{2}{p}}+\left[{}^{k}\mathcal{D}^{\xi,\eta}_{a+}g^{p}(\varkappa)\right]^{\frac{2}{p}}\\ &\quad+2\left[{}^{k}\mathcal{D}^{\xi,\eta}_{a+}f^{p}(\varkappa)\right]^{\frac{1}{p}}\left[{}^{k}\mathcal{D}^{\xi,\eta}_{a+}g^{p}(\varkappa)\right]^{\frac{1}{p}}.\end{aligned}\tag{3.12}$$

using (3.10) and (3.12) we obtain (3.9). Theorem 3.2 is thus proved. □

4. Further fractional integral inequalities

Here, we use the k-Hilfer fractional derivative to establish a few new integral inequalities.

Theorem 4.1. *Let $k>0$, $p\geq 1$ and $({}^{k}\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)$ denote the k-Hilfer fractional derivative of order ξ, $0<\xi<1$, and type $0<\eta\leq 1$, $\frac{1}{p}+\frac{1}{q}=1$ and let $({}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f)$, $({}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g)$ be two positive function on $[0,\infty)$, such that ${}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\varkappa)<\infty$, ${}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\varkappa)<\infty$. If $0<\mathfrak{m}\leq\frac{{}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau)}{{}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\tau)}\leq\mathcal{M}<\infty$, $\tau\in(a,\varkappa)$, we have*

$$\left[{}^{k}\mathcal{D}^{\xi,\eta}_{a+}f(\varkappa)\right]^{\frac{1}{p}}\left[{}^{k}\mathcal{D}^{\xi,\eta}_{a+}g(\varkappa)\right]^{\frac{1}{q}}\leq\left(\frac{\mathcal{M}}{\mathfrak{m}}\right)^{\frac{1}{pq}}\left[{}^{k}\mathcal{D}^{\xi,\eta}_{a+}f(\varkappa)^{\frac{1}{p}}g(\varkappa)^{\frac{1}{q}}\right].\tag{4.1}$$

Proof. Since $\frac{{}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau)}{({}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\tau)}\leq\mathcal{M}$, $\tau\in(a,\varkappa)$ therefore

$$[({}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g)(\tau)]^{\frac{1}{q}}\geq\mathcal{M}^{\frac{-1}{q}}[({}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f)(\tau)]^{\frac{1}{q}},\tag{4.2}$$

and also,

$$\begin{aligned}&[{}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau)]^{\frac{1}{p}}[{}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\tau)]^{\frac{1}{q}}\\ &\qquad\geq\mathcal{M}^{\frac{-1}{q}}[{}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau)]^{\frac{1}{p}}[{}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau)]^{\frac{1}{q}}\\ &\qquad\geq\mathcal{M}^{\frac{-1}{q}}[{}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau)]^{\frac{1}{p}+\frac{1}{q}}\\ &\qquad\geq\mathcal{M}^{\frac{-1}{q}}[{}^{k}\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau)].\end{aligned}\tag{4.3}$$

Multiplying both sides of (4.3) by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating the resulting identity with respect to τ from a to $\varkappa$, we have

(4.4)
$$\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}[\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)]^{\frac{1}{p}}[\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)]^{\frac{1}{q}}d\tau$$
$$\geq \frac{\mathcal{M}^{\frac{-1}{q}}}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}[\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)d\tau,$$

which implies that

$$(4.5)\qquad {}^k\mathcal{D}_{a+}^{\xi,\eta}\left[f(\varkappa)^{\frac{1}{p}}g(\varkappa)^{\frac{1}{q}}\right]\geq \mathcal{M}^{\frac{-1}{q}}\left[{}^k\mathcal{D}_{a+}^{\xi,\eta}f(\varkappa)\right].$$

Consequently,

$$(4.6)\qquad \left({}^k\mathcal{D}_{a+}^{\xi,\eta}\left[[f(\varkappa)]^{\frac{1}{p}}[g(\varkappa)]^{\frac{1}{q}}\right]\right)^{\frac{1}{p}}\geq \mathcal{M}^{\frac{-1}{pq}}\left[{}^k\mathcal{D}_{a+}^{\xi,\eta}f(\varkappa)\right]^{\frac{1}{p}}.$$

On the other hand, since $\mathfrak{m}\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)\leq {}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)$, $\tau\in(a,\varkappa)$, then we have

$$(4.7)\qquad [(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f)(\tau)]^{\frac{1}{p}}\geq \mathfrak{m}^{\frac{1}{p}}[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)(\tau)]^{\frac{1}{p}},$$

multiplying equation (4.7) by $[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)(\tau)]^{\frac{1}{q}}$, we have

$$(4.8)\qquad \begin{aligned}&[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f)(\tau)]^{\frac{1}{p}}[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)(\tau)]^{\frac{1}{q}}\\ &\geq \mathfrak{m}^{\frac{1}{p}}[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)(\tau)]^{\frac{1}{q}}[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)(\tau)]^{\frac{1}{p}}\\ &= \mathfrak{m}^{\frac{1}{p}}[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)(\tau)].\end{aligned}$$

Multiplying both sides of (4.8) by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}$. Then integrating the resulting identity with respect to τ from a to $\varkappa$, we have

(4.9)
$$\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f)(\tau)]^{\frac{1}{p}}[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)(\tau)]^{\frac{1}{q}}(\tau)d\tau$$
$$\geq \frac{\mathfrak{m}^{\frac{1}{p}}}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}[(\,{}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)(\tau)]d\tau,$$

that is,

$$(4.10)\qquad {}^k\mathcal{D}_{a+}^{\xi,\eta}\left[f(\varkappa)^{\frac{1}{p}}[g(\varkappa)]^{\frac{1}{q}}\right]\geq \mathfrak{m}^{\frac{1}{p}}\left[{}^k\mathcal{D}_{a+}^{\xi,\eta}g(\varkappa)\right].$$

Hence we can write,

$$(4.11)\qquad \left({}^k\mathcal{D}_{a+}^{\xi,\eta}\left[f(\varkappa)^{\frac{1}{p}}[g(\varkappa)]^{\frac{1}{q}}\right]\right)^{\frac{1}{q}}\geq \mathfrak{m}^{\frac{1}{pq}}\left[{}^k\mathcal{D}_{a+}^{\xi,\eta}g(\varkappa)\right]^{\frac{1}{q}},$$

multiplying equation (4.6) and (4.11) we get the result (4.1). □

Theorem 4.2. *Let* ${}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\varkappa)$ *and* ${}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\varkappa)$ *be two positive functions on* $[0,\infty[$*, such that*
${}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f^{p}(\varkappa)<\infty$, ${}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g^{q}(\varkappa)<\infty$, *If* $0<\mathfrak{m}\leq\frac{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)^{p}}{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)^{q}}\leq\mathcal{M}<\infty$, $\tau\in[a,\varkappa]$. *Then we have*

$$\left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta}f^{p}(\varkappa)\right]^{\frac{1}{p}}\left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta}g^{q}(\varkappa)\right]^{\frac{1}{q}}\leq\left(\frac{\mathcal{M}}{\mathfrak{m}}\right)^{\frac{1}{pq}}\left[{}^{k}\mathcal{D}_{a+}^{\xi,\eta}f(\varkappa)g(\varkappa)\right]. \tag{4.12}$$

Where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $k>0$, ${}^{k}\mathcal{D}_{a+}^{\xi,\eta}f(\varkappa)$ *denote the k-Hilfer fractional derivative.*

Proof. Replacing ${}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)$ and ${}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)$ by ${}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)^{p}$ and ${}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)^{q}$, $\tau\in[a,\varkappa]$, in Theorem 4.1, we obtain (4.12). □

Here, we will discuss a fractional integral inequality that is connected to the Minkowski inequality.

Theorem 4.3. *Let* $k>0$ *and* $({}^{k}\mathcal{D}_{a+}^{\xi,\eta}f)(\varkappa)$ *denote k-Hilfer fractional derivative of order* ξ, $0<\xi<1$, *and type* $0<\eta\leq1$. *Suppose that* $({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f)$ *and* $({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)$ *are two integrable functions on* $[0,\infty)$ *such that* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, *and* $0<\mathfrak{m}<\frac{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)}{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)}<\mathcal{M}$, $\tau\in[a,\varkappa]$, *Then we have*

$$\begin{aligned}{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[f(\varkappa)g(\varkappa)]\leq&\frac{2^{p-1}\mathcal{M}^{p}}{p(M+1)^{p}}\left({}^{k}\mathcal{D}_{a+}^{\xi,\eta}[f^{p}+g^{p}](\varkappa)\right)\\&+\frac{2^{q-1}}{q(\mathfrak{m}+1)^{q}}\left({}^{k}\mathcal{D}_{a+}^{\xi,\eta}[f^{q}+g^{q}](\varkappa)\right),\end{aligned}\tag{4.13}$$

Proof. Since, $\frac{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)}{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)}<\mathcal{M}$, $\tau\in[a,\varkappa]$, we have

$$\left(\mathcal{M}+1\right){}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)\leq\mathcal{M}\left({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f+{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g\right)(\tau). \tag{4.14}$$

Taking the p^{th} power on both sides of (4.14) and multiplying the resulting identity by $\frac{1}{k\Gamma_{k}(\eta(n-\xi))}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating the resulting identity with respect to τ from 0 to $\varkappa$, we have

$$\begin{aligned}&\frac{(\mathcal{M}+1)^{p}}{k\Gamma_{k}(\eta(n-\xi))}\int_{a}^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f^{p})(\tau)d\tau\\&\leq\frac{\mathcal{M}^{p}}{k\Gamma_{k}(\eta(n-\xi))}\int_{a}^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}[({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}(f+g)^{p})(\tau)]d\tau,\end{aligned}\tag{4.15}$$

therefore,

$${}^{k}\mathcal{D}_{a+}^{\xi,\eta}[f^{p}(\varkappa)]\leq\frac{M^{p}}{(\mathcal{M}+1)^{p}}\,{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f+g)^{p}(\varkappa)]. \tag{4.16}$$

On other hand, $0<\mathfrak{m}<\frac{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)}{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)}$, $\tau\in[a,\varkappa]$ we can write

$$(\mathfrak{m}+1)\,{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)\leq({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f+{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)(\tau). \tag{4.17}$$

Taking q^{th} power on both side (4.17) and multiplying resulting identity by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}$, then integrate the resulting identity with respect to τ from a to $\varkappa$, we have

$$\begin{aligned}&\frac{(\mathfrak{m}+1)^q}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g^q)(\tau)d\tau\\&\leq\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}[({}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}(f+g)^q)(\tau)]d\tau,\end{aligned} \tag{4.18}$$

consequently, we obtained

$${}^{k}\mathcal{D}_{a+}^{\xi,\eta}[g^q(\varkappa)]\leq\frac{1}{(\mathfrak{m}+1)^q}\,{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f+g)^q(\varkappa)]. \tag{4.19}$$

Now, using Young's inequality,

$$[{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau)\,{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau)]\leq\frac{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f^p(\tau)}{p}+\frac{{}^{k}\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g^q(\tau)}{q}. \tag{4.20}$$

Multiplying both sides of (4.20) by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating the resulting identity with respect to τ from a to $\varkappa$, we get

$${}^{k}\mathcal{D}_{a+}^{\xi,\eta}[f(\varkappa)g(\varkappa)]\leq\frac{1}{p}\,{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[f^p(\varkappa)]+\frac{1}{q}\,{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[g^q(\varkappa)], \tag{4.21}$$

from equation (4.16), (4.19) and (4.21), we have

$${}^{k}\mathcal{D}_{a+}^{\xi,\eta}[f(\varkappa)g(\varkappa)]\leq\frac{\mathcal{M}^p}{p(\mathcal{M}+1)^p}\,{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f+g)^p(\varkappa)]+\frac{1}{q(\mathfrak{m}+1)^q}\,{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f+g)^q(\varkappa)], \tag{4.22}$$

now using the inequality $(a+b)^r\leq2^{r-1}(a^r+b^r)$, $r>1$, a, $b\geq0$, we have

$${}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f+g)^p(\varkappa)]\leq2^{p-1}\,{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f^p+g^p)(\varkappa)], \tag{4.23}$$

and

$${}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f+g)^q(\varkappa)]\leq2^{q-1}\,{}^{k}\mathcal{D}_{a+}^{\xi,\eta}[(f^q+g^q)(\varkappa)]. \tag{4.24}$$

By including (4.23), (4.24) in (4.22), we deduce the inequality(4.13).
Thus, the proof is complete. □

5. Concluding Remarks

In [22] Nale et al. obtained Minkowski-type inequalities using generalized proportional Hadamard fractional integral operators. In [14], Iqbal et al. studied certain new Gruss inequalities by using k-Hilfer fractional derivative operator. Motivated by the work in [14, 22], in this paper, we studied reverse Minkowski fractional integral inequalities and other fractional inequalities using k-Hilfer fractional derivative operator. Also, using Young's inequality, we presented fractional integral inequality that is connected to the Minkowski inequality. Using this work we obtained more general inequalities than in the classical cases. The inequalities investigated in this paper give some contribution to the fields of fractional calculus and k-Hilfer fractional derivative operator. The technique used in this study to obtain the new fractional inequalities may motivate researchers to perform future research in this area.

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