Novi Sad J. Math. Vol. 54, No. 2, 2024, 109-116
https://doi.org/10.30755/NSJOM.14853

Semi-extending ideals and st-closed ideals in lattices[1]

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Abstract. This paper aims to study some results on semi-essential ideals of lattices. It defines fully essential and fully prime lattices and studies relation between them. The concepts of the st-closed ideal and semi-extending ideal in a lattice are introduced. Some characterizations of semi-extending lattices are given with respect to semi-essential ideals and st-closed ideals.

AMS Mathematics Subject Classification (2010): 06B05; 06B10

Key words and phrases: essential ideal; semi-essential ideal; extending ideal; semi-extending ideal; fully essential lattice; fully prime lattice; st-closed ideals

1. Introduction and Preliminaries

The study of extending modules and their generalization has been studied by several authors in the last two decades. In [7], Mijbass and Abdullah introduced the concept of semi-essential submodules and semi-uniform submodules. In [1], Ahmed and Abbas studied also studied semi-essential submodules. In [6], Ahmed and Abbas studied semi-extending modules as a generalization of extending modules. They obtained conditions under which semi-extending modules can be extending. Module M is called semi-extending if every submodule of M is semi-essential in the direct summand of M. Ahmed and Abbas also studied st-closed submodules in [2]. In [4], Behboodi et. al studied fully prime module such that every proper submodule is prime and in [3], Ahmed and Dakheel called a module M to be fully essential if every nonzero semi-essential submodule of M is an essential submodule of M.

In [9], Nimbhorkar and Shroff studied the concept of essential ideals and their various properties. They defined extending ideals in a lattice and proved some important characterizations of such ideals in modular lattices. As a generalization of essential ideals, Nimbhorkar and Patil [8] defined semi-essential ideals in a lattice and studied some of their results.

[1]Author dedicates this article to Late Professor Dr. Shriram Nimbhorkar. His valuable suggestions helped to improve the manuscript. Authors are thankful for his encouragement and guidance.

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The present study introduces and studies semi-extending lattices. It defines st-closed ideals and semi-uniform ideals in a lattice and studies some of their properties. It provide conditions under which a semi-extending lattice can be extending. Let us recall some concepts of lattice theory from Grätzer [5].

Definition 1.1. A poset $(P, \leq)$ is called lattice if infimum and supremum exist for every pair a, b in P.

Definition 1.2. A non-empty subset I of lattice L is called ideal if the following conditions hold:

1. $a, b \in I$ implies $a \vee b \in I$.

2. If $x \leqslant a$ for $a \in I$, $x \in L$ then $x \in I$.

An ideal generated by an element $a \in L$ is denoted by $(a]$ and is defined as $(a] = \{x \in L : x \leqslant a\}$.

Definition 1.3. An ideal I of a lattice L is said to be a prime ideal if for $a, b \in L$, $a \wedge b \in I$ implies either $a \in I$ or $b \in I$.

It is noted that every lattice is an ideal of itself. The set $Id(L)$ of all ideals of a lattice L forms a lattice under set inclusion as the partial order. In fact, if L is a lattice with the least element 0, then $Id(L)$ is a complete lattice.

Throughout this paper, L denotes a lattice with the least element 0. The following concepts are from [9].

Definition 1.4. Let I and J be any two nonzero ideals of a lattice L. The ideal I is said to be essential in J if $I \cap K \neq (0]$, for every nonzero ideal $K \subseteq J$ of L. It is denoted by $I \leqslant_e J$.

If $I \leqslant_e J$, then J is called an essential extension of I. The ideal I is essential in lattice L if $I \cap K \neq (0]$, for every nonzero ideal K of L.

Definition 1.5. Let I and J be any two nonzero ideals of a lattice L. The ideal I is said to be closed in J if it has no proper essential extension in J. The ideal I is closed in a lattice L if it has no proper essential extension in L.

Definition 1.6. Let I and J be any two nonzero ideals of a lattice L. The ideal I is said to be uniform in J if every nonzero ideal $K \subseteq I$ is essential in I. The ideal I is uniform in lattice L if every ideal contained in I is essential.

Definition 1.7. Let $I, J, K \in Id(L)$ such that $I \cap J = (0]$ and $K = I \vee J$ then I and J are called direct summands of K.

Definition 1.8. An ideal I of a lattice L is said to be extending if every ideal J contained in I is essential in a direct summand of I.

2. Semi-essential Ideals

This section discusses some results on semi-essential ideals in lattices. The concept of a semi-essential ideal is defined as follows [8].

Definition 2.1. An ideal I of a lattice L is said to be a semi-essential ideal if $I \cap P \neq (0]$, for every nonzero prime ideal P of L. It is denoted by $I \leqslant_{se} L$.

Let us say that $I \leqslant_{se} J$ if for every prime ideal $P \subseteq J$, $I \cap P \neq (0]$.
If $I \leqslant_{se} J$, then J is called a semi-essential extension of I.

Remark 2.2. Every essential ideal is a semi-essential ideal in L. However, converse does not need to be true.

The following remark is obvious by the definition.

Remark 2.3. Let L be a lattice and I and J be ideals of L such that $I \subseteq J$. If I is prime in L, then I is prime in J.

Lemma 2.4. *Let L be a lattice and I and J be ideals of L such that $J \nsubseteq I$ and I is prime in L. Then $I \cap J$ is a prime ideal in J.*

Proof. Let I be a prime ideal of L and $J \nsubseteq I$. Let $a, b \in J \subseteq L$ such that $a \wedge b \in I \cap J$. Then $a \wedge b \in I$ and $a \wedge b \in J$. But I is prime therefore $a \in I$ or $b \in I$. Hence, $a \in I \cap J$ or $b \in I \cap J$. Therefore, $I \cap J$ is a prime ideal of J. □

Theorem 2.5. *Let L be a lattice with 0 and I, J, K be ideals of L such that $(0] \neq I \subseteq J \subseteq K$. If $I \leqslant_{se} J$ and $J \leqslant_{se} K$ then $I \leqslant_{se} K$.*

Proof. Let P be a prime ideal of K such that $I \cap P = (0]$.
Now $(0] = I \cap P = (I \cap J) \cap P = I \cap (J \cap P)$. If $J \subseteq P$, then $(0] = I \cap (J \cap P) = I \cap J$. But $I \subseteq J$, which implies $I = (0]$, a contradiction. Hence, $J \nsubseteq P$. By Lemma 2.4, $J \cap P$ is the prime ideal in J. Further, $I \leqslant_{se} J$ and $I \cap (J \cap P) = (0]$ implies $J \cap P = (0]$. But $J \leqslant_{se} K$ and P is the prime ideal in K, which implies $P = (0]$. Hence, $I \leqslant_{se} K$. □

Now, a fully prime lattice is defined as follows:

Definition 2.6. A lattice L is called fully prime if every proper ideal of L is the prime ideal of L.

Remark 2.7. Consider the lattice shown in Figure 1, in which all ideals $\{0, a\}$, $\{0, b\}$, $\{0, a, b, c\}$ are prime ideals. Therefore, the lattice is fully prime.
Consider the lattice shown in Figure 2, the ideal $\{0, a\}$ is not prime. Therefore, the lattice is not fully prime.

The following is the definition of a fully essential lattice.

Definition 2.8. A lattice L with least element 0 is said to be a fully essential lattice if every nonzero semi-essential ideal of L is an essential ideal of L.

The following remark is a direct consequence of definitions 2.6 and 2.8.

Remark 2.9. Every fully prime lattice is fully essential.

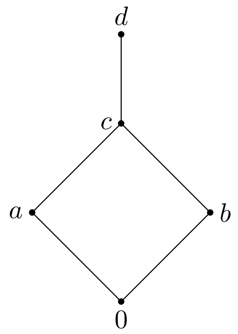

Figure 1:

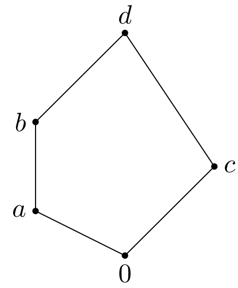

Figure 2:

Remark 2.10. Consider the lattice shown in Figure 3, lattice L has prime ideals $\{0, a\}$, $\{0, b, e\}$, $\{0, a, b, c, d, e, g\}$, $\{0, a, b, c, d, f\}$. The ideals I=$\{0, a, b, c\}$, J=$\{0, a, b, c, d\}$, K= $\{0, a, b, c, d, e, g\}$, P=$\{0, a, b, c, d, f\}$ have nonzero intersection with all prime ideals. Therefore, I, J, K and P are semi-essential ideals. Note that these ideals are essential in L. Hence, lattice L is a fully essential.

Consider the lattice shown in Figure 4, lattice L has prime ideals $\{0, a\}$, $\{0, a, b, c, d, e, g\}$. But the semi-essential ideal $\{0, a\}$ is not essential in L. Therefore, the lattice is not a fully essential lattice.

Here it is noted that not every prime ideal is semi-essential.

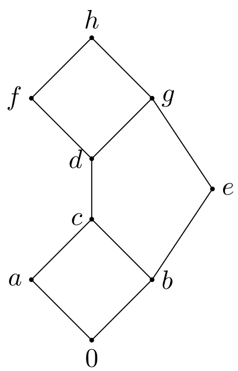

Figure 3:

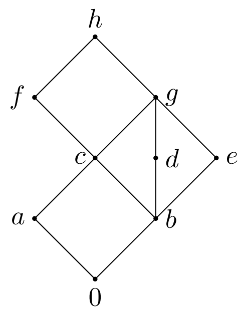

Figure 4:

Theorem 2.11. *Let L be a lattice with* 0 *and I, J be any two ideals of L such that $I \subseteq J$. Then $I \leqslant_{se} J$ if and only if L is fully prime and $I \leqslant_{e} J$.*

Proof. Suppose that $I \leqslant_{se} J$. Let K be an ideal of L such that $K \subseteq J$ and $I \cap K = (0]$.

Since L is fully prime, both I and K are prime ideals of L. By Remark 2.3,

$K \subseteq J$ implies K is the prime ideal of J. Now, $I \leqslant_{se} J$ and K is the prime ideal of J, together implies $K = (0]$. Therefore, $I \leqslant_e J$.

Converse is true by the definition. □

Now, a semi-uniform ideal in a lattice is defined and a relation between a semi-uniform and an essential lattice is given.

Definition 2.12. Let L be a lattice with 0 and J be a nonzero ideal of L. Then, J is called semi-uniform if every nonzero ideal $I \subseteq J$ is semi-essential in J.

Remark 2.13. Consider the lattice shown in Figure 5, lattice has prime ideals I=$\{0, a, b, c, j, d, e\}$, J={ 0,a,b,c,d,f,i } and K=$\{0, a, b, c, d, e, f, g, i, j\}$. Every nonzero ideal of lattice has nonzero intersection with I, J and K and therefore, is semi-essential in L. Hence, the lattice is semi-uniform.

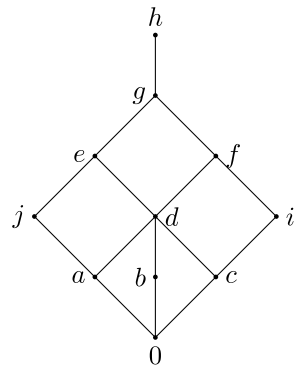

Figure 5:

Remark 2.14. Every uniform lattice is semi-uniform. But, the converse does not need to be true.

Consider the lattice L in Figure 5. It is a semi-uniform lattice but $\{0, b\}$ is not essential in L. Therefore, L is not uniform.

Theorem 2.15. *The lattice L with the least element 0 is uniform if and only if L is semi-uniform and fully essential.*

Proof. Suppose L is a fully essential lattice. Since L is semi-uniform, every nonzero ideal is semi-essential in L. Further, L is fully essential, therefore every semi-essential ideal is essential in L. Hence, every nonzero ideal is essential ideal in L and so L is uniform.

Since in a uniform lattice every ideal is essential, the converse holds. □

Theorem 2.16. *Let L be a lattice with 0 and I, J and K be ideals of L such that $I \cap K \neq (0]$. Let every prime ideal of K be prime ideal of J. If $I \leqslant_{se} J$ then $I \cap K \leqslant_{se} J \cap K$.*

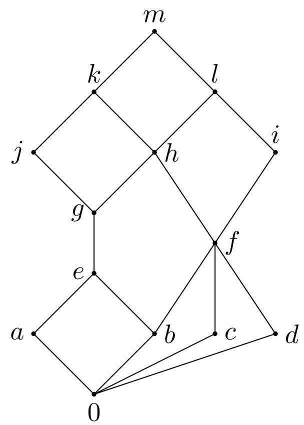

Figure 6:

Proof. Let $I \leqslant_{se} J$. Let P be the prime ideal in $J \cap K$ such that $(I \cap K) \cap P = (0]$. It is to be shown that $P = (0]$. Since $(I \cap K) \cap P = (0] \Rightarrow I \cap (K \cap P) = (0]$ and $K \nsubseteq P$, by Lemma 2.4, $K \cap P$ is prime ideal in K, therefore $K \cap P$ is prime ideal of J. Now $I \leqslant_{se} J, I \cap (K \cap P) = (0] \Rightarrow K \cap P = (0] \Rightarrow P = (0]$. □

3. Semi-extending Ideals in Lattices

This section defines the semi-extending ideal and st-closed ideal in a lattice and discusses the relation between them.

Definition 3.1. A lattice L with 0 is said to be semi-extending if every nonzero ideal I of a lattice L is semi-essential in direct summand of L.

An ideal I of L is called semi-extending if every nonzero ideal $J \subseteq I$ is semi-essential in direct summand of I.

Remark 3.2. Every extending lattice is semi-extending.

Remark 3.3. Consider the lattice shown in Figure 6, $I = \{0, a, b, c, d, e, f, g, h, i, l\}$ and $J = \{0, b\}$, $K = \{0, c\}$, $P = \{0, d\}$, $M = \{0, b, c, d, f, i\}$, $N = \{0, b, c, d, f\}$ are ideals of lattice L. Note that the ideals J, K, P are not direct summands of I and M is a direct summand of I. Further, N and M are prime ideals in M and J, K, P have nonzero intersection with M and N. Hence, J, K, P are semi-essential in M. Note that $\{0, a\}$ is also a direct summand of I, which is semi-essential in itself. Hence, I is semi-extending.

Every essential ideal is semi-essential, by Remark 2.4. Therefore, if ideal I is essential in a direct summand of L then I is semi-essential in that direct summand of L. This implies that L is semi-extending.

However the converse does not need to be true. Consider the lattice shown in Figure 6, a semi-essential ideal J is not essential in direct summand M of I. Hence, ideal I is not extending. It is already analysed in Example 3.3 that I is semi-extending.

Definition 3.4. Let L be a lattice with the least element 0. A nonzero ideal I of L is said to be a st-closed if I has no proper semi-essential extension in L.

Theorem 3.5. *Let L be a lattice with the least element* 0*. Every st-closed ideal in L is a closed ideal in L.*

Proof. Let I be a st-closed ideal in L. It is to be shown that I is a closed ideal in L. Let K be an ideal of L such that $I \leqslant_e K \subseteq L$. Since $I \leqslant_e K$, implies $I \leqslant_{se} K$. However, I is st-closed in L. Therefore, $I = K$. Hence, I is closed in L.

The following example shows that the converse of the above statement is not necessarily true. □

Remark 3.6. Consider the lattice shown in Figure 7, the ideal $J = \{0, b, c, f\}$ is closed in $I = \{0, a, b, c, d, e, f, g\}$.

Here $\{0, b, e, a\}$, $\{0, b, c, f\}$ are prime ideals in I. Hence, I is a semi-essential extension of J and therefore J is not a st-closed ideal of L. Ideal $K = \{0, a, d\}$ has no proper semi-essential extension in I. Hence, K is a st-closed in I.

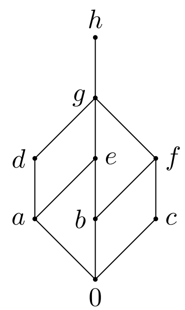

Figure 7:

Theorem 3.7. *Let L be a lattice with* 0 *and I is a nonzero ideal of L. Then there exists a st-closed ideal J of L such that* $I \leqslant_{se} J$.

Proof. Consider $V = \{K \subseteq L : K \text{ is an ideal of } L \text{ and } I \leqslant_{se} K\}$. Since $I \in V$ implies $V \neq \phi$ and it is a partial ordered set by set inclusion. Let $\mathcal{C}$ be any chain in V and let $M = \bigcup(X : X \in \mathcal{C})$. Since the union of the chain of ideals is an ideal, so, M is an ideal. Let $X \in \mathcal{C}$ and $I \leqslant_{se} X$. It is required to show that $I \leqslant_{se} M$. Let P be the prime ideal in M such that $I \cap P = (0]$. As $P \subseteq M$ implies $P \subseteq X$ for some $X \in \mathcal{C}$, since $\mathcal{C}$ is a chain. By Remark 2.3, P is prime in X and $I \leqslant_{se} X$ implies $P = (0]$. Hence, $I \leqslant_{se} M$. So, $M \in V$. Therefore, every chain in V has an upper bound. Then by Zorn's Lemma, it has a maximal element say H. Now the study aims to prove that H is st-closed. Assume that there exists an ideal D of L such that $H \leqslant_{se} D \subseteq L$. Since $I \leqslant_{se} H$ and $H \leqslant_{se} D$, by Theorem 2.5, $I \leqslant_{se} D$. $D \in V$ and $H \subseteq D$. This is a contradiction to H which is maximal. Therefore, $H = D$. Hence, H is st-closed with $I \leqslant_{se} H$. □

Theorem 3.8. *Let L be a lattice with 0. Then L is semi-extending if and only if every st-closed ideal in L is direct summand of L.*

Proof. Suppose L is a semi-extending and I is a st-closed ideal in L. Then I is semi-essential in a direct summand H of L. That is $I \leqslant_{se} H \subseteq L$. But I is st-closed in L therefore, $I = H$. Hence, I is direct summand of L.

Conversely, suppose every st-closed ideal in L is a direct summand of L. Let I be a nonzero ideal of L. By Theorem 3.7, there exists an st-closed ideal H in L such that $I \leqslant_{se} H$. Hence, L is semi-extending. □

Remark 3.9. The following are some conditions for a semi-extending lattice L with 0 to be an extending lattice.

1. Any nonzero semi-essential extension I of an ideal J in L, is fully essential.
2. L is a fully prime lattice.
3. For every ideal I of L, there exists an st-closed ideal J of L such that $I \leqslant_e J$.

Acknowledgement

Authors are also grateful to referees for valuable suggestion and kind support.

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Received by the editors June 30, 2022
First published online May 18, 2023