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# IMAGE AND ITS MATRIX, MATRIX AND ITS IMAGE

**Abstract:** The need for matrix presentation by image is clear to all who in their work use high dimension matrices. Particular problem here make matrices that could not normally be presented by images – those whose elements are arbitrary real, or even complex numbers. Here we propose a solution for visualization of arbitrary, high dimension matrix. This is illustrated by several examples, with matrices used in two-dimensional discrete linear transforms, in image processing.

Keywords: digital image, matrix, visualization, discrete cosine transform, discrete Fourier transform, JPEG compression

## **1. Introduction**

There is a relation between matrices and digital images. A digital image in a computer is presented by pixels matrix. On the other hand, there is a need (especially with high dimensions matrices) to present matrix with an image. The motive of this work is to perceive matrix elements nature and arrangement: to recognize matrix parts – positive from negative, real from complex, bigger from smaller (by magnitude).

**1.1. Image and its matrix.** A digital grayscale image is presented in the computer by pixels matrix. Each pixel of such image is presented by one matrix element – integer from the set  $\{0,1,2,\ldots,255\}$ . The numeric values in pixel presentation are uniformly changed from zero (black pixels) to 255 (white pixels).

For example, the image  $\square^1$ 

1 / 0																
or, magnified:	is	pre	sen	ited	l w	ith	the	mati	rix							
	0	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240
	1	17	33	49	65	81	97	113	129	145	161	177	193	209	225	241
	2	18	34	50	66	82	98	114	130	146	162	178	194	210	226	242
	3	19	35	51	67	83	99	115	131	147	163	179	195	211	227	243
	4	20	36	52	68	84	100	116	132	148	164	180	196	212	228	244
	5	21	37	53	69	85	101	117	133	149	165	181	197	213	229	245
	6	22	38	54	70	86	102	118	134	150	166	182	198	214	230	246
	7	23	39	55	71	87	103	119	135	151	167	183	199	215	231	247
	8	24	40	56	72	88	104	120	136	152	168	184	200	216	232	248
	9	25	41	57	73	89	105	121	137	153	169	185	201	217	233	249
	10	26	42	58	74	90	106	122	138	154	170	186	202	218	234	250
	11	27	43	59	75	91	107	123	139	155	171	187	203	219	235	251
	12	28	44	60	76	92	108	124	140	156	172	188	204	220	236	252
	13	29	45	61	77	93	109	125	141	157	173	189	205	221	237	253
	14	30	46	62	78	94	110	126	142	158	174	190	206	222	238	254
	15	31	47	63	79	95	111	127	143	159	175	191	207	223	239	255

Color images (with RGB color model) in a computer are presented with three grayscale images matrices (one for each – red, green and blue – color components).

**1.2. Matrix and its image.** Each image processing operation in a computer may be observed as an operation on image matrix. Usually, the result of such operation is matrix of original image

<sup>&</sup>lt;sup>1</sup> If it is necessary, images will have borders, which will visually demarcate them from the background.

dimension  $(m \times n)$ , whose elements need not be from the set  $\{0,1,...,255\}$ , but also they may be arbitrary real, or even complex numbers. Matrices we use here, are of very high dimensions (several tens, or hundreds of thousands, or even over million of elements). In order to get visualization about such matrix overall, the need to present it with an image arises naturally.

This idea, certainly, is not new. In programs dedicated for work with matrices, there exists a possibility to present matrix by image. However, this presentation is mainly inadequate. It does not take into consideration values out of the set  $\{0,1,2,...,255\}$ .

Matlab-like programs do this in the following steps:

- First, imaginary part is discarded (if matrix is complex)
- Then, all elements are rounded to the nearest integer
- Finally, values less than 0 become 0
- Values greater than 255 become 255

Clearly, such presentation does not show information about the contents for majority of matrices.

Good matrix presentation needs to represent nature and mutual relation between elements values. It has to show whether the matrix elements are positive or negative, real or complex numbers, and which is mutual relation between elements magnitudes.

Here we propose a different solution – hopefully better.

The following text consists of two parts (sections 2 and 3). In section 2 ("Matrices coloring") an algorithm for visualization of arbitrary matrices will be proposed. In section 3 ("Visualization of matrices used in image processing") several interesting examples of presenting high dimension matrices (real and complex) ") will be given. All these examples are related with linear transforms, which have use in image processing. The proposed algorithm of matrices visualization will hopefully be useful for linear transforms understanding.

## 2. Matrices coloring

**2.1. Presentation of real matrices.** In a matrix, in which all elements are positive or equal to zero, we may present value 0 with black, and maximum positive value with white color; in this way, all positive matrix values will be presented as grayscale nuances (from black for zero, to white for maximum positive value).

*Example 2.1:* A matrix and its image:

0	50	100	150	200	250	300	350
50	100	150	200	250	300	350	400
100	150	200	250	300	350	400	450
150	200	250	300	350	400	450	500
200	250	300	350	400	450	500	550
250	300	350	400	450	500	550	600
300	350	400	450	500	550	600	650
350	400	450	500	550	600	650	700



In this solution, matrix element values are scaled in order to be squeezed into interval [0, 255] – all matrix values are multiplied by factor (255/max. value) (in our example, this is 255/700). A matrix with all elements non-positive may be presented similarly, in nuances from black to yellow.

A matrix with positive and negative values will be presented with an image, in which positive values will be given in nuances from black to white, and negative – from black to yellow. Scaling may be done according to the biggest matrix elements magnitude.

*Example 2.3:* The matrix





*Example 2.4:* In the next figure, two white Gaussian noise matrices<sup>2</sup> may be seen. Matrix on the left is of considerably lower dimension comparing with that on the right, and its elements are presented "zoomed".



**2.2. Complex matrix presentation.** Pure imaginary matrices may be presented similarly. Negative imaginary values we present with nuances from black to green, and positive – from black to red color.

*Example 2.5:* In the next three images real and imaginary matrix, and their sum – complex matrix are presented.

<sup>&</sup>lt;sup>2</sup> Matrices whose elements have values from normal distribution with zero mean

![](_page_3_Figure_0.jpeg)

Of course, the proposed colors combination is not the only possible. However, it is not also completely arbitrary. In colors choice it is very important that every two distant points in the complex plane would be presented with different colors.

Let us explain the notion of "distant points". This demand means that between any two points in the complex plane presented with the same color must not be any point of different color. (Clearly, two close points, due to infinite number of complex plane points, and finite number of different colors, may be colored by the same color).

The suggested color combination (yellow–white& green–red) is correct in this sense. Used colors intensity increases with moving away of origin, and therefore, in none of the four complex plane quadrants two distant points colored with the same color exist.

In addition, in quadrants, colors appearance in complex numbers presentation is  $(x, y \ge 0)$ :

I quadrant: $x(r+g+b) + y \cdot r$	mostly red, and some amount of blue color
II quadrant: $x(r+g) + y \cdot r$	mostly red, and without blue color
III quadrant: $x(r+g) + y \cdot g$	mostly green, and without blue color
IV quadrant: $x(r+g+b)+y \cdot g$	mostly green, and some amount of blue color
ore two points in different quadrants r	presented with the same color. do not exist

Therefore two points in different quadrants presented with the same color, do not exist. If, instead of green, we use blue color, it will be (combination yellow–white& blue–red): III quadrant:  $x(r+g) + y \cdot b = x(r+g+b) + (y-x)b$ 

IV quadrant:  $x(r+g+b) + y_1 \cdot b$ 

Hence, in the third and in the fourth quadrant, two points presented with the same color (combination of white and blue) exist. Therefore, the combination yellow-white& blue-red is not a good one.

In the next two figures are presented these two colors combinations. In the left figure is the "good" combination (yellow–white& green–red). In the right one is the "bed" combination (yellow–white& blue–red). Here, there are marked several pairs of distant points, presented by the same color.

![](_page_4_Figure_5.jpeg)

![](_page_4_Figure_6.jpeg)

# 3. Visualization of matrices used in image processing

In this section several interesting examples will be mentioned, in which, owing to very high dimensions, it is necessary to present matrix with an image,. Examples are chosen with the goal of making easier understanding of (in image processing often used) linear transforms of matrices. Subsection 3.1 is dedicated to discrete cosine transform (DCT) of image of arbitrary dimensions. Subsection 3.2 presents blocked DCT, used in JPEG compression. Subsection 3.3 illustrates one way for image dimensions (including aspect ratio) changing. Subsection 3.4 presents Fourier transform (FT), which, in contrast to other transforms, uses complex matrices.

**3.1. Discrete cosine transform of image.** We get discrete cosine transform (DCT) of matrix S of dimension  $m \times n$ , using formula  $D = M_m^{-1} \cdot S \cdot M_n$ , where S is an image matrix in spatial domain (original image matrix), D is image matrix in transform domain; both matrices (S and D) have dimensions  $m \times n$ .  $M_m$  and  $M_n$  are square matrices of dimensions  $m \times m$  and  $n \times n$  respectively, and may be calculated with the code (here written for matrix of dimension  $m \times m$ , labeled with M):

```
c=ones(m)*sqrt(2/m); c(1)=sqrt(1/m);
for i=1:m
    for j=1:m
        M(i,j)=c(j)*cos((2*i-1)*(j-1)*pi/(2*m));
    end
end
```

These matrices are constant – they do not depend on image S content, but only on its dimension.

In the several next figures are images for matrices  $M_2$ ,  $M_3$ ,  $M_4$ ,  $M_8$  and  $M_{128}$ , for discrete cosine transform.

![](_page_5_Figure_3.jpeg)

We may observe matrices S and D as presentations of the same image in two different bases.

*Example 3.1:* If the image is in the spatial domain presented with matrix  $S = \begin{bmatrix} 0 & 50 \\ 100 & 200 \end{bmatrix}$ , then (since  $M_2 = M_2^{-1} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ ),

$$\boldsymbol{D} = \boldsymbol{M}_2^{-1} \cdot \boldsymbol{S} \cdot \boldsymbol{M}_2 = \begin{bmatrix} 175 & -75 \\ -125 & 25 \end{bmatrix}.$$

In the spatial domain, the image is presented in the standard basis  $\boldsymbol{B}_s = \{\boldsymbol{b}_s^{11}, \boldsymbol{b}_s^{12}, \boldsymbol{b}_s^{21}, \boldsymbol{b}_s^{22}\}$ where  $\boldsymbol{b}_s^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{b}_s^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{b}_s^{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \boldsymbol{b}_s^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ 

 $\boldsymbol{S} = 0 \cdot \boldsymbol{b}_s^{11} + 50 \cdot \boldsymbol{b}_s^{12} + 100 \cdot \boldsymbol{b}_s^{21} + 200 \cdot \boldsymbol{b}_s^{22}$ In the transform domain, the same image is presented with matrix  $\boldsymbol{D}$  in the basis

$$\boldsymbol{B}_{d} = \{\boldsymbol{b}_{d}^{11}, \boldsymbol{b}_{d}^{12}, \boldsymbol{b}_{d}^{21}, \boldsymbol{b}_{d}^{22}\}$$
  
where  $\boldsymbol{b}_{d}^{11} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ \boldsymbol{b}_{d}^{12} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \ \boldsymbol{b}_{d}^{21} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \ \boldsymbol{b}_{d}^{22} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \text{ and}$   
$$\boldsymbol{S} = 175 \cdot \boldsymbol{b}_{d}^{11} - 75 \cdot \boldsymbol{b}_{d}^{12} - 125 \cdot \boldsymbol{b}_{d}^{21} + 25 \cdot \boldsymbol{b}_{d}^{22}$$

In the next figure we may see base images  $B_s$  (left) and  $B_d$  (right) for matrices of dimension  $2 \times 2$ .

![](_page_6_Figure_6.jpeg)

![](_page_6_Figure_7.jpeg)

In the next two figures are presented base images  $B_d$  for matrices of dimension  $4 \times 4$  and  $8 \times 8$ .

![](_page_6_Figure_9.jpeg)

![](_page_7_Picture_1.jpeg)

*Example 3.2:* Several first steps in presentation of crest image of Faculty of Mathematics, Belgrade:

![](_page_7_Picture_3.jpeg)

Original image has dimension  $32 \times 28$ , and it is presented in base  $B_d$  with linear combination of 896 base images (each of them is multiplied by corresponding element in matrix D). Here first 18 steps in deriving original image from base images, are showed. Each step is presented by pair (base image, result image). Steps are arranged in compliance with decreasing order of coefficients magnitudes in matrix D.

**3.2. Blocked DCT and JPEG compression.** JPEG compression of grayscale image  $S_c$  is based on blocked DCT:

- Image  $S_c$  is divided in image blocks of dimensions  $8 \times 8$  pixels

– DCT is applied on each image block. Result is matrix  $D_c$ , produced with joining all  $8 \times 8$  blocks in the transform domain.

- Each element of each  $8 \times 8$  block is subjected to quantization – is divided by a number, determined in advance, and the quotient is rounded to the nearest integer. The divisor for each block element is determined by quantization table<sup>3</sup>, i.e. by position of the element in the block. This is a "lossy" step in compression – it enables that the image data may be recorded in a less memory space, with respect to the original image data.

*Example* 3.3: In the next figure we may see images for matrices S (dimension  $8 \times 8$ ),  $M_8$  and D.

![](_page_8_Figure_7.jpeg)

Block D – result of discrete cosine transform (also of dimension  $8 \times 8$ ), in contrast with block in spatial domain, has prominent differences between values in different block parts. Its elements near top-left corner (so-called DC element) are with far bigger magnitudes values comparing with block remaining part. These values decline with going to bottom-right corner.

The same result we obtain with matrix multiplication of  $S_c$  (here, "Cameraman", dimension of 128×128) and  $M_{8c}$  (presented in the middle) – result is again matrix  $D_c$  (right)

$$\boldsymbol{D}_{c} = \boldsymbol{M}_{8c}^{-1} \cdot \boldsymbol{S}_{c} \cdot \boldsymbol{M}_{8c}$$

![](_page_8_Picture_11.jpeg)

<sup>&</sup>lt;sup>3</sup> Quantization table is in advanced determined matrix of dimension  $8 \times 8$ , which contains divisors for all block elements.

Matrix  $M_{sc}$  is almost diagonal. On its diagonal, it has matrices  $M_s$ , and beyond it, all elements have values zero.

In each  $8 \times 8$  matrix  $D_c$  block, the block elements near the DC element have bigger magnitudes comparing with the rest of the block; on the other hand, they have smaller quantization coefficients in relation to remote elements. Thus, quantization is more intensive (more data will be lost) in block elements remote from DC element.

Therefore, in JPEG compression, data in bottom-right corner of each DCT block are lost much more, than in top-left corner.

Three next examples show data nullifying impact<sup>4</sup> in bottom–right image block parts on matrix  $D_c$  (Nullifying effect for the small part of each 8×8 block; a half of each image block; the most part of each image block in transform domain).

Each of the showed three examples is presented with three images:

– designation of block part that will be nullified;

- matrix  $D_c$ ' arisen by nullifying of coefficients part in all matrix blocks;
- matrix  $S_c$ ' arisen from  $D_c$ ' by applying inverse DCT.

Inverse DCT (multiplication by inverse of matrix  $M_{sc}$ ) returns us into the spatial domain with some extent distorted image. If we have not gone too far, these changes need not be visible to human eyes.

![](_page_9_Picture_10.jpeg)

**3.3. One way for image dimension changing.** Image dimension changing may also be done by applying transform (for example discrete cosine) on the whole image:

<sup>&</sup>lt;sup>4</sup> In these examples, in order to illustrate, more complicate (and not much different) quantization we replace with data nullifying in bottom-right corner of each block.

- We perform DCT overall on the whole image S (here presented, as example, image "Cameraman", of dimension  $128 \times 128$  - top left). Result of this operation (matrix D) we may see on the right (showed on top-right image). Matrix D is the same dimension ( $128 \times 128$ ).

- We add 30 columns to the right side of matrix D (elements values in these appended columns are 0). Then, we crop 30 rows in the bottom of it. The result matrix D', of dimension  $98 \times 158$  we may see on the bottom-left image<sup>5</sup>.

On matrix D' we apply inverse DCT. Result is image S' of dimension  $98 \times 158$  showed bottom right.

After inverse DCT, we have original image with aspect ratio changed.

![](_page_10_Picture_5.jpeg)

**3.4. Fourier transform.** Fourier transform (FT) of matrix S of dimension  $m \times n$  we get with formula

$$F = M_m^{-1} \cdot S \cdot M_n$$

Matrix S is an original image matrix, matrix F is FT for this image; both of them have dimension  $m \times n$ . Square matrices  $M_m$  and  $M_n$  have dimensions  $m \times m$ , and  $n \times n$ , respectively, and they may be calculated by Matlab code (here, this matrix is labeled with M, and of dimension  $m \times m$ ):

```
for il=1:m
    for jl=1:m
        M(il,jl)=exp(-2*pi*i/m)*(il-1)*(jl-1);
        end
end
```

<sup>&</sup>lt;sup>5</sup> When DCT is applied on the whole image, elements magnitudes near DC element in resultant matrix are much bigger, comparing with the rest of the matrix. So, matrix D (and D') would be presented with totally black rectangle, with little white point in top–left corner. In order to have better matrix presentation, top–right (D) and bottom–left (D') images are presented with logarithmed magnitudes.

F and  $M_m$  (and  $M_n$ ) are complex matrices. Here it will be presented several  $M_m$  matrices for Fourier transform:

![](_page_11_Figure_2.jpeg)

Graphical presentations of base images for Fourier transform:

![](_page_11_Figure_4.jpeg)

![](_page_12_Figure_1.jpeg)

For matrices S and F applies

$$S = \sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} \cdot b_{s}^{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij} \cdot b_{f}^{ij}$$
, where  $S = [s_{ij}]_{m \times n}$ ,  $F = [f_{ij}]_{m \times n}$ , and  $b_{s}^{ij}$  is base image in spatial domain that correspond with matrix  $S$  coefficient  $s_{ij}$ ,  $b_{f}^{ij}$  is base image in transform domain that corresponds with matrix  $F$  coefficient  $f_{ij}$ .

*Example 3.5: Image presentation in new basis.* In the next figure, we may see several first steps in presentation of image S by FT base images.

Original image of dimension 8×8 pixels

![](_page_13_Picture_3.jpeg)

In this example matrix S is of dimension  $8 \times 8$ , and it is presented in base  $B_d$  with linear combination of 64 base images of dimension  $8 \times 8$  (each of them is multiplied by corresponding element of matrix F). Here, first 18 steps in deriving original image from matrix F and base images are showed. Each step is presented by pair (base image, result image). Steps are arranged in compliance with decreasing order of coefficients magnitudes in matrix F.

![](_page_13_Figure_5.jpeg)

## 4. Conclusion

Here the proposed solution enables visualization of arbitrary matrices. The presented examples are mainly related with linear transforms – DCT and Fourier, and blocked DCT (which is used in JPEG compression). This examples choice is made for the reader to understand better high dimension matrices, used in image processing. This solution may be helpful, not only to people that are engaged in image processing, but also to all engaged in (real or complex) high dimension matrices.

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### СЛИКА И ЊЕНА МАТРИЦА. МАТРИЦА И ЊЕНА СЛИКА

Сажетак: Предлажмо решење за визуелизацију матрица — како реалних, тако и комплексних. Потреба за приказом матрице у облику слике јасна је свима који се баве матрицама великих димензија. Посебан проблем ту стварају матрице које не могу бити нормално приказане сликом — оне чији су елементи произвољни реални, па чак и комплексни бројеви. Дато решење илустровано је са неколико примера, матрицама коришћеним у дводимензионим дискретним линеарним трансформацијама, у обради слика.

**Кључне речи:** дигитална слика, матрица, визуелизација, дискретна косинусна трансформација, дискретна Фуријеова трансформација, ЈПЕГ компресија

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