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ALEKSANDAR KRON IN RELEVANCE LOGIC

The most important event in the career of Aleksandar Kron (1937–2000), logician and professor of philosophy at the University of Belgrade, was the visit to the University of Pittsburgh in the early 1970's. There he met Alan Ros Anderson and Nuel Belnap during the time when their Entailment: The Logic of Relevance and Necessity" was in preparation, so that Kron was present when relevance logic was in full growth. In Pitsburgh he entered the subject and remained in it up to the end of his life. This can be seen not only through the number of his papers in relevance logic, but through all other papers of his wide range of interests – the idea of relevancy is incorporated in his works in quantum logic, analysis of causality and decision theory.

Relevance logic is invented in order to avoid the paradoxes of material implication, such as $p \to (q \to p)$, and the paradoxes of strict implication, such as $p \to (q \vee \neg q)$. The modal logic and in particular the notion of the strict implication were made on the same grounds. What is wrong with these paradoxes is that antecedent and consequent of a true implication can have completely different contents. Since the logical system has nothing to do with the notion of content, it is necessary to find the formal aspects of a true implication that are ignored by the material and the strict implication. The formal principle of relevance logic, that forces theorems and inferences to preserve the content, is the variable sharing principle. This means that no implication can be proved in a propositional relevance logic if its antecedent and consequent do not have at least one propositional variable in common. No inference can be shown valid if the premises and conclusion do not share at least one propositional variable.

The implementation of variable sharing in a formal system requires a radical departure from the semantics of classical logic. Also, this principle is only the necessary, but not the sufficient condition for the logic to be understood as a relevance logic. It does not eliminate all of the paradoxes of material and strict implications. So relevance logic goes in a different way and provides a notion of relevant proof in terms of the real use of premises.

Relevance logics are proof-theoretically oriented so that it is easier to present their syntax in the form of natural deduction systems or sequent calculi, than as a Hilbert-style systems. Relevancy can be ensured by imposing certain restrictions on the rules of a natural deduction system or by removing the rules that allow the introduction of arbitrary formulae on the right or on the left side of the sequent in Gentzen calculus. The central systems of relevance logic are the logic R of relevant implication and the logic E of relevant entailment [Anderson, Belnap, 1975]. The logic R had been known long before the work of Anderson and Belnap, but they gave it the definite form and invented the logic E, and a bulk of other relevance logics close to both R and E. Among the theorems that are provable in the logic R there are, for example:

Identity:	$A \to A$,
Suffixing:	$(A \to B) \to ((B \to C) \to (A \to C))$,
Assertion:	$A \to ((A \to B) \to B)$,
Contraction:	$(A \to (A \to B)) \to (A \to B)$
Distribution:	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$,
Contraposition:	$(A \to \neg B) \to (B \to \neg A)$,
Double negation:	$\neg\neg A \to A$.

In the logic E of relevant entailment the implication was supposed to be a strict relevant implication. To achieve that, Anderson and Belnap defined the logic E such that in this logic one can prove the theorem of

$$\text{Entailment:} \quad ((A \to A) \to B) \to B.$$

Entailment is not provable in R so that R and E are different logics. If we present them as Hilbert-style systems, with modus ponens $A \to B, A \vdash B$ and adjunction $A, B \vdash A \wedge B$ as rules, then the logic E is the logic R minus assertion plus entailment. The logics R and E are variable sharing and do satisfy conditions of the real use of premises so that their implications are relevant. If we add the necessity operator $\Box$ to R, together with the axioms $\Box(A \to B) \to (\Box A \to \Box B)$ and $(\Box A \wedge \Box B) \to \Box(A \wedge B)$, and necessitation rule $A \vdash \Box A$, we obtain the modal logic of strict relevant implication NR. But the logic NR is different from E since, in some natural translation, there are theorems of NR that are not theorems of E, so that there are at least two logics that are candidates for the proper logic of strict relevant implication.

Among the problems with the logics R and E, we shall first mention the problem of the deduction theorem. After a long discussion in their Entailment, Anderson and Belnap concluded that, for the logics E and T (the relevance logic different from R and E), there is no deduction theorem in standard form. Kron [1973, 1976], found what is known as the best form of deduction theorem for the relevance logics T, E and R. The deduction theorem is crucial for the implication in every logical system so that this is one of Kron's most notable achievements in relevance logic. Charles Kielkopf in [1977] named certain derivations *Kron derivations* and deduction theorems, for relevance logics *Kron Deduction theorems.*

Another unusual property of the propositional logics R and E is that they are undecidable [Urquhart 1984]. These facts motivated Krons investigation of the subsystems of R and E as systems without distribution, without contraction or without involutive negation. Since the axiom of distribution is behind the failure of decidability of R and E, and because of the natural proof-theoretical way in which relevant distributionless logics arise, Došen [1993] believes that the central relevant logic should be R minus distribution and that its modal extension NR minus distribution, and without the axiom $(\Box A \wedge \Box B) \to \Box(A \wedge B)$, is a better candidate for the logic of strict relevance implication than the logic E. But there are semantical arguments in favor of distributivity - the semantics evaluates sentences as true or false in each world and treats conjunction and disjunction extensionally [Belnap 1993]. Kron believed in distributivity and tried to overcome undecidability of relevance logic by the removal of the contraction axiom. The presence of contraction corresponds to allowing premises to be used more than once. In the system of natural deduction with subscripts for relevance logic, subscripts serve to control the relevant use of premises and are finite sets. In the absence of contraction subscripts also serve to count the number of occurrences of premises, so that suscripts are now becoming finite multisets. These systems are not at all easy and Kron was careful and patient enough to develop and improve the specific subscript techniques for Getzen formulations of several relevant logics in the neighborhood of positive fragments of the logics R and T without contraction [Kron 1978, 1980; Giambrone and Kron 1987]. For these systems Kron proved the cut elimination theorems and obtained the decidability procedures.

Similar subscript techniques Kron used to solve the identity problem for the "most weak relevance logic" posed by Belnap in early 1960's [Kron 1985]. The logic was formulated in the language that contains propositional variables, parentheses and implication, has modus ponens as a rule, and has the following axioms

$$\text{Prefixing:}\quad (B \to C) \to ((A \to B) \to (A \to C)),$$
$$\text{Suffixing:}\quad (A \to B) \to ((B \to C) \to (A \to C)).$$

Belnap asked for the proof that no instance of the identity axiom $A \to A$ is a theorem in this logic or to say in different way, if $A \to B$ and $B \to A$ are provable in the logic with identity, prefixing and sufixing, then A and B are the same formula. The problem remained open for about twenty years, and was solved in 1983 by R. K. Meyer and E. Martin, using the algebraic and semantical methods. Kron solved the problem constructively in the sequent system, that contains the most weak relevance logic.

Among the arguments in favor of weaker relevance logics is not only the decidability. There is another feature that makes the relevance logic without contraction attractive - in the first-order predicate version of this logic, one can formulate a consistent naive set theory (with unrestricted comprehension axiom). The motivation for this logic came from the logic known as Grishins logic - the first-order predicate calculus without contraction axiom [Grishin, 1974]. Grishins logic is decidable and the distributivity is not provable in it. Kron defined a relevance variant of Grishins logic. This variant is close to quantificational R without contraction and, in contrast to Grishins logic, it is distributive. Kron was not willing to leave the distributivity. The logic has a Gentzen-style formulation and it is a decidable first-order logic with the Craig-Lindon interpolation property [Kron 1993].

The semantics of relevance logic is a Kripke-style semantics, but with the ternary accessibility relation R between worlds, and with the semantic clause for implication:

$A \to B$ holds in the world a if and only if, for all the worlds b and c,
if $R(a, b, c)$ and A holds at b, then B holds in the world c.

But the use of the ternary relation is not sufficient to avoid all the paradoxes of strict implication. This brings us to the non-classical clause for negation. It requires an operator * on the worlds such that

$\neg A$ is true in a if and only if A is false in the world a^*.

Once again, it is difficult to interpret this quite formal clause.

Kron did not study semantics, but he stimulated the investigation in this direction. Kosta Došen and Milan Božić worked with a ternary relation R such that $R(a, b, c)$ means $a \cdot b \leq c$, where $\cdot$ is a binary operation on worlds that corresponds to intensional conjunction and where $\leq$ is a partial ordering on worlds, so that the semantic clause for implication was:

$A \to B$ holds at the world b if and only if for every world a,
if A holds at a, then B holds at the world $a \cdot b$.

Using this interpretation of the relevant implication, Božić [1983] gave a transparent classification of the semantics of relevance logic - promising for some model theory of this logic.

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