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A new notion of convergence in gradual normed linear spaces

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Abstract. In this paper we introduce the notion of $\mathcal{I}_\lambda-$statistical convergence of sequences as one of the extensions of $\mathcal{I}-$ statistical convergence in the gradual normed linear spaces. We investigate some fundamental properties of the newly introduced notion and its relationship with $\mathcal{I}-$ statistical convergence. In the end, we introduce and investigate the concept of $\mathcal{I}_\lambda-$statistical limit points, cluster points and establish some implication relations.

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1. Introduction

The idea of fuzzy sets [37] was first introduced by Zadeh in the year 1965 which was an extension of classical set-theoretical concept. Nowadays it has wide applicability in different branches of science and engineering. The term "fuzzy number" plays a crucial role in the study of fuzzy set theory. Fuzzy numbers were basically the generalization of intervals, not numbers. Even fuzzy numbers do not obey a few algebraic properties of the classical numbers. So the term "fuzzy number" is debatable to many authors due to its different behavior. The term "fuzzy intervals" is often used by many authors instead of fuzzy numbers. To overcome the confusion among the researchers, in 2008, Fortin et al. [15] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are mainly known by their respective assignment function which is defined in the interval $(0, 1]$. So in some sense, every real number can be viewed as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have been used in computation and optimization problems.

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In 2011, Sadeqi and Azari [28] first introduced the concept of gradual normed linear space. They studied various properties of the space from both the algebraic and topological points of view. Further progress in this direction has occurred due to Ettefagh et al. [12, 13] and many others. An extensive study on gradual real numbers [1, 11, 23] can be addressed, where many more references can be found.

On the other hand, the notion of statistical convergence was first introduced by Fast [14] and Steinhaus [32] independently in the year 1951. Later on, it was further investigated and studied from the sequence space point of view by Fridy [17], Šalát [36], and many others. In 2000, statistical convergence was extended to $\lambda-$statistical convergence by Mursaleen [26] involving a non-decreasing sequence of positive numbers $\lambda = (\lambda_n)$ satisfying

$$\lambda_1 = 1, \quad \lambda_{n+1} - \lambda_n \leq 1 \text{ and } \lambda_n \to \infty \text{ as } n \to \infty. \tag{1.1}$$

Later on, several works have been carried out in this direction by Çolak and Bektaş [3], Savaş and Mohiuddine [29], and many others. In 2001, the idea of $\mathcal{I}-$convergence was developed by Kostyrko et al. [21] mainly as an extension of statistical convergence. They showed that many other known notions of convergence were a particular type of $\mathcal{I}-$convergence by considering particular ideals. Consequently, this direction successively gets more attention from the researchers and became one of the most active areas of research. Several investigations and extensions of $\mathcal{I}-$convergence can be found in the works of Kostyrko et al. [20], Tripathy and Hazarika [35, 33, 34], and many others.

Combining the notion of statistical convergence and $\mathcal{I}-$convergence, in 2011, Savaş and Das [31] introduced the notion of $\mathcal{I}-$statistical convergence and $\mathcal{I}_\lambda-$statistical convergence. The generalized de la Vallée-Poussin mean is defined by

$$t_n((x_k)) = \tfrac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1,\ n]$. A sequence (x_k) is said to be $\mathcal{I} - [V, \lambda]$ summable [31] to l if

$$\mathcal{I} - \lim_{n \to \infty} t_n((x_k)) = l.$$

For $\mathcal{I} = \mathcal{I}_f$, the class of all finite subsets of $\mathbb{N}$, $\mathcal{I}-[V, \lambda]-$summability reduces to $[V, \lambda]-$summability [22]. Several investigations on $\mathcal{I}-$statistical convergence, $\mathcal{I}_\lambda-$statistical convergence, and their generalized notions have been carried out by Das and Savaş [7], Das et al. [8], Savaş [30], Şengül et al. [6], and many others [2, 9, 10].

The usual convergence of sequences in gradual normed linear spaces was introduced by Ettefagh et al. [13]. Recently, Choudhury and Debnath [4, 5] have extended it to $\mathcal{I}-$convergence and $\mathcal{I}-$statistical convergence using the concept of ideals. From that point of view, the study of $\mathcal{I}_\lambda-$statistical convergence of sequences in gradual normed linear spaces is very natural.

2. Definitions and Preliminaries

Definition 2.1. [15] A gradual real number $\tilde{r}$ is defined by an assignment function $\mathcal{A}_{\tilde{r}} : (0,1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number $\tilde{r}$ is said to be non-negative, if for every $\varphi \in (0,1]$, $\mathcal{A}_{\tilde{r}}(\varphi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

In [15], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

Definition 2.2. Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $\mathcal{A}_{\tilde{r}_1}$ and $\mathcal{A}_{\tilde{r}_2}$ respectively. Then $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $\mathcal{A}_{\tilde{r}_1 * \tilde{r}_2}$ given by

$$\mathcal{A}_{\tilde{r}_1 * \tilde{r}_2}(\varphi) = \mathcal{A}_{\tilde{r}_1}(\varphi) * \mathcal{A}_{\tilde{r}_2}(\varphi),\ \forall \varphi \in (0,1].$$

Then, the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r}(c \in \mathbb{R})$ are defined by

$$\mathcal{A}_{\tilde{r}_1 + \tilde{r}_2}(\varphi) = \mathcal{A}_{\tilde{r}_1}(\varphi) + \mathcal{A}_{\tilde{r}_2}(\varphi) \quad \text{and} \quad \mathcal{A}_{c\tilde{r}}(\varphi) = c\mathcal{A}_{\tilde{r}}(\varphi),\ \ \forall \varphi \in (0,1].$$

For any real number $p \in \mathbb{R}$, the constant gradual real number $\tilde{p}$ is defined by the constant assignment function $\mathcal{A}_{\tilde{p}}(\varphi) = p$ for any $\varphi \in (0,1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $\mathcal{A}_{\tilde{0}}(\varphi) = 0$ and $\mathcal{A}_{\tilde{1}}(\varphi) = 1$ respectively. One can easily verify that $G(\mathbb{R})$ with the gradual addition and gradual scalar multiplication forms a real vector space [15].

Definition 2.3. [28] Let X be a real vector space. The function $\|\cdot\|_G : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X if, for every $\varphi \in (0,1]$, the following conditions hold for any $x, y \in X$:
(i) $\mathcal{A}_{\|x\|_G}(\varphi) = \mathcal{A}_{\tilde{0}}(\varphi)$ if and only if $x = 0$;
(ii) $\mathcal{A}_{\|cx\|_G}(\varphi) = |c|\mathcal{A}_{\|x\|_G}(\varphi)$ for any $c \in \mathbb{R}$;
(iii) $\mathcal{A}_{\|x+y\|_G}(\varphi) \leq \mathcal{A}_{\|x\|_G}(\varphi) + \mathcal{A}_{\|y\|_G}(\varphi)$.

The pair $(X, \|\cdot\|_G)$ is called a gradual normed linear space (GNLS).

Definition 2.4. [28] Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then, (x_k) is said to be gradual convergent to $x \in X$, if for every $\varphi \in (0,1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\varphi)) \in \mathbb{N}$ such that $\mathcal{A}_{\|x_k - x\|_G}(\varphi) < \varepsilon$, $\forall k \geq N$. Symbolically, $x_k \xrightarrow{\|\cdot\|_G} x$.

Example 2.5. [28] Let $X = \mathbb{R}^m$ and for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m, \varphi \in (0,1]$, define $\|\cdot\|_G$ by

$$\mathcal{A}_{\|x\|_G}(\varphi) = e^{\varphi} \sum_{i=1}^{m} |x_i|.$$

Then, $\|\cdot\|_G$ is a gradual norm on $\mathbb{R}^m$, and $(\mathbb{R}^m, \|\cdot\|_G)$ is a GNLS.

Definition 2.6. [21] Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X, if
(i) for each $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, and (ii) for each $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$.

Definition 2.7. [21] Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X, if
(i) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, and (ii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. The filter $\mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

Definition 2.8. [21] Let $\mathcal{I} \subset P(\mathbb{N})$ be a non-trivial ideal on $\mathbb{N}$. A real-valued sequence (x_k) is said to be $\mathcal{I}$-convergent to l, if for each $\varepsilon > 0$, the set

$$C(\varepsilon) = \{k \in \mathbb{N} : \mid x_k - l \mid \geq \varepsilon\}$$

belongs to $\mathcal{I}$. In this case, l is called the $\mathcal{I}$-limit of the sequence (x_k) and it is written as $\mathcal{I} - \lim_{k} x_k = l$ or $x_k \xrightarrow{\mathcal{I}} l$.

Definition 2.9. [19] Let $K \subseteq \mathbb{N}$ and K_n denote the set $\{k \in K : k \in I_n\}$. Then, the $\mathcal{I}_\lambda-$density of K is defined by

$$d_\lambda^{\mathcal{I}}(K) = \mathcal{I} - \lim_{n\to\infty} \tfrac{|K_n|}{\lambda_n},$$

provided that the limit exists, where $I_n = [n - \lambda_n + 1,\ n]$, $n \in \mathbb{N}$.

Definition 2.10. [31] A real-valued sequence (x_k) is said to be $\mathcal{I}_\lambda-$statistically convergent to l, if for every $\varepsilon > 0, \delta > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{\lambda_n}|\{k \in I_n : |x_k - l| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

Equivalently, $d_\lambda^{\mathcal{I}}(\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}) = 0$. Symbolically, $x_k \xrightarrow{\mathcal{I}_\lambda^{st}} l$.

If (x_{m_k}) is a subsequence of the sequence (x_k) and $d_\lambda^{\mathcal{I}}(\{m_1, m_2, \cdots\}) = 0$, then (x_{m_k}) is said to be an $\mathcal{I}_\lambda-$thin subsequence of (x_k).

On the other hand, if $d_\lambda^{\mathcal{I}}(\{m_1, m_2, \cdots\}) > 0$ or the set $\{m_1, m_2, \cdots\}$ fails to have $\mathcal{I}_\lambda-$density, then (x_{m_k}) is said to be an $\mathcal{I}_\lambda-$nonthin subsequence of (x_k).

Definition 2.11. [24] A number $x_0 \in \mathbb{R}$ is said to be an $\mathcal{I}_\lambda-$statistical limit point of a real-valued sequence (x_k), if there exists an $\mathcal{I}_\lambda-$nonthin subsequence of (x_k) that converges to x_0.

Definition 2.12. [24] A number $x_0 \in \mathbb{R}$ is said to be an $\mathcal{I}_\lambda-$statistical cluster point of a real-valued sequence (x_k), if for every $\varepsilon > 0$,

$$d_\lambda^{\mathcal{I}}(\{k \in \mathbb{N} : |x_k - x_0| < \varepsilon\}) \neq 0.$$

Remark 2.13. (i) If we take $\lambda_n = n$, then Definition 2.9, Definition 2.10, Definition 2.11, and Definition 2.12 reduces to the definition of $\mathcal{I}-$natural density [25], $\mathcal{I}-$statistical convergence [31], $\mathcal{I}-$statistical limit point [10], and $\mathcal{I}-$statistical cluster point [27] respectively.
(ii) If we take $\lambda_n = n$ and $\mathcal{I} = \mathcal{I}_f$, then Definition 2.9, Definition 2.10, Definition 2.11, and Definition 2.12 turns to the definition of natural density [16], statistical convergence [17], statistical limit point [18], and statistical cluster point [18] respectively.

Definition 2.14. [4] Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then, (x_k) is said to be gradual $\mathcal{I}-$convergent (in short, $\mathcal{I} - \|\cdot\|_G$ convergent) to $x \in X$, if for every $\varphi \in (0,1]$ and $\varepsilon > 0$, the set

$$B(\varphi, \varepsilon) = \{k \in \mathbb{N} : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\} \in \mathcal{I}.$$

Symbolically, $x_k \xrightarrow{\mathcal{I}-\|\cdot\|_G} x$.

Definition 2.15. [5] Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then, (x_k) is said to be gradually $\mathcal{I}-$statistical convergent (in short, $\mathcal{I}st-\|\cdot\|_G$ convergent) to $x \in X$, if for every $\varphi \in (0,1]$ and $\varepsilon > 0, \delta > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{n} \left|\left\{k \leq n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Symbolically, $x_k \xrightarrow{\mathcal{I}st-\|\cdot\|_G} x$.

The set of all gradual $\mathcal{I}-$statistical convergent sequences is denoted by $\mathcal{I}^{st}(G)$.

Throughout the paper, we use $\mathbf{0}$, $\lambda = (\lambda_n)$, and $\mathcal{I}$ to denote the $m-$tuple $(0, 0, \cdots, 0, 0)$, a non-decreasing sequence of positive numbers that satisfies (1.1), and a non-trivial admissible ideal in $\mathbb{N}$ respectively.

3. Main Results

Definition 3.1. Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then, (x_k) is said to be $\mathcal{I} - [V, \lambda]_G$ summable to $x \in X$, if for every $\varphi \in (0,1]$ and $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \left(\sum_{k \in I_n} \mathcal{A}_{\|x_k - x\|_G}(\varphi)\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

In this case, we write $x_k \xrightarrow{\mathcal{I}-[V,\lambda]_G} x$.

We shall denote the set of all $\mathcal{I} - [V, \lambda]_G$ summable sequences by $\mathcal{I} - [V, \lambda]_G$.

Definition 3.2. Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then, (x_k) is said to be gradual $\mathcal{I}_\lambda-$statistical convergent (in short, $\mathcal{I}_\lambda^{st} - \|\cdot\|_G$ convergent) to $x \in X$, if for every $\varphi \in (0,1]$ and $\varepsilon > 0, \delta > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Equivalently, $d_\lambda^{\mathcal{I}}\left(\left\{k\in\mathbb{N}:\mathcal{A}_{\|x_k-x\|_G}(\varphi)\geq\varepsilon\right\}\right)=0$. In this case, we write, $x_k\xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G}x$.

We shall denote the set of all gradual $\mathcal{I}_\lambda-$statistical convergent sequences by $\mathcal{I}_\lambda^{st}(G)$.

Example 3.3. Let $X=\mathbb{R}^m$ and $\|\cdot\|_G$ be the norm defined in Example 2.5. Take a fixed $S\in\mathcal{I}$. Then, the sequence (x_k) defined by

$$x_k=\begin{cases}(0,0,\cdots,0,k), & n-[\sqrt{\lambda_n}]+1\leq k\leq n, n\notin S\\ (0,0,\cdots,0,k), & n-\lambda_n+1\leq k\leq n, n\in S\\ \mathbf{0}, & otherwise\end{cases}$$

is $\mathcal{I}_\lambda^{st}-\|\cdot\|_G$ convergent to $\mathbf{0}$.
Justification. For any $\varepsilon>0$ $(0<\varepsilon<1)$, since

$$\frac{1}{\lambda_n}\left|\left\{k\in I_n:\mathcal{A}_{\|x_k-\mathbf{0}\|_G}(\varphi)\geq\varepsilon\right\}\right|=\frac{[\sqrt{\lambda_n}]}{\lambda_n}\to 0$$

as $n\to\infty$ and $n\notin S$, so for every $\delta>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\left|\left\{k\in I_n:\mathcal{A}_{\|x_k-\mathbf{0}\|_G}(\varphi)\geq\varepsilon\right\}\right|\geq\delta\right\}\subset S\cup\{1,2,\cdots,k_1\} \tag{3.1}$$

for some $k_1\in\mathbb{N}$. Since $\mathcal{I}$ is admissible, it follows from (3.1) that $x_k\xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G}\mathbf{0}$.

Theorem 3.4. *Let (x_k) be a sequence in the GNLS $(X,\|\cdot\|_G)$. Then,*
(i) $x_k\xrightarrow{\mathcal{I}-[V,\lambda]_G}x$ implies $x_k\xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G}x$ but the converse is not true;
(ii) If (x_k) is gradually bounded and $x_k\xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G}x$ then, $x_k\xrightarrow{\mathcal{I}-[V,\lambda]_G}x$.

Proof. (i) Let $x_k\xrightarrow{\mathcal{I}-[V,\lambda]_G}x$. Then, for any $\varepsilon>0$ and $\varphi\in(0,1]$, the following inequation holds

$$\begin{aligned}\sum_{k\in I_n}\mathcal{A}_{\|x_k-x\|_G}(\varphi)&\geq\sum_{\substack{k\in I_n\\ \mathcal{A}_{\|x_k-x\|_G}(\varphi)\geq\varepsilon}}\mathcal{A}_{\|x_k-x\|_G}(\varphi)\\ &\geq\varepsilon\left|\left\{k\in I_n:\mathcal{A}_{\|x_k-x\|_G}(\varphi)\geq\varepsilon\right\}\right|.\end{aligned}$$

Consequently for a given $\delta>0$,

$$\frac{1}{\lambda_n}\left|\left\{k\in I_n:\mathcal{A}_{\|x_k-x\|_G}(\varphi)\geq\varepsilon\right\}\right|\geq\delta\Rightarrow\frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{A}_{\|x_k-x\|_G}(\varphi)\geq\varepsilon\delta.$$

This implies that

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\left|\left\{k\in I_n:\mathcal{A}_{\|x_k-x\|_G}(\varphi)\geq\varepsilon\right\}\right|\geq\delta\right\}\\ &\qquad\subseteq\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{A}_{\|x_k-x\|_G}(\varphi)\geq\varepsilon\delta\right\}\in\mathcal{I}.\end{aligned}$$

Hence, $x_k \xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G} x$.

For the converse part, consider Example 3.3. It was shown that $x_k \xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G} \mathbf{0}$. But clearly

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{A}_{\|x_k-\mathbf{0}\|_G}(\varphi)\to\infty \text{ as } n\to\infty.$$

This shows that (x_k) is not $\mathcal{I}-[V,\lambda]_G$ summable to $\mathbf{0}$.

(ii) Let $x_k \xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G} x$ and (x_k) is gradually bounded, say

$$\mathcal{A}_{\|x_k-x\|_G}(\varphi)\le M,\ \forall k\in\mathbb{N}.$$

Then, for any $\varepsilon>0$ and $\varphi\in(0,1]$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{A}_{\|x_k-x\|_G}(\varphi)&=\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \mathcal{A}_{\|x_k-x\|_G}(\varphi)\ge\varepsilon}}\mathcal{A}_{\|x_k-x\|_G}(\varphi)\\&\quad+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \mathcal{A}_{\|x_k-x\|_G}(\varphi)<\varepsilon}}\mathcal{A}_{\|x_k-x\|_G}(\varphi)\\&\le\frac{M}{\lambda_n}\left|\left\{k\in I_n:\mathcal{A}_{\|x_k-x\|_G}(\varphi)\ge\varepsilon\right\}\right|+\varepsilon.\end{aligned}$$

Let

$$B(\varphi,\varepsilon)=\left\{n\in\mathbb{N}:\tfrac{1}{\lambda_n}\left|\left\{k\in I_n:\mathcal{A}_{\|x_k-x\|_G}(\varphi)\ge\varepsilon\right\}\right|\ge\tfrac{\varepsilon}{M}\right\}.$$

Then, by hypothesis, $B(\varphi,\varepsilon)\in\mathcal{I}$ and for any $n\in\mathbb{N}\setminus B(\varphi,\varepsilon)$,

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{A}_{\|x_k-x\|_G}(\varphi)<2\varepsilon.$$

Consequently, the inclusion

$$\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{A}_{\|x_k-x\|_G}(\varphi)\ge 2\varepsilon\right\}\subseteq B(\varphi,\varepsilon)$$

holds and the rest follows from the hereditary property of $\mathcal{I}$. □

Theorem 3.5. *Let (x_k) be a sequence in the GNLS $(X,\|\cdot\|_G)$ such that $x_k \xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G} x$. Then x is unique.*

Proof. If possible suppose $x_k \xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G} x$ and $x_k \xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G} y$ holds for $x,y\in X$ with $x\neq y$. Then, for any $\varepsilon>0$ and $\varphi\in(0,1]$, we have

$$d_\lambda^{\mathcal{I}}(B_1(\varphi,\varepsilon))=d_\lambda^{\mathcal{I}}(B_2(\varphi,\varepsilon))=1,$$

where

$$B_1(\varphi,\varepsilon)=\left\{k\in\mathbb{N}:\mathcal{A}_{\|x_k-x\|_G}(\varphi)<\varepsilon\right\}$$

and

$$B_2(\varphi,\varepsilon)=\left\{k\in\mathbb{N}:\mathcal{A}_{\|x_k-y\|_G}(\varphi)<\varepsilon\right\}.$$

Choose $m\in B_1(\varphi,\varepsilon)\cap B_2(\varphi,\varepsilon)$, then $\mathcal{A}_{\|x_m-x\|_G}(\varphi)<\varepsilon$ and $\mathcal{A}_{\|x_m-y\|_G}(\varphi)<\varepsilon$. Consequently,

$$\mathcal{A}_{\|x-y\|_G}(\varphi)\leq\mathcal{A}_{\|x_m-x\|_G}(\varphi)+\mathcal{A}_{\|x_m-y\|_G}(\varphi)<\varepsilon+\varepsilon=2\varepsilon.$$

Since ε is arbitrary, so $\mathcal{A}_{\|x-y\|_G}(\varphi)=\mathcal{A}_{\tilde{0}}(\varphi)$ and so we must have $x=y$. □

Theorem 3.6. *Let (x_k) be a sequence in the GNLS $(X,\|\cdot\|_G)$ such that $x_k\xrightarrow{\lambda st-\|\cdot\|_G}x$. Then, $x_k\xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G}x$.*

Proof. $x_k\xrightarrow{\lambda st-\|\cdot\|_G}x$ implies that for every $\varphi\in(0,1]$ and $\varepsilon>0$,

$$\lim_n\frac{1}{\lambda_n}\left|\left\{k\in I_n:\mathcal{A}_{\|x_k-x\|_G}(\varphi)\geq\varepsilon\right\}\right|=0.$$

So for every $\varphi\in(0,1]$, $\varepsilon>0$, and $\delta>0$, the set

$$\left\{n\in\mathbb{N}:\tfrac{1}{\lambda_n}\left|\left\{k\in I_n:\mathcal{A}_{\|x_k-x\|_G}(\varphi)\geq\varepsilon\right\}\right|\geq\delta\right\}$$

is a finite set and eventually becomes a member of $\mathcal{I}$, as $\mathcal{I}$ is admissible. □

Remark 3.7. The converse of the above theorem is not necessarily true. One can easily verify the fact by considering Example 3.3.

Remark 3.8. For a sequence (x_k) in the GNLS $(X,\|\cdot\|_G)$, $x_k\xrightarrow{\mathcal{I}-\|\cdot\|_G}x$ implies $x_k\xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G}x$. But the converse is not necessarily true.

Example 3.9. Let $X=\mathbb{R}^m$ and $\|\cdot\|_G$ be the norm defined in Example 2.5. Consider $\lambda_n=n$ and the ideal $\mathcal{I}=\mathcal{I}_f$, ideal consisting of all finite subsets of $\mathbb{N}$. Define the sequence (x_k) as

$$x_k=\begin{cases}\mathbf{0}, & k=n^3,n\in\mathbb{N}\\(0,0,\cdots,0,1), & otherwise\end{cases}.$$

Then, (x_k) is $\mathcal{I}_\lambda^{st}-\|\cdot\|_G$ convergent to $(0,0,\cdots,0,1)$ but not $\mathcal{I}-\|\cdot\|_G$ convergent to $(0,0,\cdots,0,1)$.

Theorem 3.10. *Let (x_k) be a sequence in the GNLS $(X,\|\cdot\|_G)$. If every subsequence of (x_k) is $\mathcal{I}_\lambda^{st}-\|\cdot\|_G$ convergent to $x\in X$, then (x_k) is also $\mathcal{I}_\lambda^{st}-\|\cdot\|_G$ convergent to x.*

Proof. If possible suppose (x_k) is not $\mathcal{I}_\lambda^{st} - \|\cdot\|_G$ convergent to x in spite of having all the subsequences $\mathcal{I}_\lambda^{st} - \|\cdot\|_G$ converging to x. Then, by definition, there exists particular $\varepsilon > 0$ and $\delta > 0$ such that the set

$$B = B(\varphi, \varepsilon, \delta) = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\}\right| \geq \delta\right\} \notin \mathcal{I}.$$

Now admissibility of $\mathcal{I}$ ensures that B contains an infinite number of elements. Put $B = \{n_1 < n_2 < \cdots < n_j < \cdots\}$ and define $y_j = x_{k_j}, j \in \mathbb{N}$. Then, (y_j) is a subsequence of (x_k) that is not $\mathcal{I}_\lambda^{st} - \|\cdot\|_G$ converging to x, which contradicts our assumption. □

Remark 3.11. The converse of the above theorem is not necessarily true. One can easily verify this fact by considering Example 3.9.

Theorem 3.12. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, \|\cdot\|_G)$ such that $x_k \xrightarrow{\mathcal{I}_\lambda^{st} - \|\cdot\|_G} x$ and $y_k \xrightarrow{\mathcal{I}_\lambda^{st} - \|\cdot\|_G} y$. Then,*
(i) $x_k + y_k \xrightarrow{\mathcal{I}_\lambda^{st} - \|\cdot\|_G} x + y$ and
(ii) $cx_k \xrightarrow{\mathcal{I}_\lambda^{st} - \|\cdot\|_G} cx$, $c \in \mathbb{R}$.

Proof. (i) From the hypothesis, we can conclude that for every $\varphi \in (0, 1]$ and $\varepsilon > 0, \delta > 0$, the two sets $C_1 = C_1(\varphi, \varepsilon, \delta), C_2 = C_2(\varphi, \varepsilon, \delta) \in \mathcal{I}$, where

$$C_1 = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\delta}{2}\right\}$$

and

$$C_2 = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\left\{k \in I_n : \mathcal{A}_{\|y_k - y\|_G}(\varphi) \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\delta}{2}\right\}.$$

Then, $(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \in \mathcal{F}(\mathcal{I})$ and so $(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \neq \emptyset$. Choose $n \in (\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2)$. Then, the following inequality

$$\begin{aligned} &\frac{1}{\lambda_n} \left|\{k \in I_n : \mathcal{A}_{\|(x_k + y_k) - (x + y)\|_G}(\varphi) \geq \varepsilon\}\right| \\ &\qquad \leq \frac{1}{\lambda_n} \left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \frac{\varepsilon}{2}\right\}\right| \\ &\qquad\qquad + \frac{1}{\lambda_n} \left|\left\{k \in I_n : \mathcal{A}_{\|y_k - y\|_G}(\varphi) \geq \frac{\varepsilon}{2}\right\}\right|, \end{aligned}$$

holds and consequently, we have the following inclusion:

$$(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\{k \in I_n : \mathcal{A}_{\|(x_k + y_k) - (x + y)\|_G}(\varphi) \geq \varepsilon\}\right| < \delta\right\}. \tag{3.2}$$

Now as $(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \in \mathcal{F}(\mathcal{I})$, so the set in the right-hand side of (3.2) also belongs to $\mathcal{F}(\mathcal{I})$ which means that $x_k + y_k \xrightarrow{\mathcal{I}_\lambda^{st} - \|\cdot\|_G} x + y$.

(ii) If $c = 0$, there is nothing to prove. So let us assume $c \neq 0$. Then, for every $\varphi \in (0,1]$ and $\varepsilon > 0$, the following inequation

$$\begin{aligned}\frac{1}{\lambda_n}\left|\left\{k \in I_n : \mathcal{A}_{\|cx_k - cx\|_G}(\varphi) \geq \varepsilon\right\}\right| &= \frac{1}{\lambda_n}\left|\left\{k \in I_n : |c|\mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \\ &\leq \frac{1}{\lambda_n}\left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \frac{\varepsilon}{|c|}\right\}\right|\end{aligned}$$

holds good and the result follows. □

Theorem 3.13. *$\mathcal{I}_\lambda^{st}(G) \supseteq \mathcal{I}^{st}(G)$ provided that $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$.*

Proof. For any $\varepsilon > 0$ and $\varphi \in (0,1]$,

$$\begin{aligned}\frac{1}{n}\left|\left\{k \leq n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| &\geq \frac{1}{n}\left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \\ &\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right|\end{aligned}$$

Now if $\liminf_{n\to\infty} \frac{\lambda_n}{n} = p$, then by definition, the set $\left\{n \in \mathbb{N} : \frac{\lambda_n}{n} < \frac{p}{2}\right\}$ contains a finite number of elements and consequently the following inclusion holds for any $\delta > 0$:

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \geq \frac{p\delta}{2}\right\} \cup \left\{n \in \mathbb{N} : \frac{\lambda_n}{n} < \frac{p}{2}\right\}.\end{aligned}$$

Now if $x_k \xrightarrow{\mathcal{I}st-\|\cdot\|_G} x$ holds, then the set on the right-hand side belongs to $\mathcal{I}$ due to the admissibility of $\mathcal{I}$ and as a consequence, the set on the left-hand side also belongs to $\mathcal{I}$. Hence, $x_k \xrightarrow{\mathcal{I}_\lambda^{st}-\|\cdot\|_G} x$. □

Theorem 3.14. *$\mathcal{I}_\lambda^{st}(G) \subseteq \mathcal{I}^{st}(G)$ provided that $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$.*

Proof. Let $\delta > 0$ be given. Since $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$, we can have a $m \in \mathbb{N}$ satisfying

$$\left|\frac{\lambda_n}{n} - 1\right| < \frac{\delta}{2}\ \forall n \geq m.$$

Now for any $\varepsilon > 0$ and $\varphi \in (0,1]$,

$$\begin{aligned}&\frac{1}{n}\left|\left\{k \leq n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \\ &= \frac{1}{n}\left|\left\{k \leq n - \lambda_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| + \frac{1}{n}\left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \\ &\leq \frac{n - \lambda_n}{n} + \frac{1}{\lambda_n}\left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \\ &\leq 1 - \left(1 - \frac{\delta}{2}\right) + \frac{1}{\lambda_n}\left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \\ &= \frac{\delta}{2} + \frac{1}{\lambda_n}\left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right|,\end{aligned}$$

holds for all $n \geq m$. Therefore, the following inclusion holds:

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \geq \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \geq \frac{\delta}{2}\right\} \cup \{1, 2, 3, \cdots, m\}.$$

Now if $x_k \xrightarrow{\mathcal{I}_\lambda^{st} - \|\cdot\|_G} x$ holds, then the set on the right-hand side belongs to $\mathcal{I}$ due to the admissibility of $\mathcal{I}$ and as a consequence, the set on the left-hand side also belongs to $\mathcal{I}$. Hence, $x_k \xrightarrow{\mathcal{I}st - \|\cdot\|_G} x$. □

Definition 3.15. Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then, $x_0 \in X$ is said to be gradual $\mathcal{I}_\lambda^{st}-$limit point of (x_k), if there exists an $\mathcal{I}_\lambda-$nonthin subsequence of (x_k) that gradual converges to x_0.

For any sequence (x_k), the set of all gradual $\mathcal{I}_\lambda^{st}-$limit points is denoted by

$$\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(x_k)}\right).$$

Definition 3.16. Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then, $x_0 \in X$ is said to be gradual $\mathcal{I}_\lambda^{st}-$cluster point of (x_k), if for any $\varepsilon > 0$ and $\varphi \in (0, 1]$,

$$d_\lambda^{\mathcal{I}}\left(\left\{k \in \mathbb{N} : \mathcal{A}_{\|x_k - x_0\|_G}(\varphi) < \varepsilon\right\}\right) \neq 0.$$

For any sequence (x_k), the set of all gradual $\mathcal{I}_\lambda^{st}-$cluster points is denoted by

$$\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(x_k)}\right).$$

Theorem 3.17. *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$ such that $x_k \xrightarrow{\mathcal{I}_\lambda^{st} - \|\cdot\|_G} x$. Then, $\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(x_k)}\right) = \{x\}$.*

Proof. If possible suppose $\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(x_k)}\right)$ contains one more element y such that $y \neq x$. Then, by definition, there exists a set $M \subset \mathbb{N}$ with $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \notin \mathcal{I}$ such that $x_{m_k} \xrightarrow{st - \|\cdot\|_G} x$. Let

$$B = B(\varphi, \varepsilon, \delta) = \left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \in I_n : \mathcal{A}_{\|x_k - y\|_G}(\varphi) \geq \varepsilon\right\}\right| \geq \delta\right\}.$$

Then B is a finite set, so $\mathbb{N} \setminus B \in \mathcal{F}(\mathcal{I})$. Now, since $x_k \xrightarrow{\mathcal{I}_\lambda^{st} - \|\cdot\|_G} x$, so for any $\varphi \in (0, 1]$ and $\varepsilon > 0, \delta > 0$, the set

$$C = C(\varphi, \varepsilon, \delta) = \left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}).$$

Put

$$D = D(\varphi, \varepsilon, \delta) = \left\{n \in M : \frac{1}{n} \left|\left\{k \in I_n : \mathcal{A}_{\|x_k - x\|_G}(\varphi) \geq \varepsilon\right\}\right| \geq \delta\right\}.$$

Then since $\mathbb{N}\setminus D \supset C$, so $\mathbb{N}\setminus D \in \mathcal{F}(\mathcal{I})$. Thus we have, $(\mathbb{N}\setminus B)\cap(\mathbb{N}\setminus D) \in \mathcal{F}(\mathcal{I})$ and eventually $(\mathbb{N} \setminus B) \cap (\mathbb{N} \setminus D) \neq \emptyset$. Let $j \in (\mathbb{N} \setminus B) \cap (\mathbb{N} \setminus D)$ and take $\varepsilon = \mathcal{A}_{\left\|\frac{x-y}{2}\right\|_G}(\varphi)$. Then we have,

$$\frac{1}{j}\left|\left\{k \leq j : \mathcal{A}_{\|x_j - x\|_G}(\varphi) \geq \varepsilon\right\}\right| < \delta$$

and

$$\frac{1}{j}\left|\left\{k \leq j : \mathcal{A}_{\|x_j - y\|_G}(\varphi) \geq \varepsilon\right\}\right| < \delta.$$

Now choosing δ sufficiently small we can have an element say

$$p \in \left\{k \leq j : \mathcal{A}_{\|x_j - x\|_G}(\varphi) \geq \varepsilon\right\} \cap \left\{k \leq j : \mathcal{A}_{\|x_j - y\|_G}(\varphi) \geq \varepsilon\right\}.$$

But then,

$$\varepsilon = \mathcal{A}_{\left\|\frac{x-y}{2}\right\|_G}(\varphi) \leq \frac{1}{2}\left(\mathcal{A}_{\|x_p - x\|_G}(\varphi) + \mathcal{A}_{\|x_p - y\|_G}(\varphi)\right) < \frac{1}{2}(\varepsilon + \varepsilon) = \varepsilon,$$

which is a contradiction. □

Theorem 3.18. *For any sequence (x_k) in the GNLS $(X, \|\cdot\|_G)$,*

$$\mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Lambda_{(x_k)}\right) \subseteq \mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Gamma_{(x_k)}\right).$$

Proof. Let $x_0 \in \mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Lambda_{(x_k)}\right)$. Then, there exists a $\mathcal{I}_\lambda-$nonthin subsequence (x_{m_k}) such that $x_{m_k} \xrightarrow{\|\cdot\|_G} x_0$, where $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subseteq \mathbb{N}$. Let $\varepsilon > 0$ be given. Since, $x_{m_k} \xrightarrow{\|\cdot\|_G} x_0$, so for any $\varphi \in (0, 1]$, the set $B = \left\{m_k : \mathcal{A}_{\|x_{m_k} - x_0\|_G}(\varphi) \geq \varepsilon\right\}$ is a finite set. Consequently, $d_\lambda^{\mathcal{I}}(B) = 0$. Now as the inclusion

$$M \subseteq \left\{k \in \mathbb{N} : \mathcal{A}_{\|x_k - x_0\|_G}(\varphi) < \varepsilon\right\} \cup B$$

holds and $d_\lambda^{\mathcal{I}}(M) \neq 0$, so we must have

$$d_\lambda^{\mathcal{I}}\left(\left\{k \in \mathbb{N} : \mathcal{A}_{\|x_k - x_0\|_G}(\varphi) < \varepsilon\right\}\right) \neq 0.$$

This means that $x_0 \in \mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Gamma_{(x_k)}\right)$. Since, $x_0 \in \mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Lambda_{(x_k)}\right)$ is arbitrary, so

$$\mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Lambda_{(x_k)}\right) \subseteq \mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Gamma_{(x_k)}\right)$$

holds and the proof is complete. □

Theorem 3.19. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, \|\cdot\|_G)$ such that $d_\lambda^{\mathcal{I}}(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$. Then, (i) $\mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Lambda_{(x_k)}\right) = \mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Lambda_{(y_k)}\right)$ and (ii) $\mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Gamma_{(x_k)}\right) = \mathcal{I}_\lambda^{st} - \|\cdot\|_G\left(\Gamma_{(y_k)}\right)$.*

Proof. (i) Let $x_0 \in \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(x_k)}\right)$. Then, there exists a $\mathcal{I}_\lambda-$nonthin subsequence (x_{m_k}) such that $x_{m_k} \xrightarrow{\|\cdot\|_G} x_0$, where $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subseteq \mathbb{N}$. Let $\varepsilon > 0$ be given. Since $d_\lambda^{\mathcal{I}}\left(\{k \in \mathbb{N} : x_k \neq y_k\}\right) = 0$ holds, so we must have $d_\lambda^{\mathcal{I}}\left(\{m_k \in \mathbb{N} : x_{m_k} = y_{m_k}\}\right) \neq 0$. Therefore from the latter set we have an $\mathcal{I}_\lambda-$nonthin subsequence (y_{m_k}) such that $y_{m_k} \xrightarrow{\|\cdot\|_G} x_0$. Consequently, $x_0 \in \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(y_k)}\right)$. As $x_0 \in \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(x_k)}\right)$ is arbitrary, $\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(x_k)}\right) \subseteq \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(y_k)}\right)$. Applying similar technique, we can prove that $\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(x_k)}\right) \supseteq \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(y_k)}\right)$. Hence, $\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(x_k)}\right) = \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Lambda_{(y_k)}\right)$.

(ii) Suppose $x_0 \in \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(x_k)}\right)$. Then by definition, for any $\varepsilon > 0$ and $\varphi \in (0,1]$,

$$d_\lambda^{\mathcal{I}}\left(\left\{k \in \mathbb{N} : \mathcal{A}_{\|x_k - x_0\|_G}(\varphi) < \varepsilon\right\}\right) \neq 0.$$

Let B denote the set $\{k \in \mathbb{N} : x_k = y_k\}$. Then, $d_\lambda^{\mathcal{I}}(B) = 1$ and eventually

$$d_\lambda^{\mathcal{I}}\left(\left\{k \in \mathbb{N} : \mathcal{A}_{\|x_k - x_0\|_G}(\varphi) < \varepsilon\right\} \cap B\right) \neq 0.$$

This implies that

$$x_0 \in \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(y_k)}\right).$$

Since $x_0 \in \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(x_k)}\right)$ is arbitrary, so we have

$$\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(x_k)}\right) \subseteq \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(y_k)}\right).$$

Applying similar technique we can show that

$$\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(x_k)}\right) \supseteq \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(y_k)}\right).$$

Hence, $\mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(x_k)}\right) = \mathcal{I}_\lambda^{st} - \|\cdot\|_G \left(\Gamma_{(y_k)}\right)$. □

Conclusion

In this paper, we have investigated a few fundamental properties of $\mathcal{I}_\lambda-$statistical convergence in the gradual normed linear spaces. We also introduced $\mathcal{I}-[V, \lambda]_G$ summability in the gradual normed linear spaces and established Theorem 3.4 to reveal the interrelationship between the notions. Finally, we have introduced the concept of $\mathcal{I}_\lambda-$statistical limit points, cluster points and established Theorem 3.18 and Theorem 3.19 to study their interrelationship and several properties.

Summability theory and the convergence of sequences have wide applications in various branches of mathematics particularly, in mathematical analysis. Research in this direction based on gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. The obtained results may be useful for future researchers to explore various notions of convergences in the gradual normed linear spaces in more detail.

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