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SCR-WARPED PRODUCT LIGHTLIKE SUBMANIFOLDS OF GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we introduce SCR-warped product lightlike submanifold of golden semi-Riemannian manifold. We obtain characterization theorem of SCR-warped product lightlike submanifold of the type $M_T \times_\lambda M_\perp$ of golden semi-Riemannian manifold. Further, we show that for a proper SCR-warped product lightlike submanifold of golden semi-Riemannian manifold, induced connection ∇ is not a metric connection. Finally, we find necessary and sufficient conditions for SCR lightlike submanifold of golden semi-Riemannian manifold to be a SCR-warped product lightlike submanifold in terms of the canonical structures P and F. Moreover, we give two non-trivial examples of SCR-warped product lightlike submanifold of golden semi-Riemannian manifold.

1. Introduction

The study of warped product manifolds was introduced by Bishop and O'Neill in [2]. These manifolds are generalizations of Riemannian product manifolds. In [13], warped product manifolds were used as models for spacetime near black holes or bodies with large gravitational fields. In [3], Chen B.Y. studied warped product CR-submanifolds in Kähler manifolds and introduced the notion of CR-warped products. In [3], they proved several fundamental properties of CR-warped products in Kähler manifolds and established a general inequality for an arbitrary CR-warped product in a Kähler manifold. In [5], the author studied the fundamental problem of finding a warped function such that the degenerate metric g admits a constant scalar curvature on M. In [19], the author introduced warped product lightlike submanifolds of semi-Riemannian manifolds. In [19], they also showed that the null geometry of M reduces to the corresponding non-degenerate geometry of its semi-Riemannian submanifold.

Geometry of degenerate submanifolds differs from the geometry of non-degenerate submanifolds. The main difference between lightlike submanifolds and non-degenerate

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submanifolds arises from the fact that, in the first case, the vector bundle intersects with the tangent bundle of the submanifold. Thus, one cannot use the theory of non-degenerate submanifolds to define the induced geometric objects on lightlike submanifolds. In [6], Duggal and Bejancu introduced a non-degenerate screen distribution to construct a non-intersecting lightlike transversal vector bundle of the tangent bundle and studied the geometry of arbitrary lightlike submanifolds of semi-Riemannian manifolds. Many authors have studied lightlike submanifolds in various spaces [8, 21]. In [9], the authors introduced screen Cauchy-Riemann lightlike submanifolds of indefinite Kähler manifolds. In [9], they also showed that SCR lightlike submanifolds include invariant (complex) and screen real subcases of lightlike submanifolds and studied some properties of proper totally umbilical SCR-lightlike submanifolds, as well as their invariant (complex) and screen real subcases. In [20], Sangeet K. studied SCR warped product lightlike submanifolds of indefinite Kähler manifolds and obtained several results on warped product lightlike submanifolds.

The golden proportion ψ is the real positive root of the equation $x^2-x-1=0$ (thus $\psi=\frac{1+\sqrt{5}}{2}\approx 1.618\ldots$). Inspired by the golden proportion, Crasmareanu and Hretcanu defined a golden structure $\tilde{P}$, which is a tensor field satisfying $\tilde{P}^2-\tilde{P}-I=0$ on the manifold $\overline{M}$ [4]. A Riemannian manifold $\overline{M}$ with a golden structure $\tilde{P}$ is called a golden Riemannian manifold and was studied in [4, 12]. In [18], Ozkan M. investigated complete and horizontal lifts of the golden structure in the tangent bundle. Lightlike hypersurfaces of golden semi-Riemannian manifolds were studied by Poyraz and Yasar [16]. In [17], the authors proved that there is no radical anti-invariant lightlike submanifold of golden semi-Riemannian manifolds. In [10], the author studied the geometry of screen transversal lightlike submanifolds, radical screen transversal lightlike submanifolds, and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds, investigating the geometry of distributions. Screen pseudo-slant and golden GCR-lightlike submanifolds of a golden semi-Riemannian manifold were studied in [1, 15]. In [14], N. Onen Poyraz introduced screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds and found the conditions for the integrability of distributions. In [14], the author obtained some results for totally umbilical screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds.

The purpose of this paper is to study SCR warped product lightlike submanifolds of golden semi-Riemannian manifolds. The paper is arranged as follows. In Section 2, some definitions and basic results about lightlike submanifolds and golden semi-Riemannian manifolds are given. In Section 3, we study SCR warped product lightlike submanifolds of golden semi-Riemannian manifolds, providing examples. We also prove that for SCR warped product lightlike submanifolds of golden semi-Riemannian manifolds, the induced connection ∇ is not a metric connection. In Section 4, we find characterization theorems in terms of the canonical structures P and F on an SCR lightlike submanifold of golden semi-Riemannian manifolds, forcing it to be an SCR warped product lightlike submanifold, and establish an inequality for the squared norm of the second fundamental form h in terms of the warping function λ of golden semi-Riemannian manifolds.

2. Preliminaries

Let $\overline{M}$ be a C^{∞}-differentiable manifold. If a $(1,1)$-type tensor field $\tilde{P}$ on $\overline{M}$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I, \tag{1}$$

then $\tilde{P}$ is called a golden structure on $\overline{M}$, where I is the identity transformation.

Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and let $\tilde{P}$ be a golden structure on $\overline{M}$. If $\tilde{P}$ satisfies the following equation

$$\overline{g}(\tilde{P}U, W) = \overline{g}(U, \tilde{P}W), \tag{2}$$

then $(\overline{M}, \overline{g}, \tilde{P})$ is called a golden semi-Riemannian manifold [18]. Also, if $\tilde{P}$ is integrable, then we have [4]

$$\overline{\nabla}_U \tilde{P}W = \tilde{P}\overline{\nabla}_U W. \tag{3}$$

Now, from (2), we get

$$\overline{g}(\tilde{P}U, \tilde{P}W) = \overline{g}(\tilde{P}U, W) + \overline{g}(U, W), \tag{4}$$

for all $U, W \in \Gamma(T\overline{M})$.

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\overline{M}$, where g is the induced metric of $\overline{g}$ on M. If $\overline{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold [6] of $\overline{M}$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is $TM = RadTM \oplus_{orth} S(TM)$. Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and Rad(TM) in $S(TM^{\perp})^{\perp}$, respectively. Then

$$\begin{aligned} tr(TM) &= ltr(TM) \oplus_{orth} S(TM^{\perp}), \\ T\overline{M}|_M &= TM \oplus tr(TM), \\ T\overline{M}|_M &= S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

THEOREM 2.1 ([6]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Suppose U is a coordinate neighbourhood of M and $\{\xi_i\}$, $i \in \{1, 2, \ldots, r\}$ is a basis of $\Gamma(Rad(TM|_U))$. Then, there exist a complementary vector subbundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis $\{N_i\}$, $i \in \{1, 2, \ldots, r\}$ of $\Gamma(ltr(TM|_U))$ such that $\overline{g}(N_i, \xi_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, for any $i, j \in \{1, 2, \ldots, r\}$.*

Following are four cases of a lightlike submanifold $\big(M, g, S(TM), S(TM^{\perp})\big)$:

Case 1. r-lightlike if $r < \min(m, n)$,

Case 2. co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,

Case 3. isotropic if $r = m < n$, $S\left(TM\right) = \{0\}$, Case 4. totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(tr(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. From (5) and (6), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{9}$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M, respectively.

Also by using (5), (7)-(9) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{10}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X).$$

Now, denote the projection of TM on $S(TM)$ by S. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, we have

$$\nabla_X SY = \nabla^*_X SY + h^*(X, SY), \tag{11}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi.$$

By using above equations, we obtain $\overline{g}(h^l(X, SY), \xi) = g(A^*_{\xi} X, SY)$.

It is important to note that in general ∇ is not a metric connection on M. Since $\overline{\nabla}$ is metric connection, by using (7), we get

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y), \tag{12}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$.

Definition 2.2 ([7]). A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, is said to be totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that for any $X, Y \in \Gamma(TM)$,

$$h(X, Y) = g(X, Y)H. \tag{13}$$

In case $H = 0$, M is called totally geodesic. Using (7) and (13), we conclude that M is totally umbilical if and only if there exists smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that $h^l(X, Y) = g(X, Y)H^l$, $h^s(X, Y) = g(X, Y)H^s$ and $D^l(X, W) = 0$, for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$.

Definition 2.3 ([2]). Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F, respectively and λ be a positive differentiable function on B. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_\lambda F$ is the manifold $B \times F$ equipped with Riemannian metric g such that $g = g_B + \lambda^2 g_F$. More explicitly, if X is tangent to $M = B \times_\lambda F$ at point (p,q), then

$$\|X\|^2 = \|\pi_*(X)\|^2 + \lambda^2(\pi(p,q))\|\eta_*(X)\|^2.$$

Here, function λ is called the warping function of the warped product and a warped product manifold is said to be trivial, if λ is constant. For a differentiable function λ on a manifold M, the gradient $\nabla\lambda$ is defined by

$$g(\nabla\lambda, X) = X\lambda, \quad \forall X \in \Gamma(TM).$$

Theorem 2.4 ([2]). *Let $M = B \times_\lambda F$ be a warped product manifold. If $X, Y \in \Gamma T(B)$ and $U, V \in \Gamma T(F)$, then*

$$\nabla_X Y \in \Gamma T(B), \quad \nabla_X V = \nabla_V X = \left(\frac{X\lambda}{\lambda}\right) V, \quad \nabla_U V = -\frac{g(U,V)}{\lambda}\nabla\lambda. \tag{14}$$

Corollary 2.5 ([2]). *On a warped product manifold, $M = B \times_\lambda F$,*
(i) B is totally geodesic in M,

(ii) F is totally umbilical in M.

Definition 2.6 ([19]). Let (M_1, g_1) be a totally lightlike submanifold of dimension r and (M_2, g_2) be a semi-Riemannian submanifold of dimension m of a semi-Riemannian $\overline{M}$. Then the product manifold $M = M_1 \times_\lambda M_2$ is said to be a warped product lightlike submanifold of $\overline{M}$ with the degenerate metric g defined by

$$g(X,Y) = g_1(\pi_* X, \pi_* Y) + (\lambda \circ \pi)^2 g_2(\eta_* X, \eta_* Y),$$

for every $X, Y \in \Gamma(TM)$, where $*$ is the symbol for the tangent map. Here, $\pi : M_1 \times M_2 \to M_1$ and $\eta : M_1 \times M_2 \to M_2$ denote the projection maps given by $\pi(p,q) = p$ and $\eta(p,q) = q$ for $(p,q) \in M_1 \times M_2$, respectively.

3. SCR-warped product lightlike submanifolds

In this section, we study SCR warped product lightlike submanifold of golden semi-Riemannian manifolds, which are warped products of the type $M = M_T \times_\lambda M_\perp$, where M_T is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of $\overline{M}$.

Definition 3.1. Let M be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$. Then we say that M is a SCR-lightlike submanifold of $\overline{M}$ if following conditions are satisfied:
(i) There exists a non-null distribution $D \subseteq S(TM)$ such that

$$S(TM) = D \oplus D^\perp, \; \tilde{P}(D) = D, \; \tilde{P}(D^\perp) \subseteq S(TM^\perp), \; D \cap D^\perp = \{0\},$$

where $D^\perp$ is orthogonal complementary to D in $S(TM)$.

(ii) $RadTM$ is invariant with respect to $\tilde{P}$, i.e. $\tilde{P}RadTM = RadTM$.
Then, we have

$$\tilde{P}ltr(TM) = ltr(TM), \quad TM = D' \oplus D^{\perp},\ D' = D \perp Rad(TM).$$

Thus it follows that D' is also invariant with respect to $\tilde{P}$. We indicate the orthogonal complementary to $\tilde{P}(D^{\perp})$ in $S(TM^{\perp})$ by μ. Then we obtain $tr(TM) = ltr(TM) \perp \tilde{P}(D^{\perp}) \perp \mu$.

A SCR-lightlike submanifold is said to be proper if $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$.

Let P' and Q' be the projections on D' and $D^{\perp}$, respectively. Then for any $X \in \Gamma(TM)$, we have $X = P'X + Q'X$. Applying $\tilde{P}$ to this, we obtain

$$\tilde{P}X = \tilde{P}P'X + \tilde{P}Q'X, \tag{15}$$

and we can write equation (15) as

$$\tilde{P}X = PX + FX, \tag{16}$$

where PX and FX are tangential and transversal parts of $\tilde{P}X$, respectively. Also for any $V \in \Gamma(tr(TM))$, we write

$$\tilde{P}V = BV + CV, \tag{17}$$

where BV and CV are tangential and transversal parts of $\tilde{P}V$, respectively.

Differentiating (16), using (7), (9), (16) and (17), we get

$$\begin{aligned}(\nabla_X P)Y &= A_{FY}X + Bh^s(X,Y),\\ \nabla^s_X FY &= F\nabla_X Y + Ch^s(X,Y) - h^s(X,PY),\\ D^l(X,FY) &= Ch^l(X,Y) - h^l(X,PY),\end{aligned}$$

for any $X, Y \in \Gamma(TM)$.

Proposition 3.2. *There exist no isotropic or totally lightlike proper SCR lightlike submanifold of a golden semi-Riemannian manifold* $(\overline{M}, \overline{g}, \tilde{P})$.

Proof. We suppose that M is isotropic or totally lightlike, then $S(TM) = 0$, hence $D = 0$ and $D^{\perp} = 0$. □

Lemma 3.3. *Let* (M, g) *be a SCR lightlike submanifold of a golden semi-Riemannian manifold* $(\overline{M}, \overline{g}, \tilde{P})$. *Then we have*

$$(\nabla_X P)Y = A_{FY}X + Bh(X,Y), \tag{18}$$

$$(\nabla^t_X F)Y = Ch(X,Y) - h(X,PY), \tag{19}$$

$$g(PX,Y) - g(X,PY) = g(X,FY) - g(FX,Y), \tag{20}$$

$$\begin{aligned}g(PX,PY) &= g(PX,Y) + g(X,Y) + g(FX,Y)\\ &\quad - g(PX,FY) - g(FX,PY) - g(FX,FY),\end{aligned} \tag{21}$$

for all $X, Y \in \Gamma(TM)$, *where*

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y \text{ and } (\nabla^t_X F)Y = \nabla^t_X FY - F\nabla_X Y. \tag{22}$$

Proof. Using (3), (5), (6), (16) and (17), on comparing tangential and transversal parts of the resulting equation, we obtain (18) and (19). Finally, using (2), (4) and (16), we obtain (20) and (21). □

Example 3.4. Let $(\mathbb{R}^{12}_2, \overline{g}, \tilde{P})$ be a golden semi-Riemannian manifold, where metric $\overline{g}$ is of signature $(-,-,+,+,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1, \partial x^2, \partial x^3, \partial x^4, \partial x^5, \partial x^6, \partial x^7, \partial x^8, \partial x^9, \partial x^{10}, \partial x^{11}, \partial x^{12}\}$ and $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12})$ be standard coordinate system of $\mathbb{R}^{12}_2$.

Taking, $\tilde{P}(\partial x^1, \dots, \partial x^{12}) = (\partial x^1 + \partial x^2, \partial x^1, \partial x^3 + \partial x^4, \partial x^3, \partial x^5 + \partial x^6, \partial x^5, \partial x^7 + \partial x^8, \partial x^7, \partial x^9 + \partial x^{10}, \partial x^9, \partial x^{11} + \partial x^{12}, \partial x^{11})$. Thus, $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{12}_2$. Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = u^2$, $x^2 = u^1$, $x^3 = u^2$, $x^4 = u^1$, $x^5 = -u^4$, $x^6 = -u^3$, $x^7 = u^4$, $x^8 = u^3$, $x^9 = 0$, $x^{10} = -\sin u^5 \cosh u^6$, $x^{11} = 0$, $x^{12} = \cos u^5 \sinh u^6$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where $Z_1 = \partial x^2 + \partial x^4$, $Z_2 = \partial x^1 + \partial x^3$, $Z_3 = -\partial x^6 + \partial x^8$, $Z_4 = -\partial x^5 + \partial x^7$, $Z_5 = -\cos u^5 \cosh u^6 \partial x^{10} - \sin u^5 \sinh u^6 \partial x^{12}$, $Z_6 = -\sin u^5 \sinh u^6 \partial x^{10} + \cos u^5 \cosh u^6 \partial x^{12}$.

Hence, $RadTM = span\,\{Z_1, Z_2\}$ and $S(TM) = span\,\{Z_3, Z_4, Z_5, Z_6\}$.

Now, $ltr(TM)$ is spanned by $N_1 = \frac{1}{2}(-\partial x^2 + \partial x^4)$, $N_2 = \frac{1}{2}(-\partial x^1 + \partial x^3)$ and $S(TM^\perp)$ is spanned by $W_1 = -\cos u^5 \cosh u^6 \partial x^9 - \sin u^5 \sinh u^6 \partial x^{11}$, $W_2 = -\sin u^5 \sinh u^6 \partial x^9 + \cos u^5 \cosh u^6 \partial x^{11}$, $W_3 = \partial x^6 + \partial x^8$, $W_4 = \partial x^5 + \partial x^7$.

It follows that $\tilde{P}Z_1 = Z_2$ and $\tilde{P}Z_2 = Z_1 + Z_2$, this implies $Rad(TM)$ is invariant, $Rad(TM) = Span\{Z_1, Z_2\}$ and $\tilde{P}Z_3 = Z_4$, $\tilde{P}Z_4 = Z_3 + Z_4$. Hence $\tilde{P}(D) = D$ i.e., D is invariant and $D = Span\{Z_3, Z_4\}$. Now $\tilde{P}Z_5 = W_1$, $\tilde{P}Z_6 = W_2$, $D^\perp = Span\{Z_5, Z_6\}$ and $\tilde{P}N_1 = N_2$, $\tilde{P}N_2 = N_1 + N_2$ which shows $ltr(TM)$ is invariant, $ltr(TM) = Span\{N_1, N_2\}$ and $\tilde{P}W_3 = W_4$, $\tilde{P}W_4 = W_3 + W_4$. Hence $\tilde{P}\mu = \mu$ i.e., μ is invariant and $\mu = Span\{W_3, W_4\}$. Hence M is a proper SCR 2-lightlike submanifold of $\mathbb{R}^{12}_2$. Now, if the leaves of D' and $D^\perp$ are, respectively denoted by M_T and $M_\perp$, then the induced metric tensor of $M = M_T \times_\lambda M_\perp$ is given by $ds^2 = 0((du^1)^2 + (du^2)^2) + 2((du^3)^2 + (du^4)^2) + (\cos^2 u^5 + \sinh^2 u^6)((du^5)^2 + (du^6)^2)$.

Hence, M is a SCR warped product lightlike submanifold of the type $M = M_T \times_\lambda M_\perp$ in $\mathbb{R}^{12}_2$, with warping function $\lambda = \sqrt{(\cos^2 u^5 + \sinh^2 u^6)}$.

Example 3.5. Let $(\mathbb{R}^{16}_4, \overline{g}, \tilde{P})$ be a golden semi-Riemannian manifold, where metric $\overline{g}$ is of signature $(+,+,-,-,+,+,-,-,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1, \partial x^2, \partial x^3, \partial x^4, \partial x^5, \partial x^6, \partial x^7, \partial x^8, \partial x^9, \partial x^{10}, \partial x^{11}, \partial x^{12}, \partial x^{13}, \partial x^{14}, \partial x^{15}, \partial x^{16}\}$ and $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12}, x^{13}, x^{14}, x^{15}, x^{16})$ be standard coordinate system of $\mathbb{R}^{16}_4$.

Taking, $\tilde{P}(\partial x^1, \dots, \partial x^{16}) = (\partial x^1 + \partial x^2, \partial x^1, \partial x^3 + \partial x^4, \partial x^3, \partial x^5 + \partial x^6, \partial x^5, \partial x^7 + \partial x^8, \partial x^7, \partial x^9 + \partial x^{10}, \partial x^9, \partial x^{11} + \partial x^{12}, \partial x^{11}, \partial x^{13} + \partial x^{14}, \partial x^{13}, \partial x^{15} + \partial x^{16}, \partial x^{15})$. Thus, $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{16}_4$. Suppose M is a submanifold of $\mathbb{R}^{16}_4$ given by $x^1 = u^4$, $x^2 = u^1$, $x^3 = u^4$, $x^4 = u^1$, $x^5 = u^2$, $x^6 = u^3$, $x^7 = u^2$, $x^8 = u^3$, $x^9 = -u^5 \cos\alpha$, $x^{10} = -u^6 \cos\alpha$, $x^{11} = u^5 \sin\alpha$, $x^{12} = u^6 \sin\alpha$, $x^{13} = 0$, $x^{14} = \cos u^7 \cosh u^8$, $x^{15} = 0$, $x^{16} = \sin u^7 \sinh u^8$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where $Z_1 = \partial x^2 + \partial x^4$, $Z_2 = \partial x^5 + \partial x^7$, $Z_3 = \partial x^6 + \partial x^8$, $Z_4 = \partial x^1 + \partial x^3$, $Z_5 = -\cos\alpha \partial x^9 + \sin\alpha \partial x^{11}$, $Z_6 = -\cos\alpha \partial x^{10} + \sin\alpha \partial x^{12}$, $Z_7 = -\sin u^7 \cosh u^8 \partial x^{14} + \cos u^7 \sinh u^8 \partial x^{16}$, $Z_8 = \cos u^7 \sinh u^8 \partial x^{14} + \sin u^7 \cosh u^8 \partial x^{16}$.

Hence, $RadTM = Span\,\{Z_1, Z_2, Z_3, Z_4\}$ and $S(TM) = Span\,\{Z_5, Z_6, Z_7, Z_8\}$.

Now, $ltr(TM)$ is spanned by $N_1 = \frac{1}{2}(\partial x^2 - \partial x^4)$, $N_2 = \frac{1}{2}(\partial x^5 - \partial x^7)$, $N_3 = \frac{1}{2}(\partial x^6 - \partial x^8)$, $N_4 = \frac{1}{2}(\partial x^1 - \partial x^3)$ and $S(TM^{\perp})$ is spanned by $W_1 = \cos u^7 \sinh u^8 \partial x^{13} + \sin u^7 \cosh u^8 \partial x^{15}$, $W_2 = -\sin u^7 \cosh u^8 \partial x^{13} + \cos u^7 \sinh u^8 \partial x^{15}$, $W_3 = \sin\alpha \partial x^9 + \cos\alpha \partial x^{11}$, $W_4 = \sin\alpha \partial x^{10} + \cos\alpha \partial x^{12}$.

It follows that $\tilde{P}Z_1 = Z_4$ and $\tilde{P}Z_3 = Z_2$, this implies $Rad(TM)$ is invariant, $Rad(TM) = Span\{Z_1, Z_2, Z_3, Z_4\}$ and $\tilde{P}Z_6 = Z_5$, $\tilde{P}Z_5 = Z_5 + Z_6$. Hence $\tilde{P}(D) = D$, i.e. D is invariant and $D = Span\{Z_5, Z_6\}$. Now $\tilde{P}Z_7 = W_2$, $\tilde{P}Z_8 = W_1$, $D^{\perp} = Span\{Z_7, Z_8\}$ and $\tilde{P}N_1 = N_4$, $\tilde{P}N_3 = N_2$ which shows $ltr(TM)$ is invariant, $ltr(TM) = Span\{N_1, N_2, N_3, N_4\}$ and $\tilde{P}W_4 = W_3$, $\tilde{P}W_3 = W_3 + W_4$. Hence $\tilde{P}\mu = \mu$, i.e. μ is invariant and $\mu = Span\{W_3, W_4\}$. Hence M is a proper SCR 4-lightlike submanifold of $\mathbb{R}^{16}_4$. Now, if the leaves of D' and $D^{\perp}$ are, respectively denoted by M_T and $M_{\perp}$, then the induced metric tensor of $M = M_T \times_{\lambda} M_{\perp}$ is given by $ds^2 = 0((du^1)^2 + (du^2)^2 + (du^3)^2 + (du^4)^2) + ((du^5)^2 + (du^6)^2) + (\sin^2 u^7 + \sinh^2 u^8)((du^7)^2 + (du^8)^2)$.

Hence, M is a SCR warped product lightlike submanifold of the type $M = M_T \times_{\lambda} M_{\perp}$ in $\mathbb{R}^{16}_4$, with warping function $\lambda = \sqrt{(\sin^2 u^7 + \sinh^2 u^8)}$.

Theorem 3.6. *For a proper SCR warped product lightlike submanifold $M = M_T \times_{\lambda} M_{\perp}$ of a golden semi-Riemannian manifold $\overline{M}$, the induced connection ∇ is not a metric connection.*

Proof. If possible, then let ∇ is a metric connection on M, from (12) we have $h^l = 0$. We know that $\overline{\nabla}$ is a metric connection on $\overline{M}$, for all $X \in \Gamma(Rad(TM))$ and $Z, W \in \Gamma(D^{\perp})$, we have $\overline{g}(\overline{\nabla}_Z W, X) = -\overline{g}(W, \overline{\nabla}_Z X)$, using (7) and (14), we obtain

$$\overline{g}(h^l(Z, W), X) = -X(\ln\lambda) g(Z, W). \tag{23}$$

Since $h^l = 0$, therefore (23) becomes, $X(\ln\lambda)g(Z, W) = 0$, which implies that $X(\ln\lambda) = 0$ or $g(Z, W) = 0$, but this a contradiction as M is a proper SCR-warped product lightlike submanifold and $D^{\perp}$ is non-degenerate. □

Lemma 3.7. *Let $M = M_T \times_{\lambda} M_{\perp}$ be a SCR warped product lightlike submanifold of golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then we have*
(i) $\overline{g}(h^s(X, \tilde{P}Y), \tilde{P}Z) = 0$,

(ii) $\overline{g}(h^s(Z, W), \tilde{P}V) = \tilde{P}W(\ln\lambda) g(Z, V)$,
for all $X, Y \in \Gamma(D)$, $W \in \Gamma(D')$ and $Z, V \in \Gamma(D^{\perp})$.

Proof. Since $\overline{M}$ is a golden semi-Riemannian manifold, then for all $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, we have from (3), $\overline{\nabla}_X \tilde{P}Z = \tilde{P}\overline{\nabla}_X Z$. Using (5) and (9), we have $\tilde{P}\nabla_X Z + \tilde{P}h(X, Z) = -A_{\tilde{P}Z}X + D^l(X, \tilde{P}Z) + \nabla^s_X \tilde{P}Z$, then taking scalar product with $\tilde{P}Y$, for any $Y \in \Gamma(D)$, we have $\overline{g}(\tilde{P}\nabla_X Z, \tilde{P}Y) = -\overline{g}(A_{\tilde{P}Z}X, \tilde{P}Y)$. From (10) and (14), we get $\overline{g}(h^s(X, \tilde{P}Y), \tilde{P}Z) = 0$. Now for any $W \in \Gamma(D')$ and $Z, V \in \Gamma(D^{\perp})$, from (2), (7) and (14), we have $\overline{g}(h^s(Z, W), \tilde{P}V) = \overline{g}(\overline{\nabla}_Z W, \tilde{P}V) = \overline{g}(\tilde{P}\overline{\nabla}_Z W, V) = \tilde{P}W(\ln\lambda) g(Z, V)$. □

Lemma 3.8. *Let $M = M_T \times_{\lambda} M_{\perp}$ be a SCR warped product lightlike submanifold of golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then we have $h^l(X, Z) = 0$ and $h^*(X, Z) = 0$, for any $X \in \Gamma(D')$ and $Z \in \Gamma(D^{\perp})$.*

Proof. For any $X, Y\in\Gamma(D')$ and $Z\in\Gamma(D^{\perp})$, from (7) and (14), we have $\overline{g}(h^l(X,Z),Y) = \overline{g}(\overline{\nabla}_X Z, Y)$. Since $\overline{\nabla}$ is a metric connection and using (5), we obtain $\overline{g}(h^l(X,Z),Y) = -\overline{g}(\overline{\nabla}_X Y, Z) = -g(\nabla_X Y, Z)$. Since M is a SCR warped product lightlike submanifold, then D' defines a totally geodesic foliation in M, hence we get $\overline{g}(h^l(X,Z),Y) = 0$, which gives $h^l(X,Z) = 0$. For all $X \in \Gamma(D')$, $Y \in \Gamma(Rad(TM))$ and $Z \in \Gamma(D^{\perp})$ and using (11), we obtain $h^*(X,Z) = 0$. □

THEOREM 3.9. *Let M be a totally umbilical SCR-lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. If $M = M_{\perp} \times_{\lambda} M_T$ is a SCR warped product lightlike submanifold such that $M_{\perp}$ is a totally real submanifold and M_T is a holomorphic submanifold of $\overline{M}$, then*
(i) $D^{\perp}$ defines a totally geodesic foliation in M,

(ii) D' does not defines a totally geodesic foliation in M.

Proof. Let $M = M_{\perp} \times_{\lambda} M_T$ be a SCR warped product lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then for all $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_{\perp})$ and using (14), we obtain

$$\nabla_X Z = \nabla_Z X = (Z \ln \lambda) X. \tag{24}$$

Now for any $X, Y \in \Gamma(D^{\perp})$ and from (18), we obtain $P\nabla_X Y = -A_{FY}X - Bh(X,Y)$, then for all $Z \in \Gamma(D)$, using (6) and (24), we get $g(P\nabla_X Y, Z) = -g(A_{FY}X, Z) = -g(\tilde{P}Y, \nabla_X Z) = 0$, then using non-degeneracy of D, we obtain $P\nabla_X Y = 0$, which implies $\nabla_X Y \in \Gamma(D^{\perp})$, this show $D^{\perp}$ defines a totally geodesic foliation in M.

Let h^T and A^T, respectively denote the second fundamental form and shape operator of M_T in M, then for all $X, Y \in \Gamma(D')$ and $Z \in \Gamma(D^{\perp})$, we obtain from (5), $g(h^T(X,Y),Z) = g(\nabla_X Y, Z) = -g(Y, \nabla_X Z)$, using (24), we get $g(h^T(X,Y),Z) = -(Z \ln \lambda) g(X,Y)$. Now let $\hat{h}$ be the second fundamental form of M_T in $\overline{M}$, then

$$\hat{h}(X,Y) = h^T(X,Y) + h^l(X,Y) + h^s(X,Y), \tag{25}$$

for all $X, Y \in \Gamma(TM_T)$. Then taking scalar product with $Z \in \Gamma(D^{\perp})$, we get

$$g(\hat{h}(X,Y),Z) = g(h^T(X,Y),Z) = -(Z \ln \lambda) g(X,Y). \tag{26}$$

Since M_T is a holomorphic (invariant) submanifold of $\overline{M}$, then we have

$$\hat{h}(X, \tilde{P}Y) = \hat{h}(\tilde{P}X, Y) = \tilde{P}\hat{h}(X,Y). \tag{27}$$

Using (26) and (27) in (25), we obtain $g(h(X,Y), \tilde{P}Z) = -(Z \ln \lambda) g(X,Y)$. Thus, $g(h(D', D'), \tilde{P}Z) \neq 0$, for all $Z \in \Gamma(D^{\perp})$, which implies that D' does not defines a totally geodesic foliation in M. □

THEOREM 3.10. *Let (M, g) be a SCR lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$ such that totally real distribution $D^{\perp}$ being integrable and $\tilde{P}X = X$ for all $X \in \Gamma(D')$. Then (M, g) is locally a SCR warped product lightlike submanifold if and only if $A_{\tilde{P}Z}X = -(\tilde{P}X)(\mu)Z$, for each $X \in \Gamma(D')$, $Z \in \Gamma(D^{\perp})$ and μ is a C^{∞}-function on M such that $Z\mu = Z(\ln \lambda) = 0$, for all $Z \in \Gamma(D^{\perp})$.*

Proof. Let (M, g) be a locally SCR warped product lightlike submanifold of $\overline{M}$. Since $\overline{M}$ is a golden semi-Riemannian manifold, then for each $X \in \Gamma(D')$ and $Z \in \Gamma(D^{\perp})$, from (2.3), we have $\overline{\nabla}_X \tilde{P}Z = \tilde{P}\overline{\nabla}_X Z = \overline{\nabla}_Z \tilde{P}X$, using (5) and (9), we obtain $-A_{\tilde{P}Z}X + \nabla^s_X \tilde{P}Z + D^l(X, \tilde{P}Z) = \nabla_Z \tilde{P}X + h(Z, \tilde{P}X)$. On equating tangential components on both sides and using (14), we get $A_{\tilde{P}Z}X = \nabla_Z \tilde{P}X = -\tilde{P}X(\ln\lambda)Z$. As $\mu = \ln\lambda$ is a function on M_T, therefore $Z\mu = Z(\ln\lambda) = 0$, for all $Z \in \Gamma(D^\perp)$.

Conversely, let $A_{\tilde{P}Z}X = -(\tilde{P}X)(\mu)Z$, for any $X \in \Gamma(D')$ and $Z \in \Gamma(D^\perp)$, taking scalar product with $Y \in \Gamma(D')$, we get $g(A_{\tilde{P}Z}X, Y) = -g((\tilde{P}X)(\mu)Z, Y) = 0$, then using (2.14), we obtain $\overline{g}(h^s(X,Y), \tilde{P}Z) + \overline{g}(Y, D^l(X, \tilde{P}Z)) = 0$. Using (4), (9) and $\tilde{P}X = X$ for each $X \in \Gamma(D')$, this implies $\overline{g}(h^s(D', D'), \tilde{P}Z) = 0$ and also $\overline{g}(h^l(D', D'), \tilde{P}Z) = 0$, for all $Z \in \Gamma(D^\perp)$. Thus we obtain $\overline{g}(h(D', D'), \tilde{P}Z) = 0$, i.e. $h(D', D')$ has no components in $\tilde{P}Z$, which implies that D' defines a totally geodesic foliation in M.

Let $A_{\tilde{P}Z}X = -(\tilde{P}X)(\mu)Z$, for any $X \in \Gamma(D')$ and $Z \in \Gamma(D^\perp)$, taking scalar product with $W \in \Gamma(D^\perp)$, we get

$$g((\tilde{P}X)(\mu)Z, W) = -g(A_{\tilde{P}Z}X, W). \tag{28}$$

Using the definition of gradient $g(\nabla\phi, X) = X\phi$ and (10) in (28), we obtain

$$g(\nabla\mu, \tilde{P}X)g(W, Z) = -g(h^s(X, W), \tilde{P}Z). \tag{29}$$

From (3), (5) and using $\overline{\nabla}$ is metric connection in (29), we get

$$g(\nabla\mu, \tilde{P}X)g(W, Z) = -g(\nabla_W Z, \tilde{P}X). \tag{30}$$

Let h' be the second fundamental form of $D^\perp$ in M and let ∇' be the induced connection of $D^\perp$ in M, then for $X \in \Gamma(D)$ and $W, Z \in \Gamma(D^\perp)$, we get

$$g(h'(W, Z), \tilde{P}X) = g(\nabla_W Z - \nabla'_W Z, \tilde{P}X) = g(\nabla_W Z, \tilde{P}X). \tag{31}$$

Then from (30) and (31), we get

$$g(h'(W, Z), \tilde{P}X) = -g(\nabla\mu, \tilde{P}X)g(W, Z). \tag{32}$$

Using non-degeneracy of D from (32), we obtain

$$h'(W, Z) = -\nabla\mu g(W, Z), \tag{33}$$

which implies that the distribution $D^\perp$ is totally umbilical in M. By hypothesis, the totally real distribution $D^\perp$ is integrable, using (33) and the condition $Z\mu = 0$ for any $Z \in \Gamma(D^\perp)$ implies that each leaf of $D^\perp$ is an intrinsic sphere in M. From [11], we have: "If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non-trivial vector sub-bundles such that E_1 is spherical and it's orthogonal complement E_0 is auto-parallel, then the manifold M is locally isometric to a warped product $M_0 \times_f M_1$".

Thus, we conclude that M is locally a SCR warped product lightlike submanifold of the type $M_T \times_\lambda M_\perp$ in $\overline{M}$, where $\lambda = e^\mu$. □

4. SCR-warped product lightlike submanifolds and canonical structures

In this section, we find characterizations in terms of the canonical structures P and F on a SCR-lightlike submanifold of a golden semi-Riemannian manifold.

LEMMA 4.1. *Let $M = M_T \times_\lambda M_\perp$ be a SCR warped product lightlike submanifold of golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then we have*
(i) $(\nabla_Z P)X = PX(\ln \lambda)Z$,

(ii) $(\nabla_U P)Z = P(\nabla \ln \lambda)g(U, Z)$,
for any $U \in \Gamma(TM)$, $X \in \Gamma(D')$ and $Z \in \Gamma(D^\perp)$, where $\nabla \ln \lambda$ denotes the gradient of $\ln \lambda$.

Proof. Let $M = M_T \times_\lambda M_\perp$ be a SCR warped product lightlike submanifold of golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then for all $X \in \Gamma(D')$ and $Z \in \Gamma(D^\perp)$, from (22) and (14), we obtain $(\nabla_Z P)X = \nabla_Z PX = PX(\ln \lambda)Z$. From (22), for $U \in \Gamma(TM)$ and $Z \in \Gamma(D^\perp)$, we have $(\nabla_U P)Z = -P\nabla_U Z$, which implies that $(\nabla_U P)Z \in \Gamma(D')$. Then for any $X \in \Gamma(D)$, we obtain $g((\nabla_U P)Z, X) = g(Z, \nabla_U PX) = PX(\ln \lambda)g(Z, U)$. Using the definition of gradient of λ and non-degeneracy of D, we get the required result. □

THEOREM 4.2. *Let (M, g) be a SCR-lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$ with totally real distribution $D^\perp$ being integrable, then M is locally a SCR warped product lightlike submanifold if and only if $(\nabla_U P)V = ((PV)\mu)Q'U + g(Q'U, Q'V)P(\nabla\mu)$, for each $U, V \in \Gamma(TM)$, where μ is a C^∞-function on M satisfying $Z\mu = 0$, for all $Z \in \Gamma(D^\perp)$.*

Proof. Assume that $M = M_T \times_\lambda M_\perp$ be a SCR warped product lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then for all $U, V \in \Gamma(TM)$, we have $(\nabla_U P)V = (\nabla_{P'U} P)P'V + (\nabla_{Q'U} P)P'V + (\nabla_U P)Q'V$. Since D' defines a totally geodesic foliation in M, from (18), we obtain

$$(\nabla_{P'U} P)P'V = 0. \tag{34}$$

Using Lemma 4.1, we get

$$(\nabla_{Q'U} P)P'V = P(P'V)(\ln \lambda)Q'U, \tag{35}$$

and

$$(\nabla_U P)Q'V = g(U, Q'V)P(\nabla \ln \lambda) = g(Q'U, Q'V)P(\nabla \ln \lambda). \tag{36}$$

From (34)-(36), we get $(\nabla_U P)V = ((PV)\mu)Q'U + g(Q'U, Q'V)P(\nabla\mu)$, for each $U, V \in \Gamma(TM)$. Since $\mu = \ln \lambda$ is a function on M_T, therefore $Z(\mu) = Z(\ln \lambda) = 0$, for all $Z \in \Gamma(D^\perp)$.

Conversely, let M be a SCR-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ satisfying $(\nabla_U P)V = ((PV)\mu)Q'U + g(Q'U, Q'V)P(\nabla\mu)$, for each $U, V \in \Gamma(TM)$. Let $U, V \in \Gamma(D')$, this implies that $(\nabla_U P)V = 0$, from (18), we get $Bh(U, V) = 0$, this shows that $h(U, V)$ has no component in $\tilde{P}D^\perp$, for any $U, V \in \Gamma(D')$, which implies D' defines a totally geodesic foliation in M.

Now, $(\nabla_U P)V = ((PV)\mu)Q'U + g(Q'U, Q'V)P(\nabla\mu)$, for any $U, V \in \Gamma(D^\perp)$, which implies $(\nabla_U P)V = g(Q'U, Q'V)P\nabla\mu$. Taking scalar product of this with $X \in \Gamma(D)$, we get

$$g((\nabla_U P)V, X) = g(Q'U, Q'V)g(P\nabla\mu, X) = g(Q'U, Q'V)g(\nabla\mu, \tilde{P}X). \tag{37}$$

For any $U, V \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$, using (18), we obtain

$$g((\nabla_U P)V, X) = g(A_{FV}U, X) = -\overline{g}(\overline{\nabla}_U \tilde{P}V, X) = -g(\nabla_U V, \tilde{P}X). \tag{38}$$

From (37) and (38), we get

$$g(\nabla_U V, \tilde{P}X) = -g(Q'U, Q'V)g(\nabla\mu, \tilde{P}X). \tag{39}$$

Let h' be the second fundamental form of $D^\perp$ in M and let ∇' be the induced connection of $D^\perp$ in M, then for $X \in \Gamma(D)$ and $U, V \in \Gamma(D^\perp)$, we get

$$g(h'(U,V), \tilde{P}X) = g(\nabla_U V - \nabla'_U V, \tilde{P}X) = g(\nabla_U V, \tilde{P}X). \tag{40}$$

Then from (39) and (40), we get

$$g(h'(U,V), \tilde{P}X) = -g(\nabla\mu, \tilde{P}X)g(Q'U, Q'V). \tag{41}$$

Using non-degeneracy of D from (41), we obtain

$$h'(U,V) = -\nabla\mu g(Q'U, Q'V), \tag{42}$$

which implies that the distribution $D^\perp$ is totally umbilical in M. By hypothesis, the totally real distribution $D^\perp$ is integrable , using (42) and the condition $Z\mu = 0$ for any $Z \in \Gamma(D^\perp)$ implies that each leaf of $D^\perp$ is an intrinsic sphere in M. Thus, by similar result in Theorem 3.10, M is locally a SCR warped product lightlike submanifold of the type $M_T \times_\lambda M_\perp$ in $\overline{M}$, where $\lambda = e^\mu$. □

Theorem 4.3. *Let (M, g) be a SCR-lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$ with totally real distribution $D^\perp$ being integrable, then M is locally a SCR warped product lightlike submanifold if and only if $\overline{g}((\nabla^t_U F)V, \tilde{P}W) = -V(\ln\lambda)g(U,W) + \tilde{P}U(\ln\lambda)g(V,W)$, for each $U, V \in \Gamma(TM)$ and $W \in \Gamma(D^\perp)$, where μ is a C^∞-function on M satisfying $W\mu = 0$, for all $W \in \Gamma(D^\perp)$.*

Proof. Assume that M be a SCR warped product lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Therefore the distribution D' defines a totally geodesic foliation in M and from (22) for all $U, V \in \Gamma(D')$ and $W \in \Gamma(D^\perp)$, we have

$$\overline{g}((\nabla^t_U F)V, \tilde{P}W) = \overline{g}(F\nabla_U V, \tilde{P}W) = -g(\nabla_U V, W) = 0. \tag{43}$$

For any $U, W \in \Gamma(D^\perp)$ and $V \in \Gamma(D')$, from (1), (19) and Lemma 3.7, we get

$$\begin{aligned}\overline{g}((\nabla^t_U F)V, \tilde{P}W) &= \overline{g}(h^s(U,V), \tilde{P}W) - \overline{g}(h^s(U,PV), \tilde{P}W)\\ &= \tilde{P}V(\ln\lambda)g(U,W) - \tilde{P}PV(\ln\lambda)g(U,W)\\ &= -V(\ln\lambda)g(U,W).\end{aligned} \tag{44}$$

For any $V \in \Gamma(D^\perp)$, $U \in \Gamma(D')$ and $W \in \Gamma(D^\perp)$, from (19), we get

$$\overline{g}((\nabla^t_U F)V, \tilde{P}W) = \overline{g}(Ch(U,V), \tilde{P}W) = \overline{g}(h^s(U,V), \tilde{P}W) = \tilde{P}U(\ln\lambda)g(V,W). \tag{45}$$

Thus from (43)-(45), we get $\overline{g}((\nabla^t_U F)V, \tilde{P}W) = -V(\ln\lambda)g(U,W) + \tilde{P}U(\ln\lambda)g(V,W)$, for each $U, V \in \Gamma(TM)$ and $W \in \Gamma(D^\perp)$. As $\mu = \ln\lambda$ is a function on M_T, therefore $W(\mu) = W(\ln\lambda) = 0$, for all $W \in \Gamma(D^\perp)$.

Conversely, let M be a SCR-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ with totally real distribution $D^\perp$ integrable satisfying $\overline{g}((\nabla^t_U F)V, \tilde{P}W) = -V(\ln\lambda)g(U,W) + \tilde{P}U(\ln\lambda)g(V,W)$, for each $U, V \in \Gamma(TM)$ and $W \in \Gamma(D^\perp)$. For any $U, V \in \Gamma(D')$ and $W \in \Gamma(D^\perp)$, we have $\overline{g}(F\nabla_U V, \tilde{P}W) = 0$, then $g(\nabla_U V, W) = 0$, which implies that $\nabla_U V \in \Gamma(D')$, i.e. D' defines a totally geodesic foliation in M. Next for any $V \in \Gamma(D)$ and $U, W \in \Gamma(D^\perp)$, from $\overline{g}((\nabla^t_U F)V, \tilde{P}W) =$

$-V(\ln\lambda)g(U,W)+\tilde{P}U(\ln\lambda)g(V,W)$, we obtain

$$\begin{aligned}-V(\mu)g(U,W) &= \overline{g}((\nabla^t_U F)V,\tilde{P}W) = -\overline{g}((F\nabla_U V,\tilde{P}W)\\ &= -\overline{g}(\overline{\nabla}_U V,W) = g(V,\nabla_U W).\end{aligned}$$

Using the definition of gradient $g(\nabla\phi,V)=V\phi$, we obtain

$$g(\nabla_U W,V) = -g(\nabla\mu,V)g(U,W). \tag{46}$$

Let h' be the second fundamental form of $D^\perp$ in M and let ∇' be the induced connection of $D^\perp$ in M, then for $V\in\Gamma(D)$ and $U,W\in\Gamma(D^\perp)$, we get

$$g(h'(U,W),V) = g(\nabla_U W-\nabla'_U W,V) = g(\nabla_U W,V). \tag{47}$$

Then from (46) and (47), we get $g(h'(U,W),V)=-g(\nabla\mu,V)g(U,W)$. Hence, using non-degeneracy of D, we obtain

$$h'(U,W) = -\nabla\mu g(U,W), \tag{48}$$

which implies that the distribution $D^\perp$ is totally umbilical in M. By hypothesis, the totally real distribution $D^\perp$ is integrable, using (48) and the condition $Z\mu=0$ for any $Z\in\Gamma(D^\perp)$ implies that each leaf of $D^\perp$ is an intrinsic sphere in M. Thus, by similar result in Theorem 3.10, M is locally a SCR warped product lightlike submanifold of the type $M_T\times_\lambda M_\perp$ in $\overline{M}$, where $\lambda=e^\mu$. □

Theorem 4.4. *Let $M=M_T\times_\lambda M_\perp$ be a SCR warped product lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then we have*
(i) The squared norm of the second fundamental form satisfies $\|h\|^2\geq 2k\|\nabla(\ln\lambda)\|^2$, where $\nabla(\ln\lambda)$ is the gradient of $\ln\lambda$ and k is the dimension of $M_\perp$.

(ii) If the equality sign in $\|h\|^2\geq 2k\|\nabla(\ln\lambda)\|^2$ holds identically, then M_T is totally geodesic in $\overline{M}$ and $M_\perp$ is totally umbilical in $\overline{M}$.

Proof. Let $\{X_1,X_2,X_3,\ldots,X_{p+1}=\tilde{P}X_1,X_{p+2}=\tilde{P}X_2,\ldots,X_{2p}=\tilde{P}X_p,X_{2p+1}=\xi_1,X_{2p+2}=\xi_2,\ldots,X_{2p+r}=\xi_r,X_{2p+r+1}=\tilde{P}\xi_1,X_{2p+r+2}=\tilde{P}\xi_2,\ldots,X_{2p+2r}=\tilde{P}\xi_r\}$, be a local orthonormal frame of vector fields on M_T and $\{Z_1,Z_2,\ldots,Z_k\}$ a local orthonormal frame of vector fields on $M_\perp$, then we have

$$\|h\|^2 = \|h(D',D')\|^2+\|h(D^\perp,D^\perp)\|^2+2\|h(D',D^\perp)\|^2. \tag{49}$$

Using Theorem 2.1 in (49), we obtain

$$\|h\|^2 = \|h^s(D',D')\|^2+\|h^s(D^\perp,D^\perp)\|^2+2\|h^s(D',D^\perp)\|^2.$$

We have

$$\begin{aligned}\|h\|^2 &= \sum_{i,j=1}^{2p+2r}\overline{g}(h^s(X_i,X_j),h^s(X_i,X_j)) + \sum_{m,n=1}^{k}\overline{g}(h^s(Z_m,Z_n),h^s(Z_m,Z_n))\\ &\quad + 2\sum_{i=1}^{2p+2r}\sum_{m=1}^{k}\overline{g}(h^s(X_i,Z_m),h^s(X_i,Z_m)).\end{aligned}$$

Thus we get

$$\|h\|^2 \geq 2\sum_{i=1}^{2p+2r}\sum_{m=1}^{k}\overline{g}(h^s(X_i,Z_m),h^s(X_i,Z_m)). \tag{50}$$

Using Lemma 3.7 in (50), we obtain

$$\|h\|^2 \geq 2 \sum_{i=1}^{2p+2r} \sum_{m=1}^{k} (X_i \ln \lambda)^2 g(Z_m, Z_m) \geq 2k \|\nabla(\ln \lambda)\|^2,$$

which proves (i).

If the equality sign in $\|h\|^2 \geq 2k\|\nabla(\ln \lambda)\|^2$ holds, then we have

$$h^s(D', D') = 0, \; h^s(D^\perp, D^\perp) = 0 \; and \; h^s(D', D^\perp) \subset \tilde{P}D^\perp. \tag{51}$$

Since M_T is totally geodesic in M, then from first condition in (51), we get M_T is totally geodesic in $\overline{M}$. Moreover, $M_\perp$ is totally umbilical in $\overline{M}$. □

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