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# UPPER BLOW-UP TIME AND LOWER GROW-UP RATE TO SOLUTION FOR A PSEUDO-PARABOLIC EQUATION

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**Abstract**. In this paper, we consider a pseudo parabolic equation with weak-viscoelastic term, where the exponent in the source term is variable. Using a differential inequality technique, we prove that the solution with positive initial energy become unbounded at a finite time, and find an upper bound for this time. A lower grow-up rate of solution is also obtained.

## 1. Introduction

In this paper, we consider the following pseudo-parabolic equation with a weak-viscoelastic term:

$$\begin{cases} u_t - \Delta u - \Delta u_t + a(t) \int_0^t h(t-s) \Delta u(s)\, ds = |u|^{p(x)-2} u, & \text{in } (0,T] \times \Omega, \\ u(t,x) = 0, & \text{on } (0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, $\Delta$ is the Laplace operator, and the viscoelastic term is represented as $\int_0^t h(t-s) \Delta u(s)\, ds$. This term is called "weak-viscoelastic" because it is coupled with the time-weighted function $a(t)$. The term with a variable exponent, $|u|^{p(x)-2} u$, plays the role of a source, and the dissipative term $\Delta u_t$ is a linear strong damping term. Additionally, we assume that $u_0 \in H_0^1(\Omega) \cap L^{p(x)}(\Omega)$.

The exponent $p(\cdot)$ is a given continuous function defined on $\overline{\Omega}$ and satisfies:

$$\begin{cases} 2 < p_- \leq p(x) \leq p_+ < \infty, & \text{if } n = 1, 2, \\ 2 < p_- \leq p(x) \leq p_+ \leq \dfrac{2n}{n-2}, & \text{if } n \geq 3, \end{cases} \tag{2}$$

where $p_+ = \operatorname{ess\,sup} p(x)$, $p_- = \operatorname{ess\,inf} p(x)$, and the Zhikov-Fan conditions hold:

$$p(x) - p(y) \leq \frac{-A}{\log|x-y|}, \text{ for all } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{3}$$

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where $A > 0$ and $0 < \delta < 1$.

The relaxation function $h$ and the weight function $a$ are given functions satisfying the following assumptions:

(A1) $h : [0, \infty) \to [0, \infty)$ is a $C^1$ decreasing function satisfying:

$$1 - a(t) \int_0^t h(s)\, ds \geq 1 - \|a\|_\infty \int_0^\infty h(s)\, ds = K > 0. \tag{4}$$

(A2) $a : [0, \infty) \to [0, \infty)$ is a $C^1$ decreasing function ($a'(t) < 0$).

The problem in (1) arises from many important mathematical models in engineering and physical sciences. For example, see the bidirectional nonlinear shallow water waves in [20], the vibration of a nonlinear viscoelastic rod in [10], and the references therein. The equation in (1) can be used in the analysis of nonstationary processes in semiconductors in the presence of sources. Additionally, equation (1) appears as a nonclassical diffusion equation in fluid mechanics, solid mechanics, and heat conduction, as discussed in [3, 11, 18, 23] and the references therein.

Obviously, in the absence of the weak-viscoelastic term and if $p(x) = p =$ constant, the equation in (1) reduces to the following pseudo-parabolic equation with a constant exponent:

$$u_t - \Delta u - \Delta u_t = |u|^{p-2} u, \quad \text{in } (0, \infty) \times \Omega. \tag{5}$$

For equation (5), many results have been obtained, such as existence and uniqueness [1, 21], blow-up [17, 25], asymptotic behavior [16, 25], and so on. In [22], the authors obtained the global existence of solutions under suitable assumptions and provided a blow-up result with arbitrary energy levels when $a(t) = 1$ and $p(x) = p =$ constant in problem (1).

With the rapid development of mathematical theory, much attention has been paid to the study of mathematical nonlinear models of pseudo-parabolic, hyperbolic, and parabolic equations with variable exponents of nonlinearity. For instance, in [5], the following nonlinear pseudo-parabolic equation with variable exponents was considered:

$$u_t - \Delta_{m(x)} u - \Delta u_t = |u|^{p(x)-2} u, \text{ in } (0, \infty) \times \Omega, \tag{6}$$

and it was proved that any solutions of this equation with nontrivial initial data blow up in finite time in the $H^1(\Omega)$-norm. They also obtained an upper bound and a lower bound for the blow-up time of the solution with negative initial energy. In another study [13], Liao considered the same problem treated in [5] and obtained the blow-up of the solution to (6), but with positive initial energy. Also, Himadan [9] considered the following weak-viscoelastic parabolic equation with variable exponents:

$$u_t - \operatorname{div}\left(|\nabla u|^{q(x)-2} \nabla u\right) + \sigma(t) \int_0^t h(t-s)\, \Delta u(s)\, ds = |u|^{p(x)-2} u, \text{ in } (0, \infty) \times \Omega. \tag{7}$$

By using differential inequality techniques, he proved that the solution to the problem (7) blows up in finite time in the $L^2(\Omega)$-norm with positive initial energy. It is worth mentioning some other literature concerning the theory of our type of equation, namely, several studies [4, 12, 15].

Let us mention that equations with variable exponents and weak-viscoelastic terms are usually referred to as equations with a nonstandard growth condition. This type of equation has an extensive physical background, which appears in the study of heat conduction, viscous flow in materials with memory, electric signals in telegraph lines with nonlinear damping [1], the vibration of a nonlinear viscoelastic rod [10], bidirectional nonlinear shallow water waves [20], and the velocity evolution of ion-acoustic waves in a collisionless plasma [19], and so on.

The purposes of this paper are twofold. The first is to study blow-up phenomena, and the second is to give lower bounds for the growth rate of solutions to the problem (1), which involves pseudo-parabolic, variable exponent, and weak-viscoelastic terms. As far as we know, there are few results concerning viscoelastic pseudo-parabolic equations. Specifically, this paper is organized as follows: In Section 2, we present the necessary notation and background material needed for our work. In Section 3, we prove, using differential inequality techniques, the blow-up phenomena for the solutions to problem (1), assuming that the initial energy satisfies $0 < E(0) < E_1$. In Section 4, we prove that the solution to (1) grows in the $H^1$-norm on $(0, T_{\max})$, and we provide lower bounds for the growth rate.

## 2. Preliminary results and tools

Throughout this paper, we denote by $\|.\|_p$ the $L^p(\Omega)$ norm for $1 \leq p \leq \infty$, we also denote the inner product on the Hilbert space $L^2(\Omega)$ by $\langle .,. \rangle$. We will equip $H_0^1(\Omega)$ with the norm

$$\|u\|_{H_0^1(\Omega)} = \sqrt{\|u\|_2^2 + \|\nabla u\|_2^2},$$

and the inner product

$$\langle u, v \rangle_{H_0^1(\Omega)} = \langle u, v \rangle + \langle \nabla u, \nabla v \rangle, \ \forall u, v \in H_0^1(\Omega).$$

Firstly, let us recall some definitions, properties, and important lemmas related to Lebesgue and Sobolev spaces with a variable exponent to state the main results of this paper.

Definition 2.1. Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $p : \Omega \to [1, \infty)$ be a measurable function. The Lebesgue space $L^{p(.)}(\Omega)$, with variable exponent $p(.)$ is defined by

$$L^{p(.)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable in } \Omega \text{ and } \int |\lambda u(x)|^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The Luxemburg norm is given by

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0, \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

We notice that variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many aspects: they are Banach spaces, the Holder inequality holds, they are reflexive if $1 < p(x) < \infty$. The variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ is defined by $W^{1,p(.)}(\Omega) = \{u \in L^{p(.)}(\Omega) : \nabla u \text{ exists and } \nabla u \in L^{p(.)}(\Omega)\}$. This is a Banach space with respect to the norm $\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}$.

The space $W_0^{1,p(.)}(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. The definition of the space $W_0^{1,p(.)}(\Omega)$ in the constant-exponent case is usualy different. However, under the condition (3) the two definitions coincide (see [6]). The dual space $W_0^{-1,p'(.)}(\Omega)$ of $W_0^{1,p(.)}(\Omega)$ is defined in the same way as in the case of classical Sobolev spaces, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

LEMMA 2.2 ([6, Poincaré's inequality]). *Suppose that $p(.)$ satisfies the condition (3). Then $\|u\|_{p(.)} \le C\|\nabla u\|_{p(.)}$, $u \in W_0^{1,p(.)}(\Omega)$, where $C$ is a constant that depends only on $p(.)$ and $\Omega$.*

LEMMA 2.3 ([6, Embedding proprety]). *If $p : \Omega \to [1,\infty)$ is continuous and satisfies the condition (2), then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact. So, there exists a $C > 0$ such that $\|u\|_{p(.)} \le C\|u\|_{H_0^1(\Omega)}$, $\forall u \in H^1(\Omega)$.*

DEFINITION 2.4 (Weak solution). We say that the function $u$ is a weak solution of problem (1) on $[0,T]$ if $u \in L^\infty\left(0,T; H_0^1(\Omega) \cap L^{p(.)}(\Omega)\right)$, $u_t \in L^2\left(0,T; H_0^1(\Omega)\right)$, and $u$ satisfies (1) in the following sense:

1. $u$ satisfies the distributional identity

$$\langle u_t, v\rangle_{H_0^1(\Omega)} + \langle \nabla u, \nabla v\rangle = \int_0^t a(t)\, h(t-s)\, \langle \nabla u(s), \nabla v\rangle\, ds + \left\langle |u|^{p(x)-2} u, v\right\rangle,$$

for all test functions $v \in H_0^1(\Omega) \cap L^{p(.)}(\Omega)$.

2. $u$ satisfies the initial condition $u(0,x) = u_0(x)$.

DEFINITION 2.5 (Finite time blow-up). Let $u(t,x)$ be a weak solution of problem (1). We say $u(t,x)$ blows-up in finite time if the maximal existence time $T_{\max}$ is finite and $\lim\limits_{t\to T_{\max}} \|u\|_{H_0^1(\Omega)} = +\infty$.

By using a Faedo-Galerkin method and a Contraction the Mapping Principle as in [1, 14], we have the following theorem.

THEOREM 2.6 (Local existence). *Let $u_0 \in H_0^1(\Omega) \cap L^{p(.)}(\Omega)$ and assume that the conditions (A1)-(A2) and (2) hold. Then the problem (1) admits a unique local weak solution such that $u \in L^\infty\left(0,T; H_0^1(\Omega) \cap L^{p(.)}(\Omega)\right)$, $u_t \in L^2\left(0,T; H_0^1(\Omega)\right)$.*

## 3. Upper bound of the blow-up time

In this section, we derive an upper bound on the finite time blow-up of the solution to (1). For this purpose, we need to consider the following energy functional:

$$E(t) = \frac{1}{2}\left(1 - a(t)\int_0^t h(s)\, ds\right)\|\nabla u\|_2^2 - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx + \frac{1}{2}a(t)\,(h \circ \nabla u)(t), \quad (8)$$

where

$$(h \circ \nabla u)(t) = \int_0^t h(t-s) \|\nabla u(t) - \nabla u(s)\|_2^2 \, ds, \ \forall t \in [0,T).$$

The following lemma states one of the most important properties of the functional $E$.

LEMMA 3.1. *Let the assumptions (A1)-(A2) hold. Then we have the following estimate:*

$$E'(t) = -\|u_t\|_2^2 - \|\nabla u_t\|_2^2 + \frac{1}{2}a(t)(h' \circ \nabla u)(t) - \frac{1}{2}a(t)h(t)\|\nabla u\|_2^2 + \frac{1}{2}a'(t)(h \circ \nabla u)(t) + \frac{1}{2}a'(t)\int_0^t h(s)\,ds\,\|\nabla u(t)\|_2^2 \leq 0, \tag{9}$$

*in particular, the functional $E$ is decreasing.*

*Proof.* Multiplying equation (1) by $u_t$, integrating over $\Omega$, and using the following identity:

$$\begin{aligned}
&\int_\Omega a(t)\int_0^t h(t-s)\,\Delta^k u(s)\,u_t(t)\,dsdx \\
=&\frac{(-1)^{k+1}}{2}\frac{d}{dt}\left[a(t)\left(h\circ\nabla^k u\right)(t)\right] + \frac{(-1)^k}{2}\frac{d}{dt}\left[a(t)\int_0^t h(s)\,ds\int_\Omega \left|\nabla^k u(t)\right|^2 dx\right] \\
&+\frac{(-1)^k}{2}a(t)\left(h'\circ\nabla^k u\right)(t) + \frac{(-1)^{k+1}}{2}a(t)h(t)\int_\Omega\left|\nabla^k u(t)\right|^2 dx \\
&+\frac{(-1)^k}{2}a'(t)\left(h\circ\nabla^k u\right)(t) + \frac{(-1)^k}{2}a'(t)\int_0^t h(s)\,ds\int_\Omega\left|\nabla^k u(t)\right|^2 dx
\end{aligned}$$

we deduce that

$$\begin{aligned}
E'(t) = &-\|u_t\|_{H_0^1(\Omega)} + \frac{1}{2}a(t)(h'\circ\nabla u)(t) - \frac{1}{2}a(t)h(t)\|\nabla u\|_2^2 + \frac{1}{2}a'(t)(h\circ\nabla u)(t) \\
&+\frac{1}{2}a'(t)\int_0^t h(s)\,ds\,\|\nabla u\|_2^2 \leq 0.
\end{aligned}$$

The desired conclusions can be deduced immediately from the above estimate. □

We put

$$\psi(t) = \sqrt{\left(1 - a(t)\int_0^t h(s)\,ds\right)\|\nabla u\|_2^2 + a(t)(h\circ\nabla u)(t)}.$$

Since (A1) holds, we have

$$\psi(t) \geq \sqrt{K}\,\|\nabla u\|_2. \tag{10}$$

It follows from (8) and Lemma 3.1 that

$$E(0) \geq E(t) = \frac{1}{2}(\psi(t))^2 - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx.$$

We define the sets,

$$\Omega_+ = \{x\in\Omega : |u| \geq 1\} \text{ and } \Omega_- = \{x\in\Omega : |u| < 1\}. \tag{11}$$

It follows from the Sobolev embedding inequality (Lemma 2.3) and (10) that

$$
\begin{aligned}
E(0) \geq & E(t) \geq \frac{1}{2}(\psi(t))^2 - \frac{1}{p_-}\left[\int_{\Omega_-}|u|^{p(x)}\,dx + \int_{\Omega_+}|u|^{p(x)}\,dx\right] \\
\geq & \frac{1}{2}(\psi(t))^2 - \frac{1}{p_-}\left[\int_{\Omega_-}|u|^{p_-}\,dx + \int_{\Omega_+}|u|^{p_+}\,dx\right] \\
\geq & \frac{1}{2}(\psi(t))^2 - \frac{1}{p_-}\left[\int_{\Omega}|u|^{p_-}\,dx + \int_{\Omega}|u|^{p_+}\,dx\right] \\
\geq & \frac{1}{2}(\psi(t))^2 - \frac{1}{p_-}\left[B_-^{p_-}\left(\int_{\Omega}|\nabla u|^2\,dx\right)^{\frac{p_-}{2}} + B_+^{p_+}\left(\int_{\Omega}|\nabla u|^2\,dx\right)^{\frac{p_+}{2}}\right] \\
= & \frac{1}{2}(\psi(t))^2 - \frac{1}{p_-}\left[B_-^{p_-}\|\nabla u\|_2^{p_-} + B_+^{p_+}\|\nabla u\|_2^{p_+}\right] \\
\geq & \frac{1}{2}(\psi(t))^2 - \frac{B_-^{p_-}}{p_-(K)^{\frac{p_-}{2}}}(\psi(t))^{p_-} - \frac{B_+^{p_+}}{p_-(K)^{\frac{p_+}{2}}}(\psi(t))^{p_+} = f(\psi(t)) \qquad (12)
\end{aligned}
$$

where $B_+, B_-$ are the optimal constants satisfying the Sobolev embedding inequalities

$$\|u\|_{L^{p_+}(\Omega)} \leq B_+\|\nabla u\|_2 \text{ and } \|u\|_{L^{p_-}(\Omega)} \leq B_-\|\nabla u\|_2,$$

respectively, and the function $f:[0,\infty)\to\mathbb{R}$ defined by

$$f(\lambda) = \frac{1}{2}\lambda^2 - \frac{B_-^{p_-}}{p_-(K)^{\frac{p_-}{2}}}\lambda^{p_-} - \frac{B_+^{p_+}}{p_-(K)^{\frac{p_+}{2}}}\lambda^{p_+}. \qquad (13)$$

Then, we have the following lemma, which we will use it in the proof of Lemma 3.3, and which is easily shown to hold. So we omit the proof.

LEMMA 3.2. *Let condition* (2)*)= hold. With the function* $f:[0,\infty[\ \to\mathbb{R}$ *defined by* (13)*, we have the following statements:*
*1.* $f(0)=0$ *and* $\lim\limits_{\lambda\to\infty} f(\lambda) = -\infty$.

*2. The equation* $\frac{df}{d\lambda}=0$ *has a unique positive solution* $\lambda_0$ *satisfying*

$$1 - \frac{B_-^{p_-}}{(K)^{\frac{p_-}{2}}}\lambda_0^{p_--2} - \frac{p_+B_+^{p_+}}{p_-(K)^{\frac{p_+}{2}}}\lambda_0^{p_+-2} = 0.$$

*3. The function* $f(\lambda)$ *is increasing for* $0<\lambda<\lambda_0$*, decreasing for* $\lambda>\lambda_0$*, and* $\lambda_0$ *is the absolute maximum point of* $f(\lambda)$ *satisfying*

$$f(\lambda_0) = \frac{(p_--2)B_-^{p_-}}{2p_-(K)^{\frac{p_-}{2}}}\lambda_0^{p_-} + \frac{(p_+-2)B_+^{p_+}}{2p_-(K)^{\frac{p_+}{2}}}\lambda_0^{p_+} = E_1 > 0. \qquad (14)$$

The following lemma will play an essential role in the proof of our main result. It is similar to a lemma first used by Vitillaro [24].

Lemma 3.3. *Assume that the hypotheses (A1)-(A2) are satisfied, and condition* (2) *holds. For any* $u_0 \in H_0^1(\Omega) \cap L^{p(\cdot)}(\Omega)$ *such that*

$$0 \leq E(0) < \frac{(p_- - 2) B_-^{p_-}}{2p_- (K)^{\frac{p_-}{2}}} \lambda_0^{p_-} + \frac{(p_+ - 2) B_+^{p_+}}{2p_- (K)^{\frac{p_+}{2}}} \lambda_0^{p_+} = E_1,$$

*if* $\|\nabla u_0\|_2 > \lambda_0$*, then there exists a unique* $\lambda_1 > \lambda_0$ *such that*

$$\psi(t) \geq \lambda_1 \text{ for } t > 0. \tag{15}$$

*and*

$$\int_\Omega |u|^{p(x)} dx \geq \frac{B_-^{p_-}}{K^{\frac{p_-}{2}}} \lambda_1^{p_-} + \frac{B_+^{p_+}}{K^{\frac{p_+}{2}}} \lambda_1^{p_+}. \tag{16}$$

*Proof.* First, we have $\psi(0) = \|\nabla u_0\|_2 > \lambda_0$. Lemma 3.2 informs us that

$$\begin{cases} f(\lambda) & \text{is increasing for } 0 < \lambda < \lambda_0, \\ f(\lambda) & \text{is decreasing for } \lambda > \lambda_0, \end{cases}$$

and $\lim_{\lambda \to \infty} f(\lambda) = -\infty$, then, it follows that there exists a unique constant $\lambda_1 > \lambda_0$ such that $f(\lambda_1) = E(0)$. Thus, from (12), we deduce that $f(\lambda_1) = E(0) \geq f(\psi(0))$. Thus, we have $\psi(0) > \lambda_1$. We will claim that $\psi(t) > \lambda_1$. Conversely, suppose that there exists a $t_* \in [0, T)$, such that $\psi(t_*) < \lambda_1$. By the continuity of the function $t \to \psi(t)$, without loss of generality, we may assume that $\psi(t) \in (\lambda_0, \lambda_1)$. Recalling (12), it may be concluded that$f(\lambda_1) = E(0) \geq E(t_*)$, which contradicts (12). Hence, we have $\psi(t) > \lambda_1$ for all $t > 0$. Thus, (15) is established.

To prove (16), we use (8) and (9)

$$E(0) + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx \geq \frac{1}{2}\left(1 - a(t) \int_0^t h(s) ds\right) \|\nabla u\|_2^2 + \frac{1}{2} a(t) (h \circ \nabla u)(t)$$

which implies $\displaystyle\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx \geq \frac{1}{2}\lambda_1^2 - E(0)$.

Since $E(0) = f(\lambda_1)$, we have

$$\begin{aligned} \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx &\geq \frac{1}{2}\lambda_1^2 - \left(\frac{1}{2}\lambda_1^2 - \frac{B_-^{p_-}}{p_- (K)^{\frac{p_-}{2}}} \lambda_1^{p_-} - \frac{B_+^{p_+}}{p_- (K)^{\frac{p_+}{2}}} \lambda_1^{p_+}\right) \\ &= \frac{B_-^{p_-}}{p_- (K)^{\frac{p_-}{2}}} \lambda_1^{p_-} + \frac{B_+^{p_+}}{p_- (K)^{\frac{p_+}{2}}} \lambda_1^{p_+}. \end{aligned}$$

Because of $p_- < p(x) < p_+$, we deduce that

$$\int_\Omega |u|^{p(x)} dx \geq \frac{B_-^{p_-}}{(K)^{\frac{p_-}{2}}} \lambda_1^{p_-} + \frac{B_+^{p_+}}{(K)^{\frac{p_+}{2}}} \lambda_1^{p_+},$$

and the proof is complete. □

Remark 3.4. By combining (14) and (16), it is easily seen that

$$E_1 = \frac{(p_- - 2) B_-^{p_-}}{2p_- (K)^{\frac{p_-}{2}}} \lambda_0^{p_-} + \frac{(p_+ - 2) B_+^{p_+}}{2p_- (K)^{\frac{p_+}{2}}} \lambda_0^{p_+} \leq \frac{(p_+ - 2)}{2p_-} \left[\frac{B_-^{p_-}}{K^{\frac{p_-}{2}}} \lambda_0^{p_-} + \frac{B_+^{p_+}}{K^{\frac{p_+}{2}}} \lambda_0^{p_+}\right]$$

$$\leq \frac{(p_+-2)}{2p_-}\left[\frac{B_-^{p_-}}{K^{\frac{p_-}{2}}}\lambda_1^{p_-}+\frac{B_+^{p_+}}{K^{\frac{p_+}{2}}}\lambda_1^{p_+}\right] \leq \frac{(p_+-2)}{2p_-}\int_\Omega |u|^{p(x)}\, dx. \tag{17}$$

Now, let us prove the finite time blow-up of solutions to (1).

THEOREM 3.5. *Suppose that the assumptions (A1)-(A2) and the conditions* (2)-(3) *hold. For any* $u_0 \in H_0^1(\Omega) \cap L^{p(\cdot)}(\Omega)$ *such that* $0 < E(0) < E_1$, *if* $\|\nabla u_0\|_2 > \lambda_0$, *then, the solution* $u(t,x)$ *of the problem* (1) *exhibits blow-up in finite time* $T_{\max}$ *in the* $H_0^1(\Omega)$*-norm. Moreover, an upper bound for the blow-up time is given by*

$$T_{\max} \leq \frac{2\,(F(0))^{1-\frac{p_+}{2}}}{(p_+-2)\,C_4},$$

*where* $C_4$ *is a suitable positive constant specified below and* $F(0) = \|u_0\|_{H_0^1(\Omega)}$.

*Proof.* We define the auxiliary function

$$F(t) = \|u\|_{H_0^1(\Omega)}^2 = \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2\, dx. \tag{18}$$

Our objective is to estimate the blow-up time of $F(t)$. Multiply both sides of the differential equation in (1) by $u$ and integrate by parts, to deduce that

$$\int_\Omega u u_t dx + \int_\Omega \nabla u.\nabla u_t dx =$$

$$-\int_\Omega |\nabla u|^2\, dx + \int_\Omega a(t)\int_0^t h(t-s)\,\nabla u(t,x)\,.\nabla u(s,x)\, ds dx + \int_\Omega |u|^{p(x)}\, dx$$

By differentiating $F(t)$ with respect to $t$, we have

$$F'(t) = 2\int_\Omega u u_t dx + 2\int_\Omega \nabla u.\nabla u_t dx$$

$$= -2\int_\Omega |\nabla u|^2\, dx + 2\int_\Omega a(t)\int_0^t h(t-s)\,\nabla u(t,x)\,.\nabla u(s,x)\, ds dx + 2\int_\Omega |u|^{p(x)}\, dx. \tag{19}$$

Next, we apply Young's inequality

$$\left(\frac{1}{C_0}X\right)(C_0 Y) \leq \frac{1}{2C_0^2}X^2 + \frac{C_0^2}{2}Y^2, \text{ for all } X, Y \geq 0,$$

to obtain

$$\int_\Omega \nabla u(t,x)\,.\nabla u(s,x)\, ds dx \tag{20}$$

$$= \int_\Omega |\nabla u(t,x)|^2\, dx - \int_\Omega \nabla u(t,x)\,.\,(\nabla u(t,x) - \nabla u(s,x))\, dx$$

$$\geq \int_\Omega |\nabla u(t,x)|^2\, dx - \int_\Omega \frac{1}{C_0}\,|\nabla u(t,x)| \times C_0\,|\nabla u(t,x) - \nabla u(s,x)|\, dx$$

$$\geq \frac{2C_0^2-1}{2C_0^2}\,\|\nabla u\|_2^2 - \frac{C_0^2}{2}\,\|\nabla u(t,x) - u(s,x)\|_2^2. \tag{21}$$

It follows from (21) and (19) that

$$F'(t) \geq -2\int_\Omega |\nabla u|^2\, dx$$

$$+2\int_\Omega |u|^{p(x)}\,dx+\frac{2C_0^2-1}{C_0^2}a(t)\|\nabla u\|_2^2\int_0^t h(s)\,ds-C_0^2a(t)(h\circ\nabla u)(t).$$

Let us assume that $C_0^2\geq\frac{1}{2}$, and since $a,h\geq 0$ (Assumptions (A1)-(A2)), we obtain

$$F'(t)\geq -2\int_\Omega |\nabla u|^2\,dx+2\int_\Omega |u|^{p(x)}\,dx-C_0^2a(t)(h\circ\nabla u)(t). \tag{22}$$

We substitute for $a(t)(h\circ\nabla u)(t)$ from (8), and hence (22) becomes

$$F'(t)\geq -2\int_\Omega |\nabla u|^2\,dx+2\int_\Omega |u|^{p(x)}\,dx$$

$$-C_0^2\left[E(t)-\frac{1}{2}\left(1-a(t)\int_0^t h(s)\,ds\right)\|\nabla u\|_2^2+\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx\right].$$

It follows frome (4) that

$$F'(t)\geq -2\int_\Omega |\nabla u|^2\,dx+2\int_\Omega |u|^{p(x)}\,dx-C_0^2\left[E(t)-\frac{1}{2}K\|\nabla u\|_2^2+\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx\right]$$

$$\geq\left(2-\frac{C_0^2}{p_+}\right)\int_\Omega |u|^{p(x)}\,dx+\left(\frac{C_0^2K}{2}-2\right)\int_\Omega |\nabla u|^2\,dx-C_0^2E_1+C_0^2H(t),$$

where $H(t)=E_1-E(t)>0$.

Since $H(t)>0$, we have

$$F'(t)\geq\left(2-\frac{C_0^2}{p_+}\right)\int_\Omega |u|^{p(x)}\,dx+\left(\frac{C_0^2K}{2}-2\right)\int_\Omega |\nabla u|^2\,dx-C_0^2E_1.$$

We then use (17) to obtain

$$F'(t)\geq\left(2-\frac{C_0^2}{p_+}\right)\int_\Omega |u|^{p(x)}\,dx+\left(\frac{C_0^2K}{2}-2\right)\int_\Omega |\nabla u|^2\,dx-C_0^2\frac{(p_+-2)}{2p_-}\int_\Omega |u|^{p(x)}\,dx,$$

so

$$F'(t)\geq\left(2-\left(\frac{C_0^2}{p_+}+C_0^2\frac{(p_+-2)}{2p_-}\right)\right)\int_\Omega |u|^{p(x)}\,dx+\left(\frac{C_0^2K}{2}-2\right)\int_\Omega |\nabla u|^2\,dx. \tag{23}$$

Choose $C_0>0$ such that $\left(2-\left(\frac{C_0^2}{p_+}+C_0^2\frac{(p_+-2)}{2p_-}\right)\right)>0$ and $\left(\frac{C_0^2K}{2}-2\right)>0$ then $\frac{4}{K}<C_0^2<\frac{2}{\frac{1}{p_+}+\frac{(p_+-2)}{2p_-}}$. Taking the condition $C_0^2\geq\frac{1}{2}$ into account, we can choose a positive constant $C_0$ such that

$$\max\left(\frac{1}{2},\frac{4}{K}\right)<C_0^2<\frac{2}{\frac{1}{p_+}+\frac{(p_+-2)}{2p_-}} \tag{24}$$

holds, which implies that $F'(t)\geq 0$.

The estimate $F'(t)\geq 0$ is necessary to guarantee the blow-up of the solution but is not sufficient. As we mentioned in the introduction, we need to construct a differential inequality with respect to $t$ which leads to a blow-up in finite time.

Then, substituting $a_1=2-\frac{C_0^2}{p_+}+C_0^2\frac{(p_+-2)}{2p_-}>0, a=\frac{C_0^2K}{2}-2>0$, in (23), we obtain

$$F'(t)\geq a_1\int_\Omega |u|^{p(x)}\,dx+a\int_\Omega |\nabla u|^2\,dx.$$

Recalling the sets (11), we obtain

$$F'(t) \geq a_1 \left( \int_{\Omega_-} |u|^{p_+} \, dx + \int_{\Omega_+} |u|^{p_-} \, dx \right) + a \int_\Omega |\nabla u|^2 \, dx$$

$$\geq a_1 C_1 \left( \left( \int_{\Omega_-} |u|^2 \, dx \right)^{\frac{p_+}{2}} + \left( \int_{\Omega_+} |u|^2 \, dx \right)^{\frac{p_-}{2}} \right) + a \int_\Omega |\nabla u|^2 \, dx, \quad (25)$$

using the fact $\|u\|_2 \leq C \|u\|_r$ for all $r \geq 2$.

The inequality (25) can be written as follows

$$F'(t) \geq \quad a_1 C_1 \left( \int_{\Omega_-} |u|^2 \, dx \right)^{\frac{p_+}{2}} + \left( \left( \int_{\Omega_+} |u|^2 \, dx \right)^{\frac{p_-}{p_+}} \right)^{\frac{p_+}{2}} + a \int_\Omega |\nabla u|^2 \, dx,$$

$$\geq \quad C_2 \left( \left( \int_{\Omega_-} |u|^2 \, dx \right)^{\frac{p_+}{2}} + \left( \left( \int_{\Omega_+} |u|^2 \, dx \right)^{\frac{p_-}{p_+}} \right)^{\frac{p_+}{2}} + \int_\Omega |\nabla u|^2 \, dx \right) \quad (26)$$

where $C_2 = \min(a_1 C_1, a)$.

We shall make use of the following inequality:

$$X^\alpha + Y^\alpha + Z^\gamma \geq C (X + Y + Z)^\alpha, \ \ X, Y, Z > 0, \gamma \geq 1$$

with $X = \int_{\Omega_-} |u|^2 \, dx$, $Y = \left( \int_{\Omega_+} |u|^2 \, dx \right)^{\frac{p_-}{p_+}}$, $Z = \int_\Omega |\nabla u|^2 \, dx$ and $\alpha = \frac{p_+}{2}$, $\gamma = 1$.

Then, inequality (26) becomes

$$F'(t) \geq C_3 \left( \int_{\Omega_-} |u|^2 \, dx + \left( \int_{\Omega_+} |u|^2 \, dx \right)^{\frac{p_-}{p_+}} + \int_\Omega |\nabla u|^2 \, dx \right)^{\frac{p_+}{2}}$$

$$\geq C_3 \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\frac{p_+}{2}}. \quad (27)$$

The Poincare inequality gives $\|\nabla u\|_2^2 \geq \lambda_1 \|u\|_2^2$, where $\lambda_1$ is the first eigenvalue of the operator $-\Delta$ with zero Dirichlet condition. Thus, we have

$$\|\nabla u\|_2^2 = \frac{\lambda_1}{1+\lambda_1} \|\nabla u\|_2^2 + \frac{1}{1+\lambda_1} \|\nabla u\|_2^2 \geq \frac{\lambda_1}{1+\lambda_1} \|u\|_{H_0^1(\Omega)}^2 = \frac{\lambda_1}{1+\lambda_1} F(t). \quad (28)$$

It follows from (27) and (28) that

$$F'(t) \geq C_4 (F(t))^{\frac{p_+}{2}}, \quad (29)$$

where $C_4 = C_3 \left( \frac{\lambda_1}{1+\lambda_1} \right)^{\frac{p_+}{2}} > 0$.

By a simple integration of (29) over $(0, t)$, we easily find that

$$F(t) \geq \frac{1}{\left( (F(0))^{1-\frac{p_+}{2}} + \frac{(2-p_+)C_4 t}{2} \right)^{\frac{2}{p_+-2}}},$$

which implies that $F(t) \to \infty$ as $t \to T$ in $H_0^1(\Omega)$, where

$$T_{\max} \leq \frac{2(F(0))^{1-\frac{p_+}{2}}}{(p_+ - 2) C_4}.$$

This completes the proof. □

## 4. Lower bound the growth-rate

The purpose of this section is to prove the growth within $(0, T_{\max})$ of the solution to (1), and to give lower bounds of the growth-rate. The following lemma is important for the proof of the lower bound on the growth-rate.

LEMMA 4.1 ([8]). *Suppose that $\beta, \gamma, \delta > 0$ and $y(t)$ is a nonnegative and absolutely continuous function satisfying*

$$y'(t) + \delta y^{\beta}(t) \geq \gamma, \quad 0 < t < \infty;$$

*then*

$$y(t) \geq \min\left(y(0), \left(\frac{\gamma}{\delta}\right)^{\frac{1}{\beta}}\right).$$

THEOREM 4.2. *If all conditions of Theorem 3.5 hold, then the solution $u(t,x)$ to the problem (1) grows within $(0, T_{\max})$ in $H^1$-norm, and we have*

$$\|u(t)\|^2_{H_0^1(\Omega)} \geq \|u_0\|^2_{H_0^1(\Omega)} \exp\left(a \frac{\lambda_1}{1+\lambda_1} t\right).$$

*Moreover, a lower bound of growth-rate is given by*

$$\|u\|_{H_0^1(\Omega)} \geq \min\left(F(0), \left(\frac{\gamma}{\delta}\right)^{\frac{2}{p_+}}\right),$$

*where $\gamma, \delta$ are suitable positive constants specified below, $a = \frac{C_0^2 K}{2} - 2 > 0$ and $\lambda_1$ is the first eigenvalue of $-\Delta$ with zero Dirichlet condition.*

*Proof.* Let us consider the same function $F(t)$ as in (15), and according to (25), we obtain

$$F'(t) \geq a \int_{\Omega} |\nabla u|^2 \, dx, \ t \in (0, T_{\max}) \tag{30}$$

because of $a_2 \left( \left( \int_{\Omega_-} |u|^2 \, dx \right)^{\frac{p_+}{2}} + \left( \int_{\Omega_+} |u|^2 \, dx \right)^{\frac{p_-}{2}} \right) > 0.$

The Poincare inequality gives $\|\nabla u\|_2^2 \geq \lambda_1 \|u\|_2^2$, where $\lambda_1$ is the first eigenvalue of $(-\Delta)$. Thus, we have

$$\|\nabla u\|_2^2 = \frac{\lambda_1}{1+\lambda_1} \|\nabla u\|_2^2 + \frac{1}{1+\lambda_1} \|\nabla u\|_2^2 \geq \frac{\lambda_1}{1+\lambda_1} \|u\|^2_{H_0^1(\Omega)} = \frac{\lambda_1}{1+\lambda_1} F(t). \tag{31}$$

It follows from (30) and (31) that

$$F'(t) \geq a \frac{\lambda_1}{1+\lambda_1} F(t), \ t \in (0, T_{\max}). \tag{32}$$

Integrating (32) from 0 to $t$, it follows that

$$F(t) \geq F(0)\exp\left(a\frac{\lambda_1}{1+\lambda_1}t\right),\ t \in (0, T_{\max})$$

Thus, the solution $u(t,x)$ to the problem (1) grows within $(0, T_{\max})$ in the $H^1$-norm, and we have

$$\|u(t)\|^2_{H_0^1(\Omega)} \geq \|u_0\|^2_{H_0^1(\Omega)} \exp\left(a\frac{\lambda_1}{1+\lambda_1}t\right).$$

Finally, we prove that the solution $u(t,x)$ has a lower bound on its growth-rate in the $H^1$-norm. To this end, by applying (25) and (31), we obtain

$$F'(t)) \geq a_1 C_1 \left(\left(\int_{\Omega_-} |u|^2\, dx\right)^{\frac{p_+}{2}} + \left(\int_{\Omega_+} |u|^2\, dx\right)^{\frac{p_-}{2}}\right) + a\frac{\lambda_1}{1+\lambda_1}F(t),$$

where $a_1 = 2 - \left(\frac{C_0^2}{p_+} + C_0^2\frac{(p_+-2)}{2p_-}\right) = 2 - b_1 > 0$. Then,

$$\begin{aligned}
F'(t) &\geq C_1(2-b_1)\left[\left(\int_{\Omega_-} |u|^2\, dx\right)^{\frac{p_+}{2}} + \left(\int_{\Omega_+} |u|^2\, dx\right)^{\frac{p_-}{2}}\right] + a\frac{\lambda_1}{1+\lambda_1}F(t)\\
&\geq -C_1 b_1\left[\left(\int_{\Omega_-} |u|^2\, dx\right)^{\frac{p_+}{2}} + \left(\int_{\Omega_+} |u|^2\, dx\right)^{\frac{p_-}{2}}\right] + a\frac{\lambda_1}{1+\lambda_1}F(t)\\
&\geq -C_1 b_1\left[\left(\int_{\Omega} |u|^2\, dx\right)^{\frac{p_+}{2}} + \left(\int_{\Omega} |u|^2\, dx\right)^{\frac{p_-}{2}}\right] + a\frac{\lambda_1}{1+\lambda_1}F(t). \qquad (33)
\end{aligned}$$

Then, since $\int_\Omega |u|^2\, dx \leq \int_\Omega \left(|u|^2 + |\nabla u|^2\right) dx = F(t)$, (33) becomes

$$\begin{aligned}
F'(t) &\geq -C_1 b_1\left[(F(t))^{\frac{p_+}{2}} + (F(t))^{\frac{p_-}{2}}\right] + a\frac{\lambda_1}{1+\lambda_1}F(t)\\
&= -C_1 b_1\left[(F(t))^{\frac{p_+}{2}} + (F(t))^{\frac{p_+}{2}}(F(t))^{\frac{p_-}{2}-\frac{p_+}{2}}\right] + a\frac{\lambda_1}{1+\lambda_1}F(t)\\
&= -C_1 b_1\left[\left(1 + (F(t))^{\frac{p_-}{2}-\frac{p_+}{2}}\right)\right](F(t))^{\frac{p_+}{2}} + a\frac{\lambda_1}{1+\lambda_1}F(t)
\end{aligned}$$

Furthermore, since $\frac{p_-}{2} - \frac{p_+}{2} \leq 0$ and by using the fact that $F(t) \geq F(0) > 0$ $(F'(t) \geq 0)$, we have

$$F'(t) \geq -C_1 b_1\left[\left(1 + (F(0))^{\frac{p_-}{2}-\frac{p_+}{2}}\right)\right](F(t))^{\frac{p_+}{2}} + a\frac{\lambda_1}{1+\lambda_1}F(0) = -\delta(F(t))^{\frac{p_+}{2}} + \gamma,$$

where $\delta = C_1 b_1\left[\left(1 + (F(0))^{\frac{p_-}{2}-\frac{p_+}{2}}\right)\right] > 0$, $\gamma = a\frac{\lambda_1}{1+\lambda_1}F(0) > 0$. By Lemma 4.1, the following holds

$$F(t) = \|u\|_{H_0^1(\Omega)} \geq \min\left(F(0), \left(\frac{\gamma}{\delta}\right)^{\frac{2}{p_+}}\right).$$

This completes the proof. □

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