Zbornik radova Prirodno-matematičkog fakulteta—Univerzitet u Novom Sadu, knjiga 10 (1980)
Review of Research Faculty of Science — University of Novi Sad, Volume 10 (1980)

THE KERNEL THEOREM FOR SOME SPACES

Stevan Pilipović
Prirodno-matematički fakultet. Institut za matematiku.
21 000 Novi Sad, ul. dr Ilije Đuričića 4, Jugoslavija.

Köthe's „echelon" spaces ([2]), satisfy the K-condition ([1]). Besides the diagonal theorem this fact is the main part of the proof that the kernel theorem holds for Köthe's „echelon" spaces and for the spaces of tempered and Schwartz's distributions ([1], [2]).

In this paper we prove that the K-condition is satisfied for a class of „echelon" spaces of order greater than one ([4]). From that and the diagonal theorem, similarly as in [2], follows the kernel theorem for some „co-echelon" spaces of order $s=1/(1-1/r)$. In a special case we obtain the kernel theorem for a $\mathcal{U}_0'$-type spaces of generalized functions ([5]).

Spaces of sequences

By $X^\star$ we denote the set of sequences $A=(a_n)$, $n \in N_0=N \cup \{0\})$, such that $a_n \in X$ where X is the normed space on the field of complex numbers C, with the norm $| \; |$.

We introduce

If $\lambda \in C$, $\quad A \in X^\star$, $\quad \lambda A=(\lambda a_1, \lambda a_2, \ldots)$;

If $A, B \in X^\star$, $\quad A+B=(a_1+b_1, \quad a_2+b_2, \ldots)$

If one of the sequences A and B is with elements from C and the other is from $X^\star$, then we write

$A \cdot B=(a_1 b_1, a_2 b_2, \ldots)$; $\quad (A, B)=a_1 b_1+a_2 b_2+\ldots$

If all the elements of sequence A are real and $\neq 0$ then

$$A^{-1}=(1/a_1, 1/a_2, \ldots)$$

We make use of the following notation

$$|A|=\sup_{p\in N_0}|a_p|;\quad \|A\|_r=\sqrt[r]{\sum_{p\in N_0}|a_p|^r},\quad r\geqslant 1$$

and $\| A \|_\infty=| A |$.

By

$$T_k=(t_{k,p}),\quad p\in N_0,\quad k\in N_0$$

we denote the sequence of a real positive sequences such that

(1) for every $r, s\in N$ there exists $k\in N_0$ and $M_{r,s}>0$ such that $k\geqslant r$, $k\geqslant s$, and

$| t_{r,p}\cdot t_{s,p} | \leqslant M_{r,s} | t_{k,p} | \qquad p\in N_0;$

(2) $| T_k \overset{-k}{T}_{k+1} |<\infty$.

According to sequence T_k, we obtain a class of „echelon" spaces of order $r\geqslant 1$:

$$\lambda_r=\cap\,\lambda_{T_k,r}\quad (\lambda_{T_k,r}=\{A\in X^\star \mid \| T_kA \|_r<\infty\}).$$

The (normal) topology τ in λ_r is given by the family of seminorms

$$g_{k,r}(A)=(\sum_p \overset{r}{t}_{k,p} | a_p |^r)^{1/r},\quad k\in N_0$$

$$(g_{k,\infty}(A)=\sup_p t_{k,p} | a_p |).$$

The alfa-dual λ_r^* of λ_r

$$\lambda_r^*=\cup\,(\lambda_{T_k,r})^*\quad \left((\lambda_{T_k,r})^*=\{A\in X^\star \mid \| \overset{-1}{T_k} A \|<\infty\},\quad s=\frac{1}{1-\frac{1}{p}}\right)$$

is a topological dual of $\lambda_r(\tau)$ ([4]).

It is known:

(3) $$A_n\to A \quad\text{in}\quad \lambda_r \text{ iff for every } k\in N_0$$

$$\sum_p \overset{r}{t}_{k,p} | a_{n,p}-a_p |^r\to 0$$

(4) $$B_n\to B \text{ in } \lambda_r^* \text{ iff for some } k\in N_0$$

$$\sum_p \overset{-s}{t}_{k,p} | b_{n,p}-b_p |\to 0\qquad \left(s=\frac{1}{1-\frac{1}{r}}\right)$$

Convergence (4) and weak convergence are equivalent in λ_r^*. This may be proved directly and elementarily as it is done in [2] for the case $r=1$ and, implicitly, for the case $r=2$ in [7].

We denote $\lambda_1=\mathcal{S}$ and $\lambda_1^*=\mathcal{T}$ as in [2] and $\lambda_2=U$ and $\lambda_2^*=U'$ as in [6].

According to [1], we say that the convergence in λ_r^* satisfies the K-condition iff from $A_n\to 0$ in λ_r^*, follows that $\sum_{m=1}^{\infty} A_{n_m}\in\lambda_r^*$ for some sub-sequence (n_m).

Remark 1. (Similarly as in [2]). If sequence (T_k) is given, then λ_r and λ_r^* are determined for $r>1$. However, the converse in not true. The same spaces λ_r and λ_r^* are determined, by the sequence $\tilde{T}_k$ for which conditions (1), (2) are satisfied and for some $d>0$

$$|\tilde{T}_k\tilde{T}_{k+1}^{-1}|<d, \quad k\in N_0$$

holds. In fact, we can put $\tilde{T}_k=d^{-k}m_1,\ldots m_kT_k$ with $m_1=1$ and

$$|T_kT_{k+1}^{-1}|<m_{k+1}.$$

Remark 2. In [2] spaces $\mathcal{S}$ and $\mathcal{T}$ are defined without conditions (1). This condition is connected with spaces λ_r and λ_r^* for $r>1$. It means that we are going to observe a class of „echelon" and a class of „co-echelon" spaces of an order greater than 1, which we are going to compare with spaces λ_1 and λ_1^*

It is easy to show that if $1<r<s$

$$\mathcal{S}\subset\lambda_r\subset\lambda_s;\quad \lambda_s^*\subset\lambda_r^*\subset\mathcal{T} \tag{5}$$

also in a convergence sense.

Let $1\leqslant r_0$, s_0 such that $\frac{1}{r_0}+\frac{1}{s_0}=1$ and let $1\leqslant r\leqslant r_0$ and $s_0\leqslant s\leqslant\infty$.

THEOREM 1. *Space λ_r and $\mathcal{S}$ (respectively λ_r^* and $\mathcal{T}$) are concident with their convergence if for some $k\in N_0$*

$$\| T_k^{-1} \|\, s_0<\infty \tag{6}$$

If $t_{k,p}\to\infty$ for every $k\in N_0$ when $p\to\infty$, condition (6) *is necessary.*

Proof: Let us prove Theorem 1. for λ_r and $\mathcal{S}$, as the proof for λ_r^* and $\mathcal{T}$ is similar.

Sufficiency follows from Hölder's inequality and from (1).

Suppose that the natural number $k\in N_0$ does not exist, such that (6) holds. This means that for every $k\in N_0$

$$\| T_k^{-1} \|_{s_0} =\infty \tag{7}$$

We may assume that $|T_kT_{k+1}^{-1}|<\frac{1}{2}$ (Remark 1) and so

$$|T_iT_j^{-1}|\leqslant 2^{i-j} \text{ holds.}$$

Let Λ_n, $n \in N$, be a sequence of the finite subsets of N_o such that $\Lambda_n \cap \Lambda_{n+1} = \emptyset$ and

$$1 \leqslant \sum_{p \in \Lambda_n \setminus S_n} t_{n,p}^{-1} < 2, \; n \in N, \text{ where} \tag{8}$$

$S_n \subset \Lambda_n$ such that if $p \in S_n$, then $t_{n,p} \geqslant 1$, $p \in N_o$.

The existence of sequence (Λ_n), for which (8) holds, follows from (7).

Let $b_p = 0$ if $p \in S_n$ and $b_p = t_{n,p}^{-1}$ if $p \in \Lambda_n / S_n$. The sequence $B = (b_p)$ belongs to λ_{r_0} but does not belongs to $\mathcal{S}$. So in this way we have proved that condition (6) is necessary.

Particularly, if (6) holds fo $s_0 = 1$, then „echelon" and „co-echelon" spaces of any order greater than 1 are coincident with $\mathcal{S}$ and $\mathcal{T}$ respectively.

Remark 3. In the case when condition (6) is satisfied for some $s \geqslant 1$, then λ_r and λ_r^* are nuclear. If this condition is not satisfied these spaces are neither nulear nor coincide with space $\mathcal{S}$, respectively $\mathcal{T}$.

In the following theorem we are going to suppose that (6) is not satisfied. Before formulating this theorem we shall state some facts.

If $a, b > 0$ and $0 \leqslant x \leqslant 1$

$$(a+b) \leqslant a^x + b^x \tag{9}$$

holds. From (9), it follows (for example), that if $a_i > 0$, $1 \leqslant i \leqslant n$

$$\left(\sum_{i=1}^{n} a_i\right)^{1+x} \leqslant \sum_{i=1}^{n} a_i^{1+x} + \sum_{p+q=i}^{n} a_p^x a_q \tag{10}$$

Let $\varepsilon = (\varepsilon_p) \in \lambda_1$ and $1 \leqslant r \leqslant 2$. If

$$A_n \xrightarrow{\lambda_r} 0,$$

then there exists a sequence V_p, $p \in N$, of infinite subsets of N_0, such that $V_1 \supset V_2 \supset$ $\subset \ldots$ and

$$\sum_{n \in V_p} |a_{n,p}|^{r-[r]} < \frac{1}{pA} \varepsilon p \tag{11}$$

$$\sum_{n \in V_p} |a_{n,p}| < \frac{1}{pA} \varepsilon p \tag{11*}$$

where $A = \sup_{n \in N} |A_n|$. From $A_n \xrightarrow{\lambda_r} 0$, $n \to \infty$, follows $a_{n,p} \to 0$ if $n \to \infty$, for every $p \in N_0$, and so we get (11) and (11*).

Now, we shall prove:

THEOREM 2. *The convergence in spaces λ_r, $1 < r < \infty$ satisfies the K--condition.*

Proof: To show that the K-condition is satisfied in spaces λ_r we have to prove that for any sequence $A_n=(a_{n,p})\xrightarrow{\lambda_r}0\,(0\in\lambda_r)$ there exists a subsequence (A_{m_n}) such that

(12) $$A_{m_n}=\Big(\sum_{n=1}^{\infty} a_{m_n,1}\ \sum_{n=1}^{\infty} a_{m_n,2},\ldots\Big)\in\lambda_r$$

First, we shall prove this proposition for $r=1+x$, $0<x<1$.

Let (k_p) be a strictly increasing seguence of natural numbers, such that $k_p\in V_p$, and let (m_p) be a sub-sequence of (k_p) such that

(13) $$\sum_{p=0}^{\infty} |\, a_{m_n,p}\,|^{1+x}\, t_{n,p}^{1+x}\leqslant\frac{1}{2^{n+1}}\quad\text{holds.}$$

For the sequence A_{m_n} relation (12) holds, because for every $l\in N_0$ using (10) we have.

$$\left(\sum_{n=1}^{\infty}|\, a_{m_n,1}\,|^{1+x}\, t_{l,1}^{1+x}+\left(\sum_{n=1}^{\infty}|\, a_{m_n,2}\,|\right)^{1+x} t_{l,2}^{1+x}+\ldots+\right.$$

$$+\left(\sum_{n=1}^{\infty}|\, a_{m_n,s}\,|\right)^{1+x} t_{l,s}^{1+x}+\ldots\leqslant\left(\sum_{n=1}^{\infty}|\, a_{n_n,1}\,|\right)^{1+x} t_{l,1}^{1+x}+|\, a_{m_1,2}\,|^{1+x}\, t_{l,2}^{1+x}+$$

$$+\left(\sum_{n=1}^{\infty}|\, a_{m_n,2}\,|\right)^{1+x} t_{l,2}^{1+x}+|\, a_{m_1,2}\,|^{x}\left(\sum_{n=2}^{\infty}|\, a_{m_n,2}\,|\right) t_{l,2}^{1+x}+a_{m_1,2}\left(\sum_{n=2}^{\infty}|\, a_{m_n,2}\,|\right) t_{l,2}^{1-x}+$$

$$\leqslant\sum_{n=1}^{\infty}\left(\sum_{n=p}^{\infty}|\, a_{m_n,p}\,|\right)^{1+x} t_{l,p}^{1+x}+\sum_{p=2}^{\infty}\left(\sum_{n=p}^{\infty}|\, a_{m_n,2}\,|\right)\sum_{i=1}^{p-1}|\, a_{m_i,p}\,|^{x}\, t_{l,p}^{1+x}+$$

$$+\sum_{p=2}^{\infty}\left(\sum_{n=1}^{\infty}|\, a_{m_n,p}\,|\right)^{x}\sum_{i=1}^{p-1}|\, a_{m_i}\,|\, t_{l,q}^{1+x}+\sum_{(q,r)\in N\times N}\sum_{p=3}^{\infty}(|\, a_{m_q,p}\,|\,|\, a_{m_r,p}\,|^{x}+$$

$$+\,|\, a_{m_q,p}\,|^{x}\,|\, a_{m_r,p}\,|)\, t_{l,p}^{1+x}$$

From relations (11) and (11*) follows that the first three sums are finite, and from.

$$\sum_{p=3}^{\infty}|\, a_{m_q,p}\,|\,|\, a_{m_r,p}\,|^{x}\, t_{l,p}^{1+x}\leqslant\sum_{p=3}|\, a_{m_q,p}\,|^{1+x}\, t_{l,p}^{1+x}\sum_{p=3}^{\infty}|\, a_{m_q}\,|^{1+x}\, t_{l,p}^{1+x}\leqslant\frac{1}{2^{q+r}}$$

it follows that the last sum is finite. So we prove Theorem 2. for $r=1+x$, $0<x<1$. If $r\in N$ then we can prove this Theorem using the binom formula also. If $r=n+x$, $n\in N$ and $0<x<1$, then insteand of (10), (11), (11*) we use the similar but more complicated relation which we get by multiplying (9) by $(a+b)^n$.

This Theorem can also be proved by using the completness and metricability of space λ_r. We give direct proof of that.

Denote by ${}_1\lambda_r$ and ${}_2\lambda_r$ the spaces of sequences whch correspond to sequence ${}_1T_k=({}_1t_{k,p})$, respectively ${}_2T_k=({}_2t_{k,p})$ and suppose that these sequences satisfy condition, (6). Denote by W the space of matrices $A=(a_m)$, $m\in N_0\times N_0=P^2$, such that for every $k\in P$

$$\| T_kA \|_r=\sum_{(p,q)\in P^2} {}_1t_{k,p}^r\, {}_2t_{k,q}^r\, (a_{(p,q)})^r<\infty \quad (T_k={}_1T_k\oplus{}_2T_k).$$

The bilinear operator T which maps ${}_1\lambda_r\times{}_2\lambda_r$ into Banach spaces X is separately continuous if the next inplications hold:

$$A_n\xrightarrow{{}_1\lambda_r}A\Rightarrow T(A_n,B)\to T(A,B);\quad B_n\xrightarrow{{}_2\lambda_r}B\Rightarrow T(A,B_n)\to T(A,B)$$

when $n\to\infty$.

From Theorems 1, 2., from [1] and [2] (p. 249) follows

THEOREM 3. *For each separately continuous bilinear operator* $T:{}_1\lambda_r\times\times{}_2\lambda_r\to X$ *there exists a unique continuous linear operator* $u:W\to X$ *such that*

$$T(A,B)=u(A\oplus B)\quad (A\oplus B)=(a_p\cdot b_q),\ (p,q)\in P^2) \tag{14}$$

Conversely, for every continuous linear operator $u:W\to X$, (14) *represents a separately continuous bilinear operator which maps* ${}_1\lambda_r\times{}_2\lambda_r$ in X. *This correspodence is one—to—one.*

Remark 4. The whole preceding theory continues to apply when we replace sequences (vectors) $A=(a_p)$, $p\in N_0$, by matrices of any arbitrary, but fixed, number of dimension $A=(a_p)$, $p\in P^r$. Still more generaly, this theory continues to apply on the functions, matrices, defined on the countable set K with values in a given normed space X. It is only necessary to accomodate notations and notions to that case in the same way as it is done in [2] (p. 244). Simply, it means that in the preceding text instead of $p\in N_0$ we can put $p\in K$.

Spaces of generalized functions

Space of generalized function $\mathcal{A}'$ which elements may be expanded into a series are introduced by Zemanian [7]. In [5] we develop the theory of space $\mathcal{U}'$ in the multi dimensional case. $\mathcal{A}'$-type spaces are examples of $\mathcal{U}'$-type spaces.

From [6] it follows that the following mappings are homeomorphic

$$\mathcal{U}\ni\sum_{p\in P^r} a_p\psi_p\leftrightarrow A=(a_p)\in U\quad (U=\lambda_2)$$

$$\mathcal{U}'\ni\sum_{p\in P^r} a_p\psi_p\leftrightarrow A=(a_p)\in U'\quad (U'=\lambda_2^*)$$

$$\mathcal{W}'\ni\sum_{(p,q)\in P^{r+l}} a_{(p,q)}\,\psi_p^1\,\psi_q^2\leftrightarrow A=a_{(p,q)}\in W'$$

$\mathcal{U}$ is the space of test functions in r-dimensional case; $\mathcal{U}'$, and $\mathcal{W}'$ are the $\mathcal{U}'$--type spaces in r-dimensional, respectively in r+l-dimensional, case.

Spaces U and U' correspond to sequence $T_k=(\hat{\lambda}_n^k)$ where $n=(\nu_1,...,\nu_n)\in P^r$ and $\hat{\lambda}_n^k=\hat{\lambda}_{\nu_1}^k\cdot\hat{\lambda}_{\nu_2}^k\ldots\hat{\lambda}_{\nu_n}^k$. Spacee W corresponds to the sequence $T_k=(\hat{\lambda}_p^k\cdot\hat{\lambda}_q^k)$, $(p, q)\in P^{r+l}$.

It for some $k\in N_0$, $\sum_{p\in P^r}\frac{1}{\hat{\lambda}_k^{2k}}<\infty$ holds ($s_0=2$), we denote the correspon-ding spaces of matrices with U_0 and U_0', space of test functions with $\mathcal{U}_0$ and the space of generalized functions with $\mathcal{U}_0'$.

From Theorem 3 follows the kernel theorem for the spaces of $\mathcal{U}_0'$-type.

THEOREM 4. *If f is a bilinear, separately continuous functional in the cartesian product $\mathcal{U}_{1,0}(R^r)\times\mathcal{U}_{2,0}(R^l)$, then there is a generalized function F from $\mathcal{W}'(R^{r+l})$ such that*

$$T(\phi, \psi)=(f, \phi\oplus\psi), \tag{15}$$

where $\phi\oplus\psi=\phi(x).\ \psi(y)$ $(x\in R^r,\ y\in R^l)$ *for every* $\phi\in\mathcal{U}_{1,0}(R^r)$ *and* $\psi\in\mathcal{U}_{2,0}(R^l)$.

Conversely, for any $f\in\mathcal{W}'(R^{r+l})$ (15) represents a separately continuous bilinear functional on $\mathcal{U}_{1,0}(R^r)\times\mathcal{U}_{2,0}(R^l)$ This correspodence is one-to-one.

LITERATURE

[1] P. Antosik, *Sur les suites d'applications*, C. R. Acad. Sc. Paris, t. 287, pp. 75–77, 1978.

[2] П. Антосик, Я. Микусинский, Р. Сикорский, *Теориа обобшенн ых функций Секвенциальн ыи йодход*, Мир, Москва, 1976.

[3] I. M. Gel'fand, N. Ya. Vilenkin, *Generalized Functions Vol* 4., Acad. Press, New York and London, 1964.

[4] G. Köthe, *Topological Vector Spaces* I, New York, 1969.

[5] E. Pap S. Pilipović, *A Sequential Theory of Some Spaces of Generalized Functions*, Publ. Inst. Math., t. 25 (39), pp. 122–130, 1979.

[6] S. Pilipović, *On Convergence in Some Sub-spaces of Generalized Functions*, Mat. Vesnik Knj. 3 (16) (31), 1976, 2, pp. 209–221.

[7] А. Г. Земанян, *Интегральн ые йреобразования обобшенных функций*, Наука, Москва, 1974.

TEOREMA O JEZGRU ZA NEKE PROSTORE

S. Pilipović

REZIME

Köteov „echelon" prostor ([2]) zadovoljava K-uslov ([1]). Pored dijagonalne teoreme ta činjenica predstavlja najvažniji deo dokaza da teorema o jezgru važi z. Köteov „co-echelon" prostor kao i za prostore temperiranih i Švarcovih distribucija ([1], [2]).

U ovom radu pokazujemo da K-uslov važi za jednu klasu „echelon" prostora reda r većeg od 1 ([4]). Odatle i iz dijagonalne teoreme, slično kao u [2] sledi da teorema o jezgru važi za neke „co-echelon" prostore reda $s=(1-1/r)$. U specijalnom slučaju dobijamo teoremu o jezgru za prostore uopštenih funkcija tipa $\mathcal{U}_0'$ ([5]).