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Solution of an infinite system of third order fuzzy differential equations in sequence space c_0 and ℓ_1 via measures of noncompactness and operator type contraction

Tanweer Jalal[1] **and Asif Hussain Jan**[23]

Abstract. In this paper, the existence of solutions to third order fuzzy differential equations in the sequence space c_0 and ℓ_1 is investigated using the measure of noncompactness and operator type contraction. Suitable examples are used to demonstrate the findings.

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1. Introduction

In a dynamical setting, fuzzy differential equations (FDEs) appear to be a natural technique to characterise epistemic uncertainty propagation. The study of initial and boundary value problems for fuzzy differential equations is a recent research topic that is quite fascinating. Many authors have studied initial and boundary value problems associated with first and second order fuzzy differential equations on the metric space (E^n, D) of normal fuzzy convex sets with the distance D given by the supremum of the Hausdorff distance between the corresponding α-level sets with the distance D [1]. Under the normal assumptions of continuous and Lipschitz condition on function f, O.Kaleva [14] proved the existence and uniqueness theorem for initial value problems related with first order fuzzy differential equation

$$y(w) = f(w, y(w)).$$

Furthermore, if f is continuous and bounded, J.J.Nieto [24] established a variant of Peano's existence theorem for fuzzy differential equations. Lakshmikantham et al. [15] have published criteria for the presence and uniqueness of two-point boundary value problems.

The concept of infinite system of fuzzy differential equations generalises the concept of infinite system of ordinary differential equations, defined as

[1]Department of Mathematics, National Institute of Technology, Hazratbal, Srinagar, Jammu and Kashmir, e-mail: tjalal@nitsri.net, ORCID iD: orcid.org/0000-0001-6520-4520

[2]Department of Mathematics, National Institute of Technology, Hazratbal, Srinagar, Jammu and Kashmir, e-mail: asif_06phd20@nitsri.net, ORCID iD: orcid.org/0000-0001-6520-4520

[3]Corresponding author

differential inclusion for non-uniform upper hemicontinuity convex set with compactness in fuzzy set.

Mathematically:

$$\frac{dx_i(w)}{dw} = f_i(w, x(w), \alpha), \quad \forall \quad \alpha \in [0,1].$$

The supremum of the Hausdorff distance between the corresponding α-level sets provides solutions to the initial and boundary value problems connected with the infinite system of fuzzy differential equations on the metric space (E^n, D) of normal fuzzy convex sets with distance D.

A fuzzy differential equation with fractional differential operator is

$$\frac{dx_i^n(w)}{dw^n} = f_i(w, x(w), \alpha), \quad \forall \quad \alpha \in [0,1],$$

where n is a rational number (p/q) called the fractional derivative.

Let $I = [a,b] \subset \mathbb{R}$ and $f : I \times E^n \to E^n$ be continuous. A mapping $\phi : I \to E^n$ is a solution of initial value problem

$$y_i' = f_i(t,y), y(a) = y_0,$$

if and only if ϕ is a solution of integral equation

$$y_i(t) = y_0 + \int_a^t f_i(s, y(s))ds.$$

Now, we will give some preliminaries about the concept of measures of noncompactness.

A measure of noncompactness is a nonnegative real-valued map generated on a collection of bounded subsets of a normed (metric) space that maps the class of relatively compact sets (called kernel) to zero while other sets are transferred to a positive value.

Kuratowski [13] was the first to establish the measure of noncompactness, which is important in the study of infinite systems of differential equations. In metric and topological space, there exist various measures of noncompactness. In recent years, researchers have used the measure of noncompactnes technique to prove a number of existence results for infinite systems of differential equations in Banach spaces such as c_0, ℓ_1, ℓ_p, c etc [1, 3, 4, 6, 7, 10, 8, 11, 12, 16, 20, 19, 17, 18, 21, 22, 23, 25, 26].

Let Ω represent the space of all complex sequences $x = (x_i)_{i=1}^{\infty}$. A sequence space is a vector subspace of Ω. The set of natural, real, and positive real numbers are denoted by $\mathbb{N}, \mathbb{R}$ and $\mathbb{R}^+$ respectively.

The Kuratowski measure of noncompactness for a bounded subset P of a metric space X is defined as

$$\alpha(P) = \inf \{\delta > 0 : P \subset \cup_{i=1}^n P_i, \operatorname{diam}(P_i) \le \delta, \text{ for } 1 \le i \le m \le \infty\}$$

where $\operatorname{diam}(P_i)$ denotes diameter of the set P_i.

Another important measure of non-compactness is the Hausdorff non-compactness measure, which is defined as

$$\chi(P) = \inf\{\epsilon > 0 : P \text{ has a finite } \epsilon\text{-net in } X\}.$$

Let $(X, \|\|)$ be a Banach space, $\mathbb{R}^+ = [0, \infty)$, the symbols $\bar{X}$ and $\operatorname{conv}(X)$ denote the closure of X and convex closure of X, respectively. Let M_E denote the family of non-empty bounded subsets of E and N_E denote the family of non-empty and relatively compact subsets of E. We now define (MNC) axiomatically given by Banaś and Goebel [6].

Definition 1.1. [6] Let X be a Banach space and E be the bounded subset of X. A function $\nu : M_X \to [0, +\infty)$ is said to be measure of non-compactnes in X if it satisfies the following axioms:

1. The family $\ker \nu = \{A \in M_X : \nu(E) = 0\}$ is a nonempty and $\ker \nu \subset N_X$.
2. $E_1 \subset E_2 \Rightarrow \nu(E_1) \leq \nu(E_2)$.
3. $\nu(\operatorname{Conv}(E)) = \nu(E)$.
4. $\nu(\lambda E_1 + (1 - \lambda E_2) \leq \lambda\nu(E_1) . + (1 - \lambda)\nu(E_2)$ for all $\lambda \in (0, 1)$.
5. If (E_m) is a sequence of closed sets from M_X such that $E_{n+1} \subset A_m$ and $\lim_{m\to\infty} \nu(E_m) = 0$, then the intersection set $E_\infty = \bigcap_{m=1}^{\infty} E_m$ is non-empty.

For the proofs of our main results, we require the following preliminaries:

Theorem 1.2. *[2] (Darbo's fixed point theorem) If ψ is a closed, convex subset of a Banach space E, then every compact, continuous map $T : \psi \to \psi$ has at least one fixed point.*

Definition 1.3. A function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$is said to be a Meir-Keeler function if $\Phi(0) = 0$ and for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $w \in \mathbb{R}^+$,

$$\epsilon \leq w < \epsilon + \delta \Longrightarrow \Psi(w) < \epsilon.$$

The following concept of $O(f; .)$ and its examples was given by Altun and Turkoglu [5]. Let $F([0, \infty))$ be class of all functions $f : [0, \infty) \to [0, \infty)$ and let Θ be class of all operators

$$O(\bullet; .) : F([0, \infty)) \to F([0, \infty)), f \to O(f; .)$$

satisfying the following conditions:

(i) $O(f; w) > 0$ for $w > 0$ and $O(f; 0) = 0$,

(ii) $O(f; w) \leq O(f; s)$ for $w \leq s$,

(iii) $\lim_{n\to\infty} O(f; w_n) = O\left(f; \lim_{n\to\infty} w_n\right)$,

(iv) $O(f;\max[w,s]) = \max\{O(f,w), O(f,s)\}$ for some $f \in F([0,\infty))$.

Using the concept of $O(f;.)$, B.Hazarika et al. [11] gave the concept of generalized Meir- Keeler condensing operator.

Definition 1.4. [11] Let Ω be a nonempty subset of a Banach space E and μ is a measure of noncompactness on E. We say that an operator $T : \Omega \to \Omega$ is a generalized Meir-Keeler type function if for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any subset X of Ω

$$\epsilon \leq O(f; G(\mu(X), \varphi(\mu(X)))) < \epsilon + \delta \Longrightarrow O(f; G(\mu(T(X)), \varphi(\mu(TX)))) < \epsilon$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$is continuous function, $O(\bullet;.) \in \Theta$ and $G \in \mathbb{G}$.

Here $\mathbb{G}$ is a class of all functions $G : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) $\max[a,b] \leq G(a,b)$ for $a, b \geq 0$,

(ii) G is continuous.

Result 1.5. [11] Let Ω be a nonempty subset of a Banach space E and μ an arbitrary measure of noncompactness on E. Let $T : \Omega \to \Omega$ be a continuous and generalized Meir-Keeler condensing operator then T has at least one fixed point and the set of all fixed points of T is compact.

Result 1.6. [11] Let Ω be a nonempty, bounded and convex subset of a Banach space E and μ an arbitrary measure of noncompactness on E. Let $T : \Omega \to \Omega$ and $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be two continuous functions. If for some $k \in (0,1)$

$$O(f; G(\mu(T(X)), \varphi(\mu(TX)))) \leq kO(f; G(\mu(X), \varphi(\mu(X))),$$

is satisfied where $O(\bullet;.) \in \Theta$ and $G \in \mathbb{G}$. Then T is a generalized Meir-Keeler type function.

Remark 1.7. [11] On taking $G(a,b) = a + b, \varphi \equiv 0$ and $O(f;w) = w$ in result 1.6, then Darbo's fixed point theorem is obtained.

In order to apply result 1.5 in a given Banach space X, we need a formula expressing the measure of noncompactness by a simple formula.

The c_0 sequence space is the set of sequences converging to 0 . Norm $\|.\|_{c_0}$, on c_0 is defined as

$$\|(x_l)\|_{c_0} = \sup_{l\geq 1} \{|x_l|\}, \quad x_l \in c_0.$$

Under the norm $\|.\|_{c_0}, c_0$ is a Banach space, for $E \in M_{c_0}$, the Hausdorff measure of noncompactness in c_0 is given by

$$\chi_{c_0}(E) = \lim_{n\to\infty} \left\{ \sup_{(x_l)\in E} \left(\sup_{n\geq l} |x_n| \right) \right\}.$$

The ℓ_1 sequence space is the set of sequences whose series is absolutely convergent.

$$\|(x_l)\|_{\ell_1} = \sum_{l\geq 1}^{\infty} \{|x_l|\}, \quad x_l \in \ell_1.$$

Under the norm $\|.\|_{\ell_1}, \ell_1$ is a Banach space, for $E \in M_{\ell_1}$, the Hausdorff measure of noncompactness in ℓ_1 is given by

$$\chi_{\ell_1}(E) = \lim_{l \to \infty} \left\{ \sup_{(x_n) \in E} \left(\sup_{n \geq l} |x_n| \right) \right\}.$$

The aim of this research paper is to apply the concept of measure of noncompactness and operator type contraction to study the existence of solution of infinite system of third order fuzzy differential equations in the sequence space c_0 and ℓ_1. The solution is investigated by using the infinite system of integral equations and Green's function [9].

2. Main results

In this paper, we consider the following infinite system of third order fuzzy differential equations

$$\frac{d^3 v_i}{dw^3} = x_i\left(w, v(w), v'(w), v''(w)\right); \quad i = 1, 2, 3, \ldots \tag{2.1}$$

with boundary conditions, $v_i(0) = 0, v_i'(\xi_0) = 0, v_i^{\prime\prime}(\xi_1) = 0, w \in [0, \xi_1]$.

Let $C([0, \xi_1], \mathbb{R})$ be the space of all real valued continuous functions over $[0, \xi_1]$ and $C^3([0, \xi_1], \mathbb{R})$ be the set of all functions with the third continuous derivative on $[0, \xi_1]$. A function $v \in C^3([0, \xi_1], \mathbb{R})$ is a solution of (2.1) if and only if $v \in C([0, \xi_1], \mathbb{R})$ is a solution of the infinite system of integral equations

$$v_i(w) = \int_0^{\xi_1} Y(w, s) x_i\left(s, v(s), v'(s), v''(s)\right) ds, \text{ for } w \in [0, \xi_1], \tag{2.2}$$

where $x_i(w, v) \in C([0, \xi_1], \mathbb{R}), i = 1, 2, 3, \ldots$ is the Green's function associated with the system is given by

$$Y(w, s)_{s \in [0, \xi_0]} = \begin{cases} \dfrac{s(2\xi_0 - s) - w^2}{2}, & 0 \leq w \leq s \leq \xi_0 \\ \\ s(\xi_0 - w), & s(\xi_0 - w) \end{cases}$$

$$Y(w, s)_{s \in [\xi_0, \xi_1]} = \begin{cases} \dfrac{\xi_0^2 - w^2}{2}, & \xi_0 \leq w \leq s \leq \xi_1 \\ \\ \dfrac{\xi_0^2 + s^2}{2} - ws, & \xi_0 \leq s < w \leq \xi_1. \end{cases} \tag{2.3}$$

It is simple to demonstrate the following estimate using usual procedures

$$\max_{0 \leq w \leq \xi_1} \int_0^{\xi_1} Y(w, s) \leq \frac{\xi_0^2 \left(3\xi_1 - \xi_0\right)^2}{6}. \tag{2.4}$$

From (2.2) and (2.3) we obtain

$$v_i'''(w) = \frac{d^3}{dw^3}\int_0^{\xi_1} Y(w,s)x_i\left(s, v(s), v'(s), v''(s)\right) ds = x_i\left(w, v(w), v'(w), v''(w)\right).$$

By converting the system into an infinite system of integral equations with the help of Green's function, we analyze and establish our key results on the existence of solutions for the infinite system of third order fuzzy differential equations (2.1) with boundary conditions.

2.1. Solution of the third order fuzzy differential equation in Sequence space c_0.

The following assumptions are made in order to identify the condition under which the system (2.1) has a solution in c_0 :

(Q1) The functions x_i are defined on the set $[0, \xi_1] \times \mathbb{R}^\infty$ and take real values $(i = 1, 2, 3, \ldots)$

(Q2) The operator x defined on the space $[0, \xi_1] \times c_0$ as

$$\begin{aligned} &(w, v, v', v'') \to (x(v, v', v'')) \\ &= (x_1(w, v, v', v''), x_2(w, v, v', v''), x_3(w, v, v', v''), \ldots) \end{aligned} \tag{2.5}$$

is such that the class of all functions $((xv)(w)), w \in [0, \xi_1]$ is equicontinuous at every point of the space c_0.

(Q3) The following inequality holds:

$$|x_i\left(w, v_1, v_1', v_1'', v_2, v_2', v_2'', \ldots\right)| \leq g_i(w) + h_i(w) \sup_{i \geq 1} |v_i(w), v_i'(w), v_i''(w)|$$

where $g_i(w)$ and $h_i(w)$ are real functions defined and continuous on $[0, \xi_1]$, such that $\{g_i(w)\}_{i=1}^\infty$ converges uniformly on $[0, \xi_1]$ and the sequence $(h_i(w))$ is equibounded on $[0, \xi_1]$.

(Q4) The function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$is nondecreasing and continuous such that $\varphi(\lambda w) \leq \lambda\varphi(w)$ for $\lambda \geq 0$, and $\varphi(0) = 0, \varphi(w) > 0$ for every $w > 0$.

To prove the result, we set

$$g(w) = \{g_i(w)\}_{i=1}^\infty),$$

$$G_o = \sup\left\{g(w) : w \in [0, \xi_1]\right\},$$

$$H_o = \sup\left\{h_n(w) : n \in N, w \in [0, \xi_1]\right\}.$$

Theorem 2.1. *Under the hypotheses* $(Q_1) - (Q_3)$*, the infinite system of fuzzy differential equations* (2.1) *has at least one solution* $v(w) = (v_i(w)) \in c_0$*, for all* $w \in [0, \xi_1]$.

Proof: Consider M_0 a finite positive real number for all $v(w) = (v_i(w)) \in c_0$ for all $w \in [0, \xi_1]$ such that $\sup_{i\in\mathbb{N}} |v_i(w)| \leq M_0 < \infty$. Then from the relation (2.2) and the hypothesis $(Q2)$, we have for an arbitrary $w \in [0, \xi_1]$

$$
\begin{aligned}
\|v(w)\|_{c_0} &= \max_{i\geq 1} \left| \int_0^{\xi_1} Y(w,s) x_k\left(s, v(s), v'(s), v''(s)\right) ds \right| \\
&\leq \max_{i\geq 1} \int_0^{\xi_1} \left| Y(w,s) x_k\left(s, v(s), v'(s), v''(s)\right)\right| ds \\
&\leq \max_{i\geq 1} \int_0^{\xi_1} |Y(w,s)| \left(g_k(w) + h_k(w) \sup_{i\geq 1} |v_k(w)| \right) ds \\
&\leq \max_{i\geq 1} \int_0^{\xi_1} Y(w,s)\left(g_k(w)\right) ds + \sum_{k=1}^{\infty} \int_0^{\xi_1} Y(w,s) h_k(w) \left|v_k(w)\right| ds \\
&\leq \int_0^{\xi_1} |Y(w,s)| \left\{ \max_{i\geq 1} \left(g_k(w)\right) \right\} ds \\
&+ H_0 \int_0^{\xi_1} |Y(w,s)| \left\{ \max_{i\geq 1} |v_k(w)| \right\} ds \\
&\leq G_0 \int_0^{\xi_1} |Y(w,s)| ds + H_0 \int_0^{\xi_1} |Y(w,s)| M_0 ds \\
&\leq \frac{G_0 \xi_0^2 \xi_1 \left(3\xi_1 - \xi_0\right)}{6} + \frac{H_0 M_0 \xi_0^2 \xi_1 \left(3\xi_1 - \xi_0\right)}{6} = r_0, \text{ say.}
\end{aligned}
$$

Let $u^0(w) = \left(u_i^0(w)\right)_{i=1}^{\infty}$ where $u_i^0(w) = 0$, for all $w \in [0, \xi_1]$. Let $\bar{B} = \bar{B}\left(u^0, r_1\right)$, the closed ball centered at u^0 and radius $r_1 \leq r_0$. Then $\bar{B}$ is a nonempty, closed and convex subset of c_0.

Let us consider the operator $\Gamma = (\Gamma_i)$ on $C\left([0, \xi_1], \bar{B}\right)$, defined as follows:

For $w \in [0, \xi_1]$

$$
(\Gamma u)(w) = \left\{ \left(\Gamma_i u\right)(w) \right\} = \left\{ \int_0^{\xi_1} Y(w,s) x_i\left(s, u(s), u'(s), u''(s)\right) ds \right\}
$$

where $u(w) = (u_i(w)) \in \bar{B}$ and $u_i(w) \in C\left([0, \xi_1], \mathbb{R}\right)$.

Since $(x_i(w, u(w))) \in c_0$ for each $w \in [0, \xi_1]$, so we have

$$
\max_{i\geq 1} \left| \left(\Gamma_i v\right)(w) \right| = \max_{i\geq 1} \left| \int_0^{\xi_1} Y(w,s) x_i\left(s, u(s), u'(s), u''(s)\right) ds \right| \leq r_0 < \infty.
$$

Therefore, $(\Gamma u)(w) = \left\{ \left(\Gamma_i u\right)(w) \right\} \in c_0$ for each $w \in [0, \xi_1]$.

Also, we have

$$
\begin{aligned}
(\Gamma_i)(0) &= \int_0^{\xi_1} Y(0,s) x_i(s, u(s), u'(s), u''(s)) ds \\
&= \int_0^{\xi_1} 0.x_i(s, u(s), u'(s), u''(s)) ds = 0,
\end{aligned}
$$

and

$$\begin{aligned}(\Gamma_i)(\xi_1) &= \int_0^{\xi_1} Y(\xi_1,s)x_i(s,u(s),u'(s),u''(s))ds \\ &= \int_0^{\xi_1} 0.x_i(s,u(s),u'(s),u''(s))ds = 0.\end{aligned}$$

Therefore, each $(\Gamma_i u)(w)$ satisfies boundary conditions given in (2.1).

Since $\left\|\left(\Gamma u(w) - u^0(w)\right)\right\|_{c_0} = \|(\Gamma u(w))\|_{c_0} \leq r_0$, therefore, it follows that Γ is a self mapping on $\bar{B}$. The operator Γ is continuous $C([0,\xi_1])$ by assumption $(Q2)$. We now show that Γ is a generalized Meir-Keeler condensing operator for which for any given $\epsilon > 0$, we need to find $\delta > 0$ such that $\chi(\Gamma\bar{B}) < \epsilon$ whenever $\epsilon \leq \chi(\bar{B}) + \varphi(\bar{B}) < \epsilon + \delta$. We have

$$\begin{aligned}&\chi(\Gamma\bar{B}) + \varphi(\Gamma\bar{B}) \\ &= \lim_{n\to\infty}[\sup_{u(w)\in\bar{B}}\{\max_{i\geq1}|\int_0^{\xi_1} Y(w,s)x_k(s,v(s),v'(s),v''(s))ds|\}] \\ &\quad +\varphi[\lim_{n\to\infty}[\sup_{u(w)\in\bar{B}}\{\max_{i\geq1}|\int_0^{\xi_1} Y(w,s)x_k(s,v(s),v'(s),v''(s))ds|\}]] \\ &\leq \lim_{n\to\infty}[\sup_{u(w)\in\bar{B}}\{\max_{i\geq1}\int_0^{\xi_1} |Y(w,s)|x_k(s,v(s),v'(s),v''(s))ds\}] \\ &\quad +\varphi[\lim_{n\to\infty}[\sup_{u(w)\in\bar{B}}\{\max_{i\geq1}\int_0^{\xi_1} |Y(w,s)|x_k(s,v(s),v'(s),v''(s))ds\}]] \\ &\leq \lim_{n\to\infty}[\sup_{u(w)\in\bar{B}}\{\max_{i\geq1}\int_0^{\xi_1} |Y(w,s)|(g_k(s)+h_k(s)|v_k(s)|)ds\}] \\ &\quad +\varphi[\lim_{n\to\infty}[\sup_{u(w)\in\bar{B}}\{\max_{i\geq1}\int_0^{\xi_1} |Y(w,s)|(g_k(s)+h_k(s)|v_k(s)|)ds\}]] \\ &\leq \lim_{n\to\infty}[\sup\nolimits_{u(w)\in\overline{B}}\{\int_0^{\xi_1} |Y(w,s)|(\max_{i\geq1} g_k(s)ds \\ &\quad + \lim_{n\to\infty}[\sup_{u(w)\in\bar{B}}\{\int_0^{\xi_1} |Y(w,s)|(\max_{i\geq1} h_k(s)|v_k(s)|)ds\}] \\ &\quad +\varphi[\lim\nolimits_{n\to\infty}[\sup\nolimits_{u(w)\in\overline{B}}\{\int_0^{\xi_1} |Y(w,s)|(\max_{i\geq1} g_k(s)ds \\ &\quad + \lim_{n\to\infty}[\sup_{u_0(w)\in\bar{B}}\{\int_0^{\xi_1} |Y(w,s)|(\max_{i\geq1} h_k(s)|v_k(s)|)ds\}]] \\ &\leq \lim_{n\to\infty}[\sup\nolimits_{u_0(w)\in\overline{B}}\{\int_0^{\xi_1} |Y(w,s)|(\max_{i\geq1} g_k(s))ds \\ &\quad + \lim_{n\to\infty}[\sup_{u(w)\in\bar{B}}\{\int_0^{\xi_1} |Y(w,s)|(H_0\max_{i\geq1}|v_k(s)|)ds\}]\end{aligned}$$

$$+\varphi[\lim_{n\to\infty}[\sup_{u(w)\in\overline{B}}\{\int_0^{\xi_1}|Y(w,s)|(\max_{i\geq 1} g_k(s))ds$$

$$+\lim_{n\to\infty}[\sup_{u(w)\in\bar{B}}\{\int_0^{\xi_1}|Y(w,s)|(H_0\max_{i\geq 1}|v_k(s)|)ds\}]]$$

$$\leq \frac{H_0\xi_1\xi_0^2(3\xi_1-\xi_0)}{6}(\chi(\bar{B}))+\varphi[\frac{H_0\xi_0^2\xi_1(3\xi_1-\xi_0)}{6}(\chi(\bar{B}))]$$

$$\leq \frac{H_0\xi_0^2\xi_1(3\xi_1-\xi_0)}{6}[\chi(\bar{B})+\varphi(\chi(\bar{B})].$$

Thus, we get,

$$\chi(\Gamma\bar{B})+\varphi(\Gamma\bar{B})<\frac{H_0\xi_0^2\xi_1(3\xi_1-\xi_0)}{6}[\chi(\bar{B})+\varphi(\chi(\bar{B})]<\epsilon,$$

$$\chi(\bar{B})+\varphi(\chi(\bar{B})<\frac{6\epsilon}{H_0\xi_0^2\xi_1(3\xi_1-\xi_0)}.$$

Taking

$$\delta=\frac{6-H_0\xi_0^2\xi_1\,(3\xi_1-\xi_0)}{H_0\xi_0^2\xi_1\,(3\xi_1-\xi_0)}\epsilon,$$

we get $\epsilon\leq\chi(\bar{B})+\varphi(\bar{B})<\epsilon+\delta$. This shows that γ is a generalizeed Meir-Keeler condensing operator defined on the set $\bar{B}\subset\ell_1$ and so, it satisfies all the conditions of result 1.5 with $O(f;w)=w$ and $Y_1(a,b)=a+b$. Therefore γ has a fixed point in $\bar{B}$, which is a solution of system of equations (2.1)

The following examples exemplify the above result:

Example 2.2. Consider the following system of third order differential equations

$$\frac{d^3x_n(w)}{dw^3}-x_n\left(w,u(w),u'(w),u''(w)\right)=0 \tag{2.6}$$

where $x_n\left(w,u(w),u'(w),u''(w)\right)=e^{\frac{w}{2}}\cos\left(\frac{w}{n^3}\right)\left(u_n(w),u_n'(w),u_n''(w)\right), w\in[0,\xi_1]$ $u_i(w)\in c_0$.

Solution: Consider a positive arbitrary real number $\epsilon>0$ and $v(w)\in c_0$, we have

$$\begin{aligned}&\sum_{l=1}^{\infty}|x_n\left(w,u(w),u'(w),u''(w)\right)|\\ &\leq e^{\frac{w}{2}}\sum_{l=1}^{\infty}|u_n(w),u_n'(w),u_n''(w)|\\ &<\infty \text{ if } u(w),u'(w),u''(w)\\ &=u_i(w),u_i'(w),u_i''(w)\in c_0.\end{aligned}$$

Taking $v(w) \in c_0$ with

$$\|(u(w), u'(w), u"(w)) - (v(w), v'(w), v"(w))\|_{c_0} < \delta = \frac{\epsilon}{e^{\frac{w}{2}}},$$

$$\begin{aligned} &|x_n\left(w, u(w), u'(w), u''(w)\right) - x_n\left(w, v(w), v'(w), v''(w)\right)| \\ &= \quad | \, e^{\frac{w}{2}} \cos\left(\frac{w}{n^3}\right)(u(w), u'(w), u''(w)) - e^{\frac{w}{2}} \cos\left(\frac{w}{n^3}\right)(v(w), v'(w), v''(w)) \, | \\ &\leq \quad e^{\frac{w}{2}} \, | \, (u_n(w), u_n'(w), u_n''(w)) - (v_n(w), v_n'(w), v_n''(w)) \, | \\ &< \quad e^{\frac{w}{2}} \delta = \epsilon \end{aligned}$$

which implies the equicontinuity of $(xu)(w)_{w\in[0,\xi_1]}$ on c_0. Again we have for all $n \in N$ and $w \in [0, \xi_1]$

$$\begin{aligned} &|x_n(w, u(w), u'(w), u''(w))| \\ &\quad \leq \quad e^{\frac{w}{2}} |u_n(w), u_n'(w), u_n''(w)| \\ &\quad = \quad h_n(w)|u_n(w), u_n'(w), u_n''(w)|, \end{aligned}$$

where $h_n(w) = e^{\frac{w}{2}}$ is a real function on $[0, \xi_1]$ and the sequence $|h_n(w)|$ is equibounded on $[0, \xi_1]$. Thus by result 1.5, the system (2.6) has a unique solution in sequence space c_0.

Example 2.3. Consider the following system of third order differential equations

$$\frac{d^3 x_n(w)}{dw^3} - \frac{\sqrt{w}(\xi_1 - w)e^{-nw}}{(n+1)^6} + \sum_{m=n}^{\infty} \frac{w\,(u_m(w), u_m'(w), u_m''(w))}{(1+n^4)(m+1)^4} = 0, \tag{2.7}$$

where $n \in \mathbb{N}, w \in [0, \xi_1], (u_i(w), u_i'(w), u_i"(w)) \in c_0$.

Solution: Clearly, $\alpha_{nm}(w) = \dfrac{w}{(1+n^4)(m+1)^4}$ is continuous and $\sum\limits_{m=n}^{\infty} \dfrac{w}{(1+n^4)(m+1)^4}$ is absolutely uniformly continuous on $[0, \xi_1]$, where $m, n \in \mathbb{N}$. Since $\alpha_{nm}(w) = \sum\limits_{m=n}^{\infty} \|a_{nm}(w)\|$ is uniformly bounded on $[0, \xi_1]$, we have

$$\begin{aligned} &\sum_{m=n}^{\infty} \frac{|((u_m(w), u_m'(w), u_m''(w)) - (v_m(w), v_m'(w), v_m''(w)))|w}{(1+n^4)(m+1)^4} \\ &\qquad \leq \|((u(w), u'(w), u''(w)) - (v(w), v'(w), v''(w)))\| \frac{w\pi^4}{90}. \end{aligned}$$

If $u(w), u'(w), u''(w) = u_i(w), u_i'(w), u_i''(w) \in c_0$, then

$$x_n\left(w, u(w), u'(w), u''(w)\right) \in c_0$$

as we have

$$\begin{aligned}
&\lim_{n\to\infty} x_n\left(w, u(w), u'(w), u''(w)\right)\\
&= \lim_{n\to\infty}\left(\frac{\sqrt{w}(\xi_1 - w)e^{-nw}}{(n+1)^6} + \sum_{m=n}^{\infty}\frac{w\left(u_m(w), u'_m(w), u''_m(w)\right)}{(1+n^4)(m+1)^4}\right)\\
&\le \lim_{n\to\infty}\left(\frac{\sqrt{w}(\xi_1 - w)e^{-nw}}{(n+1)^6}\right.\\
&\quad\left.+\frac{w}{(1+n^4)}\sup_{m\ge n}\sum_{m=0}^{\infty}\left(u_m(w), u'_m(w), u''_m(w)\right)\frac{1}{(m+1)^4}\right)\\
&\le \lim_{n\to\infty}\left(\frac{\sqrt{w}(\xi_1 - w)}{(n+1)^6} + \frac{w\pi^4}{90(1+n^4)}\sup_{m\ge n}\left(u_m(w), u'_m(w), u''_m(w)\right)\right) = 0.
\end{aligned}$$

Since $(u_m(w), u'_m(w), u''_m(w)) \to 0$ for each $w \in [0, \xi_1]$, thus for all fixed w, we obtain $\lim_{n\to\infty}[(u_m(w), u'_m(w), u''_m(w))] = 0$.

Now we show that assumption $(Q2)$ is satisfied. Let us fix $\epsilon > 0$ arbitrarily and $(u(w), u'(w), u''(w)) = (u_m(w), u'_m(w), u''_m(w)) \in c_0$. Then using system (**??**)2.7) and taking $(v(w), v'(w), v''(w)) = (v_m(w), v'_m(w), v''_m(w)) \in c_0$, with $\parallel (u(w), u'(w), u''(w)) - (v(w), v'(w), v''(w)) \parallel \le \delta = \epsilon\left(\frac{90}{w\pi^4}\right)^{-1}$, we have

$$\begin{aligned}
&\left|x_n(u, u', u'')(w) - x_n(v, v', v'')(w)\right|\\
&= \sum_{m=n}^{\infty}\frac{|((u_m(w), u'_m(w), u''_m(w)) - (v_m(w), v'_m(w), v''_m(w)))|w}{(1+n^4)(m+1)^4} \le \delta\frac{w\pi^4}{90} \le \epsilon,
\end{aligned}$$

for any fixed n, which implies the continuity as assumed in $(Q2)$. Now, we show that the assumption $(Q3)$ is satisfied. Indeed for

$$x_n\left(w, u(w), u'(w), u''(w)\right),$$

we have

$$\begin{aligned}
&\left|x_n\left(w, u(w), u'(w), u''(w)\right)\right|\\
&= \left|\frac{\sqrt{w}(\xi_1 - w)}{(n+1)^6} + \frac{w\pi^4}{90(1+n^4)}\sup_{m\ge n}\left(u_m(w), u'_m(w), u''_m(w)\right)\right|\\
&= p_n(w) + q_n(w)\sup_{m\ge n}\left(u_m(w), u'_m(w), u''_m(w)\right),
\end{aligned}$$

where $p_n(w)$ vanishes identically on $[0, \xi_1]$ and $q_n(w)$ is equibounded on $[0, \xi_1]$. Thus all the conditions of $(Q3)$ are satisfied. Using result 1.5, the system (2.7) has a unique solution in sequence space c_0.

2.2. Solution of the third order fuzzy differential equation in Sequence space ℓ_1.

The following assumptions are made in order to identify the condition under which the system (2.1) has a solution in ℓ_1 :

(L1) The functions x_i are defined on the set $[0,\xi_1]\times\mathbb{R}^\infty$ and take real values $(i=1,2,3,\ldots)$.

(L2) The operator x defined on the space $[0,\xi_1]\times\ell_1$ as

$$\begin{aligned}&(w,v,v',v'')\to(x(v,v',v''))\\ &=(x_1(w,v,v',v''),x_2(w,v,v',v''),x_3(w,v,v',v''),\ldots)\end{aligned}\tag{2.8}$$

is such that the class of all functions $((xv)(w)), w\in[0,\xi_1]$ is equicontinuous at every point of the space ℓ_1.

(L3) The following inequality holds:

$$|x_i(w,v_1,v_1',v_1'',v_2,v_2',v_2'',\ldots)|\le g_i(w)+h_i(w)|v_i(w)|$$

where $g_i(w)$ and $h_i(w)$ are real functions defined and continuous on $[0,\xi_1]$, such that $\sum_{k=1}^{\infty}g_k(w)$ converges uniformly on $[0,\xi_1]$ and the sequence $(h_i(w))$ is equibounded on $[0,\xi_1]$.

(L4) The function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$is nondecreasing and continuous such that $\varphi(\lambda w)\le\lambda\varphi(w)$ for $\lambda\ge0$, and $\varphi(0)=0,\varphi(w)>0$ for every $w>0$.

To prove the result, we set

$$g(w)=\sum_{k=1}^{\infty}g_k(w),$$

$$G_1=\sup\{g(w):w\in[0,\xi_1\},$$

$$H_1=\sup\{h_n(w):n\in N,w\in[0,\xi_1]\}.$$

Theorem 2.4. *Under the hypotheses* $(Q_1)-(Q_3)$*, the infinite system of differential equations* (2.1) *has at least one solution* $v(w)=(v_i(w))\in\ell_1$*, for all* $w\in[0,\xi_1]$.

Proof. Let $S(u(w))$ denote the set of all sequences that are rearrangements of $v(w)$. If $v(w)\in S(u(w))$, then $\sum\limits_{k=1}^{\infty}|v_k(w)|\le M_1$, where M_1 is a finite positive real number for all $v(w)=(v_i(w))\in\ell_1$ for all $w\in[\xi_0,\xi_1]$. Then from the relation (2.2) and the hypothesis $(L2)$, we have for an arbitrary $w\in[0,\xi_1]$,

$$\begin{aligned}\|v(w)\|_{\ell_1}&=\sum_{k=1}^{\infty}\left|\int_0^{\xi_1}Y(w,s)x_k\left(s,v(s),v'(s),v''(s)\right)ds\right|\\ &\le\sum_{k=1}^{\infty}\int_0^{\xi_1}\left|Y(w,s)x_k\left(s,v(s),v'(s),v''(s)\right)\right|ds\end{aligned}$$

$$\leq \sum_{k=1}^{\infty} \int_0^{\xi_1} |Y(w,s)| \left(g_k(w) + h_k(w) \left| v_k(w) \right| \right) ds$$
$$\leq \sum_{k=1}^{\infty} \int_0^{\xi_1} Y(w,s) \left(g_k(w) \right) ds + \sum_{k=1}^{\infty} \int_0^{\xi_1} Y(w,s) h_k(w) \left| v_k(w) \right| ds$$
$$\leq \int_0^{\xi_1} |Y(w,s)| \left\{ \sum_{k=1}^{\infty} \left(g_k(w) \right) \right\} ds + H_1 \int_0^{\xi_1} |Y(w,s)| \left\{ \sum_{k=1}^{\infty} |v_k(w)| \right\} ds$$
$$\leq G_1 \int_0^{\xi_1} |Y(w,s)| ds + H_1 \int_0^{\xi_1} |Y(w,s)| M_0 ds$$
$$\leq \frac{G_1 \xi_0^2 \xi_1 \left(3\xi_1 - \xi_0 \right)}{6} + \frac{H_1 M_1 \xi_0^2 \xi_1 \left(3\xi_1 - \xi_0 \right)}{6} = r_0, \text{ say.}$$

Let $u^0(w) = \left(u_i^0(w) \right)_{i=1}^{\infty}$ where $u_i^0(w) = 0$, for all $w \in [0, \xi_1]$. Let $\bar{B} = \bar{B}\left(u^0, r_1 \right)$, the closed ball centered at u^0 and radius $r_1 \leq r_0$. Then $\bar{B}$ is a nonempty, closed and convex subset of ℓ_1.

Let us consider the operator $\Gamma = (\Gamma_i)$ on $C\left([0, \xi_1], \bar{B} \right)$, defined as follows: For $w \in [0, \xi_1]$

$$(\Gamma u)(w) = \left\{ \left(\Gamma_i u \right)(w) \right\} = \left\{ \int_0^{\xi_1} Y(w,s) x_i \left(s, u(s), u'(s), u''(s) \right) ds \right\},$$

where $u(w) = (u_i(w)) \in \bar{B}$ and $u_i(w) \in C\left([\xi_0, \xi_1], \mathbb{R} \right)$

Since $(x_i(w, u(w))) \in \ell_1$ for each $w \in [\xi_0, \xi_1]$, so we have

$$\sum_{i=1}^{\infty} \left| \left(\Gamma_i v \right)(w) \right| = \sum_{i=1}^{\infty} \left| \int_0^{\xi_1} Y(w,s) x_i \left(s, u(s), u'(s), u''(s) \right) ds \right| \leq r_0 < \infty$$

Therefore, $(\Gamma u)(w) = \left\{ \left(\Gamma_i u \right)(w) \right\} \in \ell_1$ for each $w \in [0, \xi_1]$.

Also, we have

$$\begin{aligned} \Gamma_i)(0) &= \int_0^{\xi_1} Y(0,s) x_i(s, u(s), u'(s), u''(s)) ds \\ &= \int_0^{\xi_1} 0.x_i(s, u(s), u'(s), u''(s)) ds = 0. \end{aligned}$$

and

$$\begin{aligned} (\Gamma_i)(\xi_1) &= \int_0^{\xi_1} Y(\xi_1, s) x_i(s, u(s), u'(s), u''(s)) ds \\ &= \int_0^{\xi_1} 0.x_i(s, u(s), u'(s), u''(s)) ds = 0. \end{aligned}$$

Therefore, each $\left(\Gamma_i u \right)(w)$ satisfies boundary conditions given in (2.1).

Since $\left\| \left(\Gamma u(w) - u^0(w) \right) \right\|_{\ell_1} = \| (\Gamma u(w)) \|_{\ell_1} \leq r_0$, therefore, it follows that Γ is self mapping on $\bar{B}$. The operator Γ is continuous $C([0, \xi_1])$ by assumption

$(L2)$. We now show that Γ is a generalized Meir-Keeler condensing operator for which for any given $\epsilon > 0$, we need to find $\delta > 0$ such that $\chi(\Gamma\bar{B}) < \epsilon$ whenever $\epsilon \leq \chi(\bar{B}) + \varphi(\bar{B}) < \epsilon + \delta$. We have

$$
\begin{aligned}
&\chi(\Gamma\bar{B}) + \varphi(\Gamma\bar{B}) \\
&= \lim_{n\to\infty}\left[\sup_{u(w)\in\bar{B}}\left\{\sum_{n=k}^{\infty}\left|\int_0^{\xi_1} Y(w,s)x_k\left(s,v(s),v'(s),v''(s)\right)ds\right|\right\}\right] \\
&\quad +\varphi\left[\lim_{n\to\infty}\left[\sup_{u(w)\in\bar{B}}\left\{\sum_{n=k}^{\infty}\left|\int_0^{\xi_1} Y(w,s)x_k\left(s,v(s),v'(s),v''(s)\right)ds\right|\right\}\right]\right] \\
&\leq \lim_{n\to\infty}\left[\sup_{u(w)\in\bar{B}}\left\{\sum_{n=k}^{\infty}\int_0^{\xi_1} |Y(w,s)|x_k\left(s,v(s),v'(s),v''(s)\right)ds\right\}\right] \\
&\quad +\varphi\left[\lim_{n\to\infty}\left[\sup_{u(w)\in\bar{B}}\left\{\sum_{n=k}^{\infty}\int_0^{\xi_1} |Y(w,s)|x_k\left(s,v(s),v'(s),v''(s)\right)ds\right\}\right]\right] \\
&\leq \lim_{n\to\infty}\left[\sup_{u(w)\in\bar{B}}\left\{\sum_{n=k}^{\infty}\int_0^{\xi_1} |Y(w,s)|\left(g_k(s)+h_k(s)\left|v_k(s)\right|\right)ds\right\}\right] \\
&\quad +\varphi\left[\lim_{n\to\infty}\left[\sup_{u(w)\in\bar{B}}\left\{\sum_{n=k}^{\infty}\int_0^{\xi_1} |Y(w,s)|\left(g_k(s)+h_k(s)\left|v_k(s)\right|\right)ds\right\}\right]\right] \\
&\leq \lim_{n\to\infty}\left[\sup\nolimits_{u(w)\in\overline{B}}\left\{\int_0^{\xi_1} |Y(w,s)|\left(\sum_{n=k}^{\infty} g_k(s)ds\ \right)\right.\right. \\
&\quad \left.\left. +\lim_{n\to\infty}\left[\sup_{u(w)\in\bar{B}}\left\{\int_0^{\xi_1} |Y(w,s)|\left(\sum_{n=k}^{\infty} h_k(s)\left|v_k(s)\right|\right)ds\right\}\right]\right.\right. \\
&\quad +\varphi\left[\lim\nolimits_{n\to\infty}\left[\sup\nolimits_{u(w)\in\overline{B}}\left\{\int_0^{\xi_1} |Y(w,s)|\sum_{n=k}^{\infty} g_k(s)ds\right)\right.\right. . \\
&\quad \left.\left. +\lim_{n\to\infty}\left[\sup_{u_0(w)\in\bar{B}}\left\{\int_0^{\xi_1} |Y(w,s)|\left(\sum_{n=k}^{\infty} h_k(s)\left|v_k(s)\right|\right)ds\right\}\right]\right]\right] \\
&\leq \lim_{n\to\infty}\left[\sup\nolimits_{u_0(w)\in\overline{B}}\left\{\int_0^{\xi_1} |Y(w,s)|\left(\sum_{n=k}^{\infty} g_k(s)\right)ds\right.\right. \\
&\quad \left.\left. +\lim_{n\to\infty}\left[\sup_{u(w)\in\bar{B}}\left\{\int_0^{\xi_1} |Y(w,s)|\left(H_1\sum_{n=k}^{\infty}\left|v_k(s)\right|\right)ds\right\}\right]\right.\right. \\
&\quad +\varphi\left[\lim\nolimits_{n\to\infty}\left[\sup\nolimits_{u(w)\in\overline{B}}\left\{\int_0^{\xi_1} |Y(w,s)|\left(\sum_{n=k}^{\infty} g_k(s)\right)ds\right.\right.\right. \\
&\quad \left.\left.\left. +\lim_{n\to\infty}\left[\sup_{u(w)\in\bar{B}}\left\{\int_0^{\xi_1} |Y(w,s)|\left(H_1\sum_{n=k}^{\infty}\left|v_k(s)\right|\right)ds\right\}\right]\right]\right]
\end{aligned}
$$

$$\leq \frac{H_1\xi_0^2\xi_1\left(3\xi_1-\xi_0\right)}{6}(\chi(\bar{B}))+\varphi\left[\frac{H_1\xi_0^2\xi_1\left(3\xi_1-\xi_0\right)}{6}(\chi(\bar{B}))\right]$$
$$\leq \frac{H_1\xi_0^2\xi_1\left(3\xi_1-\xi_0\right)}{6}[\chi(\bar{B})+\varphi(\chi(\bar{B})].$$

Thus, we get,

$$\chi(\Gamma\bar{B})+\varphi(\Gamma\bar{B})<\frac{H_1\xi_0^2\xi_1(3\xi_1-\xi_0)}{6}[\chi(\bar{B})+\varphi(\chi(\bar{B})]<\epsilon$$
$$\Rightarrow \quad \chi(\bar{B})+\varphi(\chi(\bar{B})<\frac{6\epsilon}{H_1\xi_0^2\xi_1(3\xi_1-\xi_0)}.$$

Taking

$$\delta=\frac{6-H_1\xi_0^2\xi_1\left(3\xi_1-\xi_0\right)}{H_1\xi_0^2\xi_1\left(3\xi_1-\xi_0\right)}\epsilon,$$

we get $\epsilon \leq \chi(\bar{B})+\varphi(\bar{B})<\epsilon+\delta$. This shows that γ is a generalizeed Meir-Keeler condensing operator defined on the set $\bar{B}\subset \ell_1$ and so, it satisfies all the conditions of result 1.5 with $O(f;w)=w$ and $Y_1(a,b)=a+b$. Therefore γ has a fixed point in $\bar{B}$, which is a solution of system of equations (2.1). □

The following examples exemplify the above result:

Example 2.5. Consider the following system of third order fuzzy differential equations

$$\frac{d^3x_n(w)}{dw^3}-x_n\left(w,u(w),u'(w),u''(w)\right)=0, \tag{2.9}$$

where

$$x_n\left(w,u(w),u'(w),u''(w)\right)=w^{\frac{1}{2}}\sin\left(\frac{w}{n^3}\right)\left(u_n(w),u_n'(w),u_n''(w)\right),$$
$$w\in[0,\xi_1] \text{ and } u_i(w)\in\ell_1.$$

Solution: Consider a positive arbitrary real number $\epsilon>0$ and $v(w)\in\ell_1$, we have

$$\begin{aligned}\sum_{l=1}^{\infty}\left|x_n\left(w,u(w),u'(w),u''(w)\right)\right| &\leq w^{\frac{1}{2}}\sum_{l=1}^{\infty}\left|u_n(w),u_n'(w),u_n''(w)\right|\\ &<\infty \text{ if } u(w),u'(w),u''(w)\\ &=u_i(w),u_i'(w),u_i''(w)\in\ell_1.\end{aligned}$$

Taking $v(w)\in\ell_1$ with

$$\|(u(w),u'(w),u"(w))-(v(w),v'(w),v"(w))\|_{\ell_1}<\delta=\frac{\epsilon}{w^{\frac{1}{2}}},$$

$$\left|x_n\left(w,u(w),u'(w),u''(w)\right)-x_n\left(w,v(w),v'(w),v''(w)\right)\right|$$

$$
\begin{aligned}
&=\left| w^{\frac{1}{2}} \sin\left(\frac{w}{n^3}\right)(u(w),u'(w),u''(w)) - w^{\frac{1}{2}} \sin\left(\frac{w}{n^3}\right)(v(w),v'(w),v''(w)) \right| \\
&\leq w^{\frac{1}{2}} \left|u_n(w) - v_n(w)\right| \\
&< w^{\frac{1}{2}}\delta = \epsilon
\end{aligned}
$$

which implies the equicontinuity of $(xu)(w)_{w\in[0,\xi_1]}$ on ℓ_1. Again we have for all $n \in N$ and $w \in [a_0, a_1]$

$$
\begin{aligned}
|x_n(w,u(w),u'(w),u''(w))| &\leq e^{\frac{w}{2}}|u_n(w),u'_n(w),u''_n(w)| \\
&= h_n(w)|u_n(w),u'_n(w),u''_n(w)|
\end{aligned}
$$

where $h_n(w) = w^{\frac{1}{2}}$ is a real function on $[0,\xi_1]$ and the sequence $|h_n(w)|$ is equibounded on $[0,\xi_1]$. Thus by result 1.5, the system (2.9) has a unique solution in sequence space ℓ_1.

Example 2.6. Consider the following system of third order differential equations

$$
\frac{d^3x_n(w)}{dw^3} - \frac{\sqrt[n]{w}}{n^8} + \sum_{m=n}^{\infty} \frac{w\cos(w)\,(u_m(w),u'_m(w),u''_m(w))}{m^8} = 0, \tag{2.10}
$$

where $w \in [0,\xi_1]\,, (u_i(w),u'_i(w),u_i"(w)) \in \ell_1$.

Solution: Clearly, $\frac{\sqrt[n]{w}}{n^8}$ and $\sum\limits_{m=n}^{\infty} \frac{w\cos(w)\,(u_m(w),u'_m(w),u''_m(w))}{m^8}$ are continuous on $[0,\xi_1]$, for each $n \in \mathbb{N}$.

Note that for any $w \in [0,\xi_1], x_i(w,u(w),u'(w),u"(w)) \in \ell_1$, if $(u_i(w),u'_i(w),u_i"(w)) \in \ell_1$. Moreover, we have

$$
\begin{aligned}
&\sum_{l=1}^{\infty} |x_n\left(w,u(w),u'(w),u''(w)\right)| \\
&= \sum_{l=1}^{\infty} \left| \frac{\sqrt[n]{w}}{n^8} + \sum_{m=n}^{\infty} \frac{w\cos(w)\,(u_m(w),u'_m(w),u''_m(w))}{m^8} \right| \\
&\leq \sum_{l=1}^{\infty} \frac{\sqrt[n]{w}}{n^8} + \sum_{l=1}^{\infty}\sum_{m=n}^{\infty} \left| \frac{w\cos(w)\,(u_m(w),u'_m(w),u''_m(w))}{m^8} \right| \\
&\leq \frac{\xi_1\pi^8}{9450} + \sum_{l=1}^{\infty}\sum_{m=n}^{\infty} \left| \frac{w\,(u_m(w),u'_m(w),u''_m(w))}{m^8} \right| \\
&\leq \frac{\xi_1\pi^8}{9450} + \xi_1\|\,(u(w),u'(w),u''(w))\,\|_1 < \infty.
\end{aligned}
$$

We will show that H_1 is satisfied.
Let us fix arbitrarily $\epsilon > 0$ and $(u_n(w),u'_n(w),u''_n(w)) \in \ell_1$. Then taking $(v_n(w),v'_n(w),v''_n(w)) \in \ell_1$ with

$$
\|(u(w),u'(w),u"(w)) - (v(w),v'(w),v"(w))\|_{\ell_1} < \delta = \frac{\epsilon}{\xi_1},
$$

$$|x_n(w, u(w), u'(w), u''(w)) - x_n(w, v(w), v'(w), v''(w))|$$

$$= \sum_{m=n}^{\infty} \frac{w((u_m(w), u'_m(w), u''_m(w)) - (v_m(w), v'_m(w), v''_m(w)))}{m^8} \leq \xi_1 \delta < \epsilon.$$

which implies continuity as assumed in $L2$. Now, we show that assumption $L3$ is satisfied.

$$\begin{aligned}
&|x_n(w, u(w), u'(w), u''(w))| \\
&= \left| \frac{\sqrt[n]{w}}{n^8} + \sum_{m=n}^{\infty} \frac{w \cos(w)\,(u_m(w), u'_m(w), u''_m(w))}{m^8} \right| \\
&\leq \frac{\sqrt{w}}{n^8} + \sum_{m=n}^{\infty} \frac{w}{m^8} \left|(u_m(w), u'_m(w), u''_m(w))\right| \\
&\leq g_n(w) + h_n(w) \left|(u_n(w), u'_n(w), u''_n(w))\right|
\end{aligned}$$

The function $g_n(w)$ is continuous and $\sum_{n\geq 1} g_n(w)$ converges uniformly to $\frac{\sqrt{w}\pi^8}{9450}$, also $h_n(w) = \frac{w\pi^8}{9450}$ is continuous and the sequence $(h_n(w))$ is equibounded on $[0, \xi_1]$ by $H_1 = \frac{\xi_1 \pi^8}{9450}$. Thus, by result 1.5, the system (2.10) has a unique solution in sequence space ℓ_1.

3. Conclusion

In this paper we apply the concept of measure of noncompactness and operator type contraction to study the existence of solution of third order fuzzy differential equations in the sequence space c_0 and ℓ_1. The result is supported with concrete examples. In future, one can apply the above mentioned existence of the solution in different sequence spaces.

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