Zbornik radova Prirodno-matematičkog fakulteta—Univerzitet u Novom Sadu, knjiga 10 (1980)
Review of Research Faculty of Science — University of Novi Sad, Volume 10 (1980)

SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN TOPOLOGICAL VECTOR SPACES

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In [6] Zima proved a generalization of Schauder's fixed point theorem in paranormed space and in [2] some generalizations of Zima's result for multivalued mappings in a topological vector space are obtained. So in [2], the following theorem is proved.

THEOREM 1. *Let K be a closed and convex subset of a Hausdorff topological vector space E, $\mathcal{U}$ the system of neighbourhoods of zero in E, $F:K\to 2^K$ be a compact mapping such that $\overline{\text{co}}\ F(x)=F(x)$, for every $x\in K$. If for every $V\in\mathcal{U}$ there exists $U\in\mathcal{U}$ such that*:

$$\text{co}\,(U\cap(F(K)-F(K)))\subseteq V \tag{1}$$

then there exists $x\in K$ such that $x\in F(x)$.

In [2] it is shown that in a paranormed space $(E, \|\ \|^*)$ for every set $K\subset E$ which satisfies Zima's condition, the relation (1) holds.

Using the principle of duality we shall prove, similarly as in [5] ,a fixed point theorem which is dual to Theorem 1. First we shall give some notations and definitions which we will use further in the text.

If $F:K\to 2^E$, where 2^E is the collection of all the subsets of E, then for each $y\in F(K)$:

$$F^{-1}(y)=\{x \mid x\in K,\ y\in F(x)\}$$

and so $F^{-1}:F(K)\to 2^K$. It is obvious that x is a fixed point of the mapping F, if and only if x is a fixed point of F^{-1}. Tarafdar and Husain have remarked in [5] that if X, Y and $F(X)$ are compact topological spaces ($F:X\to 2^Y$) and $F(x)=\overline{F(x)}$, for each $x\in X$, $\overline{F^{-1}(y)}=F^{-1}(y)$, for each $y\in F(X)$, then F is uppersemicontinuous if and only if the inverse mapping $F^{-1}:F(X)\to 2^X$ is uppersemicontinuous.

In the following theorem we shall denote bu $\mathcal{U}$ the family of neighbourhoods of zero in E.

THEOREM 2. *Let K be a nonempty compact subset of a Hausdorff topological vector space E such that for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ so that $\mathrm{co}\,(U \cap (K-K)) \subseteq V$. Let $F: K \to 2^E$ be an uppersemicontinuous mapping such that $K \subseteq F(K)$, $F(x) = \overline{F(x)}$, for every $x \in K$, $\overline{\mathrm{co}}\ F^{-1}(x) = F^{-1}(x)$, for every $x \in F(K)$. and $F(K) = \overline{\mathrm{co}}\, F(K)$. Then there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.*

Proof: Let us define the mapping $T: F(K) \to 2^K$ by $T(x) = F^{-1}(x)$, for every $x \in F(K)$. Since $F(x) = \overline{F(x)}\ (x \in K)$ and $\overline{F^{-1}}(y) = F^{-1}(y)\ (y \in F(K))$, the mapping T is uppersemicontinuous. Further, for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that:

$$\mathrm{co}\,(U \cap (T(F(K)) - T(F(K))) \subseteq V$$

since $\mathrm{co}\,(U \cap (K-K)) \subseteq V$. So, from Theorem 1 it follows that there exists $x_0 \in F(K)$ such that $x_0 \in T(x_0)$ and so x_0 is a fixed point of the mapping F.

Similarly as in [2], we shall prove, using Theorem 2, a fixed point theorem for the multivalued mapping $F+S$, where F is a singlevalued and S is a multivalued mapping.

THEOREM 3. *Let E be a Hausdorff topological vector space, K be a nonempty, compact subset of E such that for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ so that $\mathrm{co}\,(U \cap (K-K)) \subseteq V$. Further, suppose that $F: E \to E$ is a linear continuous mapping, $S: K \to 2^E$ is an uppersemicontinuous mapping such that $(I-F)(K) \subseteq S(K)$ and that the following conditions are satisfied:*

1. *$S(K)$ is compact and convex.*
2. *$S(x) = \mathrm{co}\, S(x)$, for every $x \in K$ and $S^{-1}(y) = \overline{\mathrm{co}}\ S^{-1}(y)$, for every $y \in S(K)$.*
3. *For every $y \in S(K)$ there exists one and only one $x(y) \in E$ so that $x(y) = Fx(y) + y$ and the set $\{x(y)\}_{y \in S(K)} \in S(K)$ is compact.* Then *$Fix\,(F+S) \neq \emptyset$.*

Proof: Since for every $y \in S(K)$ there exists $x(y) \in E$ so that $x(y) = Fx(y) + y$, we define the mapping $R: S(K) \to E$ in the following way: $Ry = x(y)$, for every $y \in S(K)$. Let us prove that the mapping R is continuous. Suppose that $\{y_\alpha\}_{\alpha \in \mathcal{A}} \subseteq S(K)$ is a convergent net and $\lim\limits_{\alpha \in \mathcal{A}} y_\alpha = y$ and let us prove that $\lim\limits_{\alpha \in \mathcal{A}} Ry_\alpha = Ry$. We have that for every $\alpha \in \mathcal{A}$, $Ry_\alpha = FRy_\alpha + y_\alpha$. Since the set $\{Ry \mid y \in S(K)\}$ is compact, there is a subnet $\{y_{\alpha_\beta}\}$ such that $\lim\limits_\beta Ry_{\alpha_\beta} = z$. Then:

$$\lim_\beta Ry_{\alpha_\beta} = F(\lim_\beta Ry_{\alpha_\beta}) + \lim_\beta y_{\alpha_\beta}$$

and so $z = Fz + y$, which implis that $z = Ry$. Since every convergent subnet of $\{Ry_\alpha\}$ has the limit Ry, it follows that $\lim\limits_\alpha Ry_\alpha = Ry$. Further, there exists $R^{-1}: R(S(K)) \to S(K)$ and since $R^{-1} z = z - Fz$, for every $z \in R(S(K))$ the mapping R^{-1} is also a continuous mapping. Let us define the mapping $R^*: K \to 2^E$ in the following way:

$$R^* x = \bigcup_{y \in Sx} Ry.$$

We shall show that the mapping R^* satisfies all the conditions of Theorem 2. First, the relation $K \subseteq R^*(K)$ follows from $Ry = FRy + y\ (y \in S(K))$ since for every $z \in K$, there exists $y \in S(K)$ such that $z - Fz = y$ and so $z = Ry \in RS(K)$. Since R is continuous and S is uppeısemicontinuous, it follows that R^* is uppersemicontinuous. Since R is an affine homeomorphism, it follows that $R(S(x))$ is closed ($x \in K$) and $R^*(K)$ is the compact and convex subset of E. It remains to prove that for every $x \in R(S(K))$ the set $(R^*)^{-1}(x)$ is closed and convex. Since $(R^*)^{-1} = S^{-1} R^{-1}$, we have that this condition is satisfied. From Theorem 2, we conclude that $\mathrm{Fix}(R^*) \neq \emptyset$ and since $\mathrm{Fix}(R^*) \subseteq \mathrm{Fix}(F+S)$, it follows that Fix $(F+S) \neq \emptyset$.

The following theorem is a generalization of the fixed point theorem from [4].

DEFINITION 1. [4] *A subset A of a topological vector space is almost convex if for every $V \in \mathcal{U}$ and every finite set $\{x_1, x_2, \ldots, x_n\} \subset A$ there exists a subset $\{z_1, z_2, \ldots, z_n\} \subset A$ such that $z_i - x_i \in V$ $(i = 1, 2, \ldots, n)$ and* co $\{z_1, z_2, \ldots, z_n\} \subset A$. We shall suppose in the following text that every V, $V \in \mathcal{U}$, is closed and symmetric.

Similarly as in [4], we shall prove the following fixed point theorem.

THEOREM 4. *Let K be a nonempty and compact subset of a Hausdorff topological vector space, $G: K \to 2^K$ be an uppersemicontinuous mapping so that $G(x) = \overline{\mathrm{co}}\, G(x)$, for every $x \in A$ where A is a dense almost convex subset of $G(K)$. If for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that*

$$\mathrm{co}\,(U \cap (G(K) - G(K)) \subseteq V$$

then there exists at least one fixed point of the mapping G.

Proof: As in [4], for every $V \in \mathcal{U}$, let $F_V = \{x \mid x \in (G(x) + V) \cap K\}$. It is obvious that $\cap \{F_V \mid V \in \mathcal{U}\} \neq \emptyset$ implies that Fix $(G) \neq \emptyset$. Since for every $V \in \mathcal{U}$ there exists $U(V) \in \mathcal{U}$ so that:

$$\mathrm{co}\,(U(V) \cap (G(K) - G(K))) \subseteq V$$

let us denote by F_V^* the set:

$$F_V^* = \{x \mid x \in K, x \in G(x) + \overline{\mathrm{co}}\,(U(V) \cap (G(K) - G(K)))\}.$$

Since $F_V^* \subseteq F_V$, for every $V \in \mathcal{U}$, it is enough to prove that:

$$\cap \{F_V^* \mid V \in \mathcal{U}\} \neq \emptyset.$$

Since K is compact and $F_{U \cap V}^* \subseteq F_V^* \cap F_U^*$, we shall prove, similarly as in [4], that F_V^* is nonempty and convex, for every $V \in \mathcal{U}$.

For every $V \in \mathcal{U}$ let:

$$G_V(x) = (G(x) + \overline{\mathrm{co}}\,(U(V) \cap (G(K) - G(K)))) \cap K, \quad x \in K$$

$$R_V(x) = (x + \overline{\mathrm{co}}\,(U(V) \cap (G(K) - G(K)))) \cap K, \quad x \in K.$$

As in [4] the mapping G_V is uppersemicontinuous on K and the graph gr G_V is closed in $K \times K$. If Δ is the diagonal in $K \times K$ then F_V^* is the projection of the compact set $\Delta \cap \mathrm{gr}\, G_V$ onto the Dom (G_V) and so F_V^* is closed. The rest of the proof is as in [4]. The following theorem is a generalization of Browder's fixed point theorem [1].

THEOREM 5. *Let K be a convex subset of topological vector space E, K_1 be a compact and convex subset of topological vector space F with $\mathcal{V}$ as the fundamental system of neighbourhcods of zero in F and T and S two mapping K into $2K_1$ so that the following conditions are satisfied:*

(i) *The mapping T is uppersemicontinuous and $T(u)$ is the nonempty closed and convex subset of K_1, for every $u \in K$.*

(ii) *For every $u \in K$, $S(u)$ is open in K_1 and $S^{-1}(v)$ is nonempty and convex subset of K, for every $v \in K_1$.*

(iii) *For every $V \in \mathcal{V}$ there exists $U \in \mathcal{V}$* sohat:

$$\mathrm{co}\,(U \cap (T(K) - T(K))) \subseteq V.$$

Then there exists $u_0 \in K$ so that:

$$T(u_0) \cap S(u_0) \neq \emptyset.$$

Proof: As in [1] let $\{u_1, u_2, \ldots, u_n\}$ be such a subset of K that:

$$K_1 \subseteq \bigcup_{i=1}^{n} S(u_i).$$

If $\{\beta_i\}_{i=1}^{n}$ is the partition of the unity subordinated to the open covering $\{S(u_i)\}_{i=1}^{n}$ let $p: K_1 \to K$ be defined by:

$$p(v) = \sum_{j=1}^{n} \beta_j(v) u_j, \quad v \in K_1.$$

Then $v \in S(p(v))$, for every $v \in K$ and let:

$$R(v) = T(p(v)), \text{ for every } v \in K_1.$$

Then $R: K_1 \to 2K_1$ and for every $v \in K_1$, $R(v)$ is the nonempty, convex and closed subset of K_1. Since p is continuous and T is uppersemicontinuous ,it follows that R is uppersemicontinuous. Further, $R(K_1) \subseteq T(K)$ and since co $(U \cap (T(K) - T(K))) \subseteq V$ we conclude that:

$$\mathrm{co}\,(U \cap (R(K_1) - R(K_1))) \subseteq \mathrm{co}\,(U \cap (T(K) - T(K))) \subseteq V.$$

So, all the conditions of Theorem 1 are satisfied for R and K_1 and so there exists $u_0 \in K_1$ such that $u_0 \in R(u_0)$. Then:

$$T(p(u_0)) \cap S(p(u_0)) \neq \emptyset.$$

Now, we shall give two examples of topological vector space E and of set K such that for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that co $(U \cap (K - K)) \subseteq V$.

Example 1. Let E be a linear space over the real or complex number field and $\|\ \|^*: E \to [0, \infty)$ so that the following conditions are satisfied:

1. $\|x\|^* = 0 \Leftrightarrow x = 0$.
2. $\|x\|^* = \|-x\|^*$, for every $x \in E$.
3. $\|x+y\|^* \leqslant \|x\|^* + \|y\|^*$, for every $x, y \in E$.

Then the pair $(E, \| \|^*)$ is a paranormed space which is also a topological vector space and where the fundamental system of the neighbourhood of zero $\mathcal{U}$ is of the form $\{V_\varepsilon\}_{\varepsilon>0}$ and:

$$V_\varepsilon=\{x \mid x \in E, \ \|x\|^*<\varepsilon\}.$$

DEFINITION 2. *Let $K \subseteq E$ where $(E, \| \|^*)$ is a paranormed space and there exists $C>0$ such that*:

$$\| \lambda x \|^* \leqslant C\lambda \| x \|^*, \qquad x \in K-K, \qquad 0 \leqslant \lambda \leqslant 1. \tag{2}$$

Then we say that the set K satisfies Zima's condition.

In [2] an example is given of E and K such that (2) is satisfied, i.e. that K satisfies Zima's condition.

Example 2. S. Kasahara proved that every real Hausdorff topological vector space E is a Φ-paranormed space $(E, \| \|, \Phi)$ over a topological semifield R_Δ (for a definition of Φ paranormed space see [3]).

DEFINITION 3. *Let K be a subset of a Φ paranormed space $(E, \| \|, \Phi)$ The set K is of Φ-type if for every $n \in N$, every $x_i \in K-K$ $(i=1, 2, \ldots, n)$ and*

$\lambda_i \in [0, 1]$ $(i=1, 2, \ldots, n)$, $\sum_{i=1}^{n} \lambda_i=1$:

$$\| \sum_{i=1}^{n} \lambda_i x_i \| \leqslant \sum_{i=1}^{n} \lambda_i \Phi(\|x_i\|).$$

In [3] it is proved that every set K of Φ-type has the property that for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that:

$$\mathrm{co}\,(U \cap (K-K)) \subseteq V.$$

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NEKE TEOREME O NEPOKRETNOJ TAČKI ZA VIŠEZNAČNA PRESLIKAVANJA U VEKTORSKO TOPOLOŠKIM PROSTORIMA

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REZIME

Koristeći metodu dualnosti koju su uveli Husain i Tarafdar dokazane su neke teoreme o nepokretnoj tački za višeznačna preslikavanja u vektorsko topološkim prostorima. Takođe je primenom rezultata rada [2] dokazana sledeća teorema o koincidenciji.

TEOREMA 5. *Neka je K konveksan podskup vektorsko topološkog prostora E, K_1 kompaktan i konveksan podskup vektorsko topološkog prostora F čiji je fundamentalni sistem okolina nule dat familijom $\mathcal{V}$ i T i S su preslikavanja K u 2^{K_1} tako da su zadovoljeni sledeći uslovi:*

(i) *Preslikavanje T je od gore poluneprekidno i $T(u)$ je neprazan i konveksan podskup od K_1, za svako $u \in K$.*

(ii) *Za svako $u \in K$, $S(u)$ je otvoren u K_1 i $S^{-1}(v)$ je neprazan i konveksan podskup od K, za svako $v \in K_1$.*

(iii) *Za svako $V \in \mathcal{V}$ postoji $U \in \mathcal{V}$ tako da je:*

$$\mathrm{co}\,(U \cap (T(K) - T(K))) \subseteq V.$$

Tada postoji $u_0 \in K$ tako da je:

$$T(u_0) \cap S(u_0) \neq \emptyset.$$