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ON THE TOPOLOGICAL STRUCTURE OF RANDOM NORMED SPACES

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ABSTRACT. From the Theorem in [1] we obtain the following Corollary: *Let* $(S, \mathcal{F}, t)$ *be the random normed space with continuous T-norm t and* $\Phi_n(x) = \underbrace{t(t(\ldots t(t}_{n\text{—time}}(x, x), x), \ldots, x)$, *for every* $n \in N$ *and every* $x \in [0, 1]$. *If the family* $\{\Phi_n(x)\}_{n \in N}$ *is equicontinuous at the point* $x = 1$, *then S is, in the* (ε, λ) *topology, a locally convex topological vector space.*

We shall give a short proof of this Corollary without the use of the Theorem on the representation of Archimedean semigroups but using a result from Krauthausen's Dissertation [2].

The notion of random normed space was introduced in [4]. By Δ we shall denote the set of all distribution functions. Let S be a vector space and $\mathcal{F}: S \to \Delta$. The triplet $(S, \mathcal{F}, t)$ where t is a T-norm [3], is a random normed space iff the following conditions are satisfied:

1. $F_p(0) = 0$, for all p in S.
2. $F_p = H$ iff $p = 0 \in S$, where:

$$H(u) = \begin{cases} 1 & u > 0 \\ 0 & u \leqslant 0 \end{cases}$$

3. If λ is a non-zero scalar then:

$$F_{\lambda p}(u) = F_p\left(\frac{u}{|\lambda|}\right)$$

for all p in S and for all $u \in R$.

4. For all $p, q \in S$ and for all $u \geqslant 0$, $v \geqslant 0$:

$$F_{p+q}(u+v) \geqslant t(F_p(u), F_q(v))$$

5. $t(u, v) \geqslant \max\{u+v-1, 0\}$ for all u, v in $[0, 1]$.

The (ε, λ) topology in S is introduced by the family $\mathcal{U}=\{U(\varepsilon, \lambda)\}_{\substack{\varepsilon>0 \\ \lambda\in(0,1)}}$ of the neighbourhood of zero:

$$U(\varepsilon, \lambda)=\{x \mid F_x(\varepsilon)>1-\lambda\}$$

and if t is a continuous T-norm then S is, in the (ε, λ)-topology, a topological vector space.

Now, we shall give some results and definitions from Krauthausen's Dissertation [2].

DEFINITION 1. [2] *Let X be a vector space. The mapping $\|\cdot\|: X\to R$ is an F-Norm iff:*

(F1) *For every $x\in X: \|x\| \geqslant 0$.*

(F2) *For every $x\in X: \|x\|=0 \Leftrightarrow x=0$.*

(F3) *For every $x\in X$ and every $a\in\mathcal{K}$ ($\mathcal{K}$ is the scalar field) $|a| \leqslant 1 \Rightarrow \|a x\| \leqslant \|x\|$.*

(F4) *For every $x, y\in X: \|x+y\| \leqslant \|x\|+\|y\|$.*

(F5) *For every $x\in X$ and every $(a_n)\in\mathcal{K}^N$:*

$$\lim_{n\to\infty} a_n=0 \Rightarrow \lim_{n\to\infty} \|a_n x\|=0.$$

Then we say that $(X, \|\cdot\|)$ is an *F normed space.*

If (X, τ) is a Hausdoıff topological vector space with a denumerable base of neighbourhoods of zero, then there exists an F-norm on X such that $(X, \|\cdot\|)$ is an F-normed space and the topology τ and the topology induced by the F-norm [2] are identical.

DEFINITION 2. *Let X be a vector space and $A\subseteq X$. The set A is p-convex $(0<p\leqslant 1)$ iff for every $n\in N$, every $x_i\in A$ $(i=1, 2, \ldots, n)$ and every $a_i\geqslant 0$ $(i=1, 2, \ldots, n)$ the following implication holds:*

$$\sum_{i=1}^{n} a_i^p=1 \Rightarrow \sum_{i=1}^{n} a_i x_i\in A.$$

DEFINITION 3. *Let X be a Hausdorff topological vector space and $0<p\leqslant 1$. We say that X is locally p-convex iff there exists a neighbourhood base of zero $\mathcal{U}$ in X such that:*

$$V=\mathrm{co}_p V, \quad \textit{for every } V\in\mathcal{U}$$

where $\mathrm{co}_p V$ is p-convex hull of V.

In [2] the following Theorem ıs proved.

THEOREM. *Let $(X, \|\cdot\|)$ be a F-normed space and $0<p\leqslant 1$ such that:*

For every $(x_i)\in X^N: \lim_{n\to\infty} x_n=0 \Rightarrow \mathrm{co}_p\{x_i \mid i\in N\}$ is bounded. Then X is a locally p-con ex space.

Now, we shall prove the following Proposition.

PROPOSITION. *Let* $(S, \mathcal{F}, t)$ *be a random normed space with continuous T-norm t such that the family* $\{\Phi_n(x)\}_{n\in N}$ *is equicontinuous at the point* $x=1$. *Then S is, in the* (ε, λ) *topology, a locally convex topological vector space.*

Proof: Suppose that $\lim_{n\to\infty} x_n=0$, $(x_i)\in S^N$ and let us prove that $\operatorname{co}\{x_n | n\in N\}$ is bounded. Since S is a topological vector space, the set $\{x_n \mid n\in N\}$ is bounded iff for every $\varepsilon>0$ and $\lambda\in(0, 1)$ there exists $\mu(\varepsilon, \lambda)>0$ such that:

$$\{x_n \mid n\in N\}\subseteq \mu\cdot U(\varepsilon, \lambda). \tag{1}$$

The relation (1) means that $F_x(\mu(\varepsilon, \lambda)\varepsilon)>1-\lambda$, for every $x\in\{x_n | n\in N\}$. Since the family $\{\Phi_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$, there exists $\delta(\varepsilon, \lambda)$ such that $x\geqslant\delta\Rightarrow\Phi_n(x)>1-\lambda$, $n\in N$. The sequence $\{x_n\}_{n\in N}$ tends to zero and so there exists $n(\varepsilon, \lambda)\in N$ such that:

$$F_{x_n}(\varepsilon)>\delta, \quad \text{for every } n\geqslant n(\varepsilon, \lambda). \tag{2}$$

Let $M_1=\{x_n \mid n\geqslant n(\varepsilon, \lambda)\}$ and $M_2=\{x_n \mid n<n(\varepsilon, \lambda)\}$. Since for every $x\in S$, F_x is a distribution function, there exists $\rho(\varepsilon, \lambda)$ such that:

$$F_x(\rho(\varepsilon, \lambda)\varepsilon)>\delta, \text{ for every } x\in M_2 \ (\rho(\varepsilon, \lambda)\geqslant 1). \tag{3}$$

Suppose now that $x\in\operatorname{co}\{x_n \mid n\in N\}$ and let us prove that:

$$F_x(2\rho(\varepsilon, \lambda)\varepsilon)>1-\lambda. \tag{4}$$

Then, from (4) we have that $\mu(\varepsilon, \lambda)=2\rho(\varepsilon, \lambda)$. Since $x\in\operatorname{co}\{x_n \mid n\in N\}$, we have that:

$$x=\sum_{k=1}^{n} r_k x_{i_k}+\sum_{s=1}^{m} p_s x_{j_s}$$

where:

$$\sum_{k=1}^{n} r_k+\sum_{s=1}^{m} p_s=1 \ (r_k, p_s\geqslant 0 \text{ for every } k=1, 2, \ldots, n;\ s=1, 2, \ldots, m)$$

and $x_{i_k}\in M_1$ $(k=1, 2, \ldots, n)$ and $x_{j_s}\in M_2$ $(s=1, 2, \ldots, m)$.

Suppose that $\sum_{k=1}^{n} r_k<1$ and $\sum_{s=1}^{m} p_s<1$ (if $\sum_{k=1}^{n} r_k=1$ or $\sum_{s=1}^{m} p_s=1$ the proof is similar).

Now, we have:

$$F_x(2\rho(\varepsilon, \lambda)\varepsilon)=F_{\sum_{k=1}^{n} r_k x_{i_k}+\sum_{s=1}^{m} p_s x_{j_s}}(2\rho(\varepsilon, \lambda)\varepsilon)\geqslant t(F_{\sum_{k=1}^{n} r_k x_{i_k}}(\rho(\varepsilon, \lambda))\varepsilon,$$

$$, F_{\sum_{s=1}^{m} p_s x_{j_s}}(\rho(\varepsilon, \lambda)\varepsilon))$$

Further:

$$F_{\sum\limits_{k=1}^{n} r_k x_{i_k}}(\rho(\varepsilon,\lambda)\varepsilon)=F_{\sum\limits_{k=1}^{n} r_k x_{i_k}}(\rho(\varepsilon,\lambda)(\sum_{k=1}^{n} r_k+1-\sum_{k=1}^{n} r_k)\varepsilon)\geqslant$$

$$\geqslant \underbrace{t(t(\ldots t}_{n\text{ times}}(t(F_{x_{i_1}}(\rho(\varepsilon,\lambda)\varepsilon),F_{x_{i_2}}(\rho(\varepsilon,\lambda)\varepsilon)),F_{x_{i_3}}(\rho(\varepsilon,\lambda)\varepsilon)),$$

$$\ldots F_0(\rho(\varepsilon,\lambda)\cdot(1-\sum_{k=1}^{n} r_k)\varepsilon))=$$

$$=\underbrace{t(t(\ldots t}_{\text{n-times}}(F_{x_{i_1}}(\rho(\varepsilon,\lambda)\varepsilon),F_{x_{i_2}}(\rho(\varepsilon,\lambda)\varepsilon)),F_{x_{i_3}}(\rho(\varepsilon,\lambda)\varepsilon),\ldots,1)=$$

$$=\underbrace{t(t(\ldots t}_{\text{(n-1) times}}(t(F_{x_{i_1}}(\rho(\varepsilon,\lambda)\varepsilon),F_{x_{i_2}}(\rho(\varepsilon,\lambda)\varepsilon)),F_{x_{i_3}}(\rho(\varepsilon,\lambda)\varepsilon)),\ldots,$$

$$,F_{x_{i_n}}(\rho(\varepsilon,\lambda)\varepsilon))\geqslant\Phi_{n-1}(\delta)$$

since (3) holds.

Similarly from (2) it follows:

$$F_{\sum\limits_{s=1}^{m} p_s x_{j_s}}(\rho(\varepsilon,\lambda)\varepsilon)\geqslant F_{\sum\limits_{s=1}^{m} p_s x_{j_s}}(\varepsilon)=F_{\sum\limits_{s=1}^{m} p_s x_{j_s}}((\sum_{s=1}^{m} p_s+1-\sum_{s=1}^{m} p_s)\varepsilon)\geqslant\Phi_{m-1}(\delta).$$

So we have:

$$F_x(2\rho(\varepsilon,\lambda)\varepsilon)\geqslant t(\Phi_{n-1}(\delta),\Phi_{m-1}(\delta))=\Phi_{m+n-1}(\delta)>1-\lambda$$

and the proof is complete.

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O TOPOLOŠKOJ STRUKTURI SLUČAJNIH NORMIRANIH PROSTORA

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REZIME

U ovom radu dat je dokaz da je slučajan normirani prostor $(S, \mathcal{F}, t)$, sa T-normom t takvom da je familija funkcija $\{\Phi_n(x)\}_{n \in N}$ u tački $x=1$ podjednako neprekidna, lokalno konveksan u (ε, λ) topologiji. Ovaj dokaz se razlikuje od ranije datog dokaza iz rada [1].