

A CHARACTERIZATION OF ULAM HYPERSTABILITY

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Abstract. The main result of this work is a simple characterization of the hyperstability of functional equations in a very general framework: for functions with the codomain endowed with a generalized homogeneous premetric. We also give some particularizations on Abelian groups equipped with generalized homogeneous norms, obtaining, among others, improvements of similar known results for Cauchy-type, Jensen-type equations, but also for equations having compound functions as solutions.

1. Introduction

In 2014, Brzdęk [2] observes an interesting property of Cauchy differences: if X and Y are normed linear spaces, r, s are real numbers such that $r + s > 0$, $D := (X \setminus \{0\}) \times (X \setminus \{0\})$ and $f : X \rightarrow Y$, then

$$\sup_{(x,y) \in D} \|x\|^r \|y\|^s \|f(x+y) - f(x) - f(y)\| \in \{0, \infty\}.$$

In other words, an arbitrary function $f : X \rightarrow Y$ is either *Cauchy on D* , i.e.

$$f(x+y) = f(x) + f(y) \quad \text{for all } (x, y) \in D \tag{1}$$

or

$$\sup_{(x,y) \in D} \|x\|^r \|y\|^s \|f(x+y) - f(x) - f(y)\| = \infty.$$

This property is a consequence of the φ -hyperstability of equation (1), where $\varphi : D \rightarrow (0, \infty)$ is a control function defined by $\varphi(x, y) := \|x\|^{-r} \|y\|^{-s}$.

The question arises whether this phenomenon also occurs for other functional equations, other control functions and what is the connection with the hyperstability.

Hyperstability of functional equations is a special case of their stability – a problem posed by S. M. Ulam [9] in metric spaces. Here we will widen the framework: instead of metric spaces we will use some special premetric spaces. We say that (Y, d) is

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a *generalized premetric space* if the function $d : Y \times Y \rightarrow \mathbb{R}_+ := [0, \infty]$ – called a *premetric* – satisfies only the condition: for all $x, y \in Y$

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y.$$

In the following lines, (Y, d) is a generalized premetric space, X and D are nonempty sets, G and $H : Y^X \rightarrow Y^D$ are two functions and

$$Gf(z) = Hf(z) \quad \text{for all} \quad z \in D \quad (2)$$

is a functional equation, where $f : X \rightarrow Y$ is the unknown. If $\varphi : D \rightarrow (0, \infty)$ is a function, then equation (2) is φ -hyperstable if and only if all the functions $f : X \rightarrow Y$ for which

$$d(Gf(z), Hf(z)) \leq \varphi(z) \quad \text{for all} \quad z \in D \quad (3)$$

are solutions of equation (2). In this context we say that φ is a *control function* for equation (2).

By $S_{(i)}$ we denote the set of solutions of equation (inequality) (i); for instance, equation (2) is φ -hyperstable if and only if $S_{(2)} = S_{(3)}$.

We will give very general conditions under which the φ -hyperstability of equation (2) is equivalent with the alternative

$$\text{either } f \in S_{(2)} \quad \text{or} \quad \sup_{z \in D} \frac{1}{\varphi(z)} d(Gf(z), Hf(z)) = \infty \quad \text{for all } f \in Y^X, \quad (4)$$

in terms of generalized homogeneity of the functions d , G and H . We apply these results within normed groups and, in particular, for Cauchy-type equations and Jensen-type equations.

Hyperstability approached in the broader sense of Ulam stability is analyzed in [3]. Information on the status of Ulam stability research can be found in [5]. Hyperstability conditions for Cauchy-type equations on restricted domains (in the meaning of equation (1), when $D = X^2$) are given in [2] and, in a more general framework (when X is part of an abelian semigroup, $D \subseteq X^2$ and the control function φ is not necessarily of Aoki-Rassias-type), in [7]. The same approach, but for Jensen-type equations, is done in [8]. Sufficient conditions under which alternatives of the form (4) hold for Cauchy equations were first given in [2] for some Aoki-Rassias-type control functions, and in [7] in an extended framework; in [8], a similar result for Jensen-type equation was proved. We can consult [6] for the behavior of generalized homogeneous functions. Details on normed groups can be found in [1].

2. Results

In the following $\mathbb{N}^*$ is the set of positive integers, $\tau \circ g$ denotes the composition of the functions $U \xrightarrow{g} Y \xrightarrow{\tau} Y$, $\tau^0 := \text{id}_Y$ and τ^i is the i -th iteration of τ for $i \in \mathbb{N}^*$.

We show first that alternative (4) is a sufficient condition for the hyperstability of equation (2).

PROPOSITION 2.1. *If $\varphi : D \rightarrow (0, \infty)$ is a function that satisfies alternative (4), then equation (2) is φ -hyperstable.*

Proof. Suppose there is a function $h \in S_{(3)} \setminus S_{(2)}$. Then $\frac{1}{\varphi(z)}d(Gh(z), Hh(z)) \leq 1$ for $z \in D$ and there exists $z_0 \in D$ for which $\frac{1}{\varphi(z_0)}d(Gh(z_0), Hh(z_0)) > 0$. Consequently $\sup_{z \in D} \frac{1}{\varphi(z)}d(Gh(z), Hh(z)) \in (0, 1]$, which is a contradiction. Therefore equation (2) is φ -hyperstable. $\square$

2.1 Hyperstability on (τ, γ) -spaces

As above, X and D are nonempty sets, (Y, d) is a generalized premetric space and $G, H : Y^X \rightarrow Y^D$ are two functions.

We introduce the concept of generalized homogeneity.

DEFINITION 2.2. Let $n \in \mathbb{N}^*$ and $\tau : Y \rightarrow Y$ be a function.

1. The function $E : Y^X \rightarrow Y^D$ is (τ, τ^n) -homogeneous if $E(\tau \circ f) = \tau^n \circ Ef$ for all $f \in Y^X$.

2. The couple (Y, d) is a (τ, γ) -space if d is a generalized premetric on Y , $\tau : Y \rightarrow Y$ is surjective function, $\gamma \in (0, \infty) \setminus \{1\}$ and the generalized premetric d is (τ, γ) -homogeneous, i.e., $d(\tau(x), \tau(y)) = \gamma d(x, y)$ for all $x, y \in Y$.

We can consult [6] for the behavior of homogeneous functions in a generalized sense.

REMARK 2.3. The study of Ulam stability of linear operators between real or complex vector spaces equipped with gauges was initiated in [4]. Recall that, if Y is a vector space over the field $K \in \{\mathbb{R}, \mathbb{C}\}$, then $\rho : Y \rightarrow [0, \infty]$ is a *gauge* if $\rho(x) = 0$ if and only if $x = 0$, and $\rho(\lambda x) = |\lambda| \rho(x)$ for all $x \in Y$ and $\lambda \in K \setminus \{0\}$. In this context, defining $d : Y^2 \rightarrow [0, \infty]$, $d(x, y) := \rho(x - y)$ and $\tau_\lambda : Y \rightarrow Y$, $\tau_\lambda(x) := \lambda x$, we remark that (Y, d) is a $(\tau_\lambda, |\lambda|)$ -space for all $\lambda \in K \setminus \{0\}$ with $|\lambda| \neq 1$.

This is the framework in which we will prove a complement of Proposition 2.1.

THEOREM 2.4. *We assume that X, D are nonempty sets, (Y, d) is a (τ, γ) -space and that $G, H : Y^X \rightarrow Y^D$ are (τ, τ^n) -homogeneous functions. If $\varphi : D \rightarrow (0, \infty)$ is a function, then dichotomy (4) holds if and only if equation (2) is φ -hyperstable.*

Proof. 1. From Proposition 2.1 it follows that alternative (4) implies the φ -hyperstability of equation (2).

2. For proving the converse, we first note that, if $x, y \in Y$ such that $\tau(x) = \tau(y)$, we have $\gamma d(x, y) = d(\tau(x), \tau(y)) = 0$ and $x = y$. Hence $\tau : Y \rightarrow Y$ is a bijective correspondence.

Since d is (τ, γ) -homogeneous, by induction on $j \in \mathbb{N}^*$ we get

$$d(\tau^j(x), \tau^j(y)) = \gamma^j d(x, y) \text{ for all } x, y \in Y, j \in \mathbb{N}^*.$$

Replacing x, y by $\tau^{-j}(x), \tau^{-j}(y)$, where $\tau^{-j} := (\tau^j)^{-1}$, we get $d(\tau^{-j}(x), \tau^{-j}(y)) = \gamma^{-j}d(x, y)$. Therefore,

$$d(\tau^j(x), \tau^j(y)) = \gamma^j d(x, y) \quad \text{for all } x, y \in Y, j \in \mathbb{Z}. \quad (5)$$

Similarly, since G and H are (τ, τ^n) -homogeneous,

$$G(\tau^j \circ f) = \tau^{nj} \circ Gf, H(\tau^j \circ f) = \tau^{nj} \circ Hf \quad \text{for all } f \in Y^X \text{ and } j \in \mathbb{Z}. \quad (6)$$

Suppose that equation (2) is φ -hyperstable and $f \in Y^X$ such that

$$c := \sup_{z \in D} \frac{1}{\varphi(z)} d(Gf(z), Hf(z)) < \infty.$$

Then,

$$d(Gf(z), Hf(z)) \leq c\varphi(z) \quad \text{for all } z \in D. \quad (7)$$

It is enough to prove that $f \in S_{(2)}$, (i.e. $Gf = Hf$) or, equivalently, $c = 0$.

Let $j \in \mathbb{Z} \setminus \{0\}$ be such that

$$c\gamma^{nj} \leq 1. \quad (8)$$

Using consecutively (6), (5), (7) and (8) we get

$$\begin{aligned} d(G(\tau^j \circ f)(z), H(\tau^j \circ f)(z)) &= d((\tau^{nj} \circ Gf)(z), (\tau^{nj} \circ Hf)(z)) \\ &= \gamma^{nj} d(Gf(z), Hf(z)) \leq c\gamma^{nj} \varphi(z) \leq \varphi(z) \end{aligned}$$

for all $z \in D$. But equation (2) is φ -hyperstable, hence $\tau^j \circ f \in S_{(2)}$ i.e.,

$$G(\tau^j \circ f) = H(\tau^j \circ f), \quad \text{or, by (6), } \tau^{nj} \circ Gf = \tau^{nj} \circ Hf.$$

Since τ^{nj} is injective we get $Gf = Hf$ and $c = 0$. $\square$

When the cardinality of Y – denoted $|Y|$ – is 1, we get $S_{(2)} = S_{(3)} = Y^X$ and equation (2) is φ -hyperstable for every $\varphi : D \rightarrow (0, \infty)$. We can analyze the hyperstability of equation (2) when Y (or D) is finite, d is finite, but (2) is not trivial ($S_{(2)} \neq Y^X$).

COROLLARY 2.5. *Assume that (Y, d) , G and H satisfy the assumptions in Theorem 2.4, d is a finite function, $|Y| < \infty$ or $|D| < \infty$, and $S_{(2)} \neq Y^X$. Then there are no functions $\varphi : D \rightarrow (0, \infty)$ for which equation (2) is φ -hyperstable.*

Proof. Suppose that $\varphi : D \rightarrow (0, \infty)$ is a control function for which equation (2) is φ -hyperstable. Let $f \in Y^X \setminus S_{(2)}$. Then there is $z_0 \in D$ such that $d(Gf(z_0), Hf(z_0)) > 0$, hence

$$\sup_{z \in D} \frac{1}{\varphi(z)} d(Gf(z), Hf(z)) \in (0, \infty)$$

which contradicts the conclusions of the previous theorem. $\square$

2.2 Hyperstability on (m, γ) -groups

A special type of (τ, γ) -space is an (m, γ) -group.

DEFINITION 2.6. We say that $(Y, +, |||)$ is an (m, γ) -group if $m \in \mathbb{N}^*$, $(Y, +)$ is a uniquely m -divisible normed Abelian group, $\gamma \in (0, \infty) \setminus \{1\}$ and $||| : Y \rightarrow \mathbb{R}_+$ is a (m, γ) -homogeneous norm on Y , i.e., $|||my|| = \gamma |||y||$ for all $y \in Y$.

We recall that:

- the commutative group $(Y, +)$ is uniquely m -divisible if and only if for all $y \in Y$, the equation $mx = y$ has a unique solution $x \in Y$; we denote $x := m^{-1}y$ or $x := \frac{y}{m}$;
- $\|\cdot\| : Y \rightarrow \mathbb{R}_+$ is a norm on the commutative group $(Y, +)$ if $\| -y \| = \|y\|$, $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in Y$ and $\|x\| = 0$ iff $x = 0$. Details on normed groups can be found in [1].

Nontrivial examples of (m, γ) -groups (e.g. normed linear spaces, valuated rings or F spaces) can be found in [6].

PROPOSITION 2.7. *If $(Y, +, \|\cdot\|)$ is an (m, γ) -group, then (Y, d) is a (τ, γ) -space, where $d(x, y) := \|x - y\|$ and $\tau(x) := mx$.*

Proof. Of course d is a translation invariant metric on Y and a (τ, γ) -homogeneous function. $\square$

Under the terms of Proposition 2.7, we note that

- $\ker \tau = \{0\}$ (since, for $x \in Y$ with $\tau(x) = 0$ we have $0 = \|mx\| = \gamma\|x\|$ and $x = 0$; hence τ is a monomorphism);
- since $(Y, +)$ is uniquely m -divisible we conclude that the function

$$\sigma : Y \rightarrow Y, \quad \sigma(y) := m^{-1}y$$

is an onto and a right inverse of the monomorphism τ (since $\tau \circ \sigma(y) = m(m^{-1}y) = y$ for all $y \in Y$); therefore τ is an automorphism of Y and $\sigma = \tau^{-1}$.

In this framework, as usual we define

$$m^k g := \tau^k \circ g \quad \text{for } k \in \mathbb{Z} \text{ and } g \in Y^X \cup Y^D.$$

In these terms we can characterize the hyperstability of the equation

$$Ef(z) = 0 \text{ for all } z \in D, \tag{9}$$

where $f : X \rightarrow Y$ is the unknown, $(Y, +, \|\cdot\|)$ is an (m, γ) -group and $E : Y^X \rightarrow Y^D$ is a fixed function; if E is (τ, τ^n) -homogeneous we also say that E is (m, m^n) -homogeneous, i.e., the function E satisfies $E(mf) = m^n Ef$ for all $f \in Y^X$.

From Proposition 2.7 and Theorem 2.4 we immediately obtain the following characterization of hyperstability.

COROLLARY 2.8. *Let X, D be nonempty sets, $(Y, +, \|\cdot\|)$ be an (m, γ) -group, $E : Y^X \rightarrow Y^D$ be an (m, m^n) -homogeneous function, where $n \in \mathbb{Z} \setminus \{0\}$ and $\varphi : D \rightarrow (0, \infty)$ be a function. Then equation (9) is φ -hyperstable if and only if*

$$\text{either } Ef = 0 \quad \text{or} \quad \sup_{z \in D} \frac{1}{\varphi(z)} \|Ef(z)\| = \infty,$$

for all $f : X \rightarrow Y$.

2.3 Applications to Cauchy-type and Jensen-type equations

Let $(S, +)$ be an Abelian semigroup.

2.3.1. If we assume that the sets

$$X \subseteq S, D \subseteq \{(x, y) \in X^2 \mid x + y \in X\} \quad (10)$$

are nonempty, $(Y, +, \|\cdot\|)$ is an (m, γ) -group and

$$E : Y^X \rightarrow Y^D, \quad Ef(x, y) := f(x + y) - f(x) - f(y),$$

then we can apply the above results to the Cauchy-type equation $Ea = 0$, i.e.,

$$a(x + y) = a(x) + a(y) \quad \text{for all } (x, y) \in D, \quad (11)$$

where $a : X \rightarrow Y$ is the unknown; in this case we say that a is *Cauchy* on D . We note that E is (m, m) -homogeneous for $m \in \mathbb{N}^*$. Applying Corollary 2.8 we obtain the following extension of [7, Theorem 2.3].

COROLLARY 2.9. *We assume that $(Y, +, \|\cdot\|)$ is an (m, γ) -group, the sets $X \subseteq S$ and $D \subseteq X^2$ satisfy (10) and $\varphi : D \rightarrow (0, \infty)$ is a function. Then equation (11) is φ -hyperstable if and only if an arbitrary function $f \in Y^X$ is Cauchy on D or*

$$\sup_{(x, y) \in D} \frac{1}{\varphi(x, y)} \|f(x + y) - f(x) - f(y)\| = \infty.$$

2.3.2. Now suppose that the homomorphism

$$S \rightarrow S, x \mapsto 2x \text{ is an automorphism;} \quad (12)$$

we denote by $y := \frac{x}{2}$ the unique solution of the equation $2y = x$ for $x \in S$. Also we assume that $(Y, +, \|\cdot\|)$ is a $(2, \gamma)$ -group, the sets

$$X \subseteq S, \quad D \subseteq \left\{ (x, y) \in X^2 \mid \frac{x + y}{2} \in X \right\} \quad (13)$$

are nonempty and the equation

$$\rho\left(\frac{x + y}{2}\right) = \frac{\rho(x) + \rho(y)}{2} \quad \text{for all } (x, y) \in D, \quad (14)$$

have the function $\rho : X \rightarrow Y$ as the unknown. Since for the function

$$E : Y^X \rightarrow Y^D, \quad E\rho(x, y) := \rho\left(\frac{x + y}{2}\right) - \frac{\rho(x) + \rho(y)}{2},$$

we have $E(m\rho) = mE\rho$ for $m \in \mathbb{N}^*$, we can apply Corollary 2.8 and we obtain a generalization of [8, Proposition 2.3].

COROLLARY 2.10. *We assume that $(S, +)$ is an Abelian semigroup that satisfies (12), $X \subseteq S$ and $D \subseteq X^2$ satisfy (13), $(Y, +, \|\cdot\|)$ is a $(2, \gamma)$ -group and $\varphi : D \rightarrow (0, \infty)$ is a function. Then equation (14) is φ -hyperstable if and only if for all $\rho \in Y^X$*

$$\text{either } \rho \in S_{(14)} \quad \text{or} \quad \sup_{(x, y) \in D} \frac{1}{\varphi(x, y)} \left\| 2\rho\left(\frac{x + y}{2}\right) - \rho(x) - \rho(y) \right\| = \infty.$$

Finally we give an concrete example of using the above results. Let $(Y, +, \|\cdot\|)$ be

a linear normed space,

$$g : \left(0, \frac{\pi}{2}\right) \rightarrow X := (-\infty, 0), \quad g(u) := \ln \sin u$$

and $D := X^2$. In [8] it was proven that the equation

$$f\left(\arcsin \sqrt{\sin u \cdot \sin v}\right) = \frac{f(u) + f(v)}{2} \text{ for all } u, v \in \left(0, \frac{\pi}{2}\right), \quad (15)$$

where $f : \left(0, \frac{\pi}{2}\right) \rightarrow Y$ is the unknown, has the solutions $S_{(15)} = \{\rho \circ g \mid \rho \in S_{(14)}\}$ (see [8, Proposition 3.11]) and that, for $p, q > 0$ and $\varphi : D \rightarrow (0, \infty)$, $\varphi(u, v) := u^p v^q$, equation (15) is φ -hyperstable (see [8, Corollary 3.13]). Applying Corollary 2.8 we obtain the following dichotomy.

COROLLARY 2.11. *Suppose that $(Y, +, \|\cdot\|)$ is a linear normed space and $f : \left(0, \frac{\pi}{2}\right) \rightarrow Y$. Then either*

$$\sup_{u, v \in \left(0, \frac{\pi}{2}\right)} u^r v^s \left\| 2f\left(\arcsin \sqrt{\sin u \cdot \sin v}\right) - f(u) - f(v) \right\| = \infty \text{ for all } r, s < 0$$

$$\text{or} \quad \rho\left(\frac{x+y}{2}\right) = \frac{\rho(x) + \rho(y)}{2} \text{ for all } x, y \in (-\infty, 0),$$

where $\rho : (-\infty, 0) \rightarrow Y$, $\rho(x) := f(\arcsin e^x)$.

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