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ON DIFFERENTIAL IDENTITIES IN σ-PRIME RINGS WITH A PAIR OF DERIVATIONS

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Abstract. The primary objective of this paper is to investigate the commutativity of σ-prime rings with the second kind involution, involving pairs of derivations that satisfy specific differential identities. Finally, we present examples to illustrate that the conditions assumed in our results are essential and cannot be omitted.

1. Introduction

Throughout this work, $\mathscr{R}$ is taken to be an associative ring with $\mathscr{J}_Z$ as its center. For any $t_1, t_2 \in \mathscr{R}$, the notation $[t_1, t_2]$ denotes the commutator, defined by $t_1t_2 - t_2t_1$, while $t_1 \circ t_2$ represents the anti-commutator, given by $t_1t_2 + t_2t_1$. We use the basic identities $[t_1t_2, t_3] = t_1[t_2, t_3] + [t_1, t_3]t_2$ and $[t_1, t_2t_3] = [t_1, t_2]t_3 + t_2[t_1, t_3]$ for all $t_1, t_2, t_3 \in \mathscr{R}$ very frequently. Recall that an involution is an anti-automorphism of order 2. A ring $\mathscr{R}$ with an involution σ is said to be σ-prime if $a\mathscr{R}b = a\mathscr{R}\sigma(b) = (0)$ or $\sigma(a)\mathscr{R}b = a\mathscr{R}b = (0)$ implies either $a = 0$ or $b = 0$. Every prime ring with an involution σ is a σ-prime ring, but the converse is not true in general. For instance, let $S = \mathscr{R} \times \mathscr{R}^0$, where $\mathscr{R}^0$ is the opposite ring of a prime ring $\mathscr{R}$. The mapping σ on S defined by $\sigma(t_1, t_2) = (t_2, t_1)$ is an involution on S. Thus, S with an involution σ is σ-prime but not a prime ring. An element $t_1 \in \mathscr{R}$ is said to be hermitian if $\sigma(t_1) = t_1$ and skew-hermitian if $\sigma(t_1) = -t_1$. Let $\mathscr{J}_H$ denote the set of all hermitian elements and $\mathscr{J}_S$ denote the set of all skew-hermitian elements of $\mathscr{R}$. An involution σ is said to be of the first kind if $\mathscr{J}_Z \subseteq \mathscr{J}_H$; otherwise, it is of the second kind, and in this case, we have $\mathscr{J}_S \cap \mathscr{J}_Z \neq (0)$. An element $t_1 \in \mathscr{R}$ is called a normal element if it commutes with its image under involution σ, and a ring $\mathscr{R}$ is called a normal ring if every element of the ring $\mathscr{R}$ is normal (see in [5]).

A mapping ψ on $\mathscr{R}$ is termed a derivation if $\psi(t_1 + t_2) = \psi(t_1) + \psi(t_2)$ and $\psi(t_1t_2) = \psi(t_1)t_2 + t_1\psi(t_2)$ hold for all $t_1, t_2 \in \mathscr{R}$. Let $b \in \mathscr{R}$ be a fixed element of $\mathscr{R}$.

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Then, the mapping ψ on $\mathscr{R}$ defined by $\psi(t_1) = [b, t_1] = bt_1 - t_1 b$ for all $t_1 \in \mathscr{R}$ is a derivation, and such a derivation is called an inner derivation induced by b. A map $f : \mathscr{R} \to \mathscr{R}$ is called centralizing on $\mathscr{R}$ if $[f(t_1), t_1] \in \mathscr{J}_Z$ holds for all $t_1 \in \mathscr{R}$. In particular, if $[f(t_1), t_1] = 0$ holds for all $t_1 \in \mathscr{R}$, then it is called commuting.

Stimulated by the description of centralizing maps, a map f from $\mathscr{R}$ into itself is called σ-centralizing if $[f(t_1), \sigma(t_1)] \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$ and is called σ-commuting if $[f(t_1), \sigma(t_1)] = 0$ for all $t_1 \in \mathscr{R}$. The narrative of centralizing and commuting maps dates back to 1955, when Divinsky proved that if a simple Artinian ring has commuting non-trivial automorphisms, then it is commutative. A few years later, Posner [14] established that the presence of a nonzero centralizing derivation on a prime ring implies the commutativity of the ring. The study of centralizing (resp. commuting) derivations and various generalizations of the concept of centralizing (resp. commuting) maps are the main concepts emerging directly from Posner's result, with many applications in various areas. Recently, a number of algebraists have demonstrated the commutativity theorem for prime and semi-prime rings with or without an involution, accepting identities on automorphisms, derivations, left centralizers, and generalized derivations (for example) [1, 2, 4, 8, 9, 11].

In 2014, Ali and Dar [1] began the study of σ-centralizing derivations on prime rings with an involution and proved a σ-version of the classical results of Posner [14], under certain assumptions. They proved that if $\mathscr{R}$ is a prime ring with an involution σ such that char$(\mathscr{R}) \neq 2$, and ψ is a nonzero derivation on $\mathscr{R}$ such that $[\psi(t_1), \sigma(t_1)] \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$ and $\psi(\mathscr{J}_S \cap \mathscr{J}_Z) \neq (0)$, then $\mathscr{R}$ is commutative. Furthermore, this result was extended by Najjar et al. [10] for the second kind involution instead of the condition $\psi(\mathscr{J}_S \cap \mathscr{J}_Z) \neq (0)$. Recently, Alahmadi et al. [2] generalized the above results for generalized derivations and proved that "Let $\mathscr{R}$ be a prime ring with an involution σ of the second kind such that char$(\mathscr{R}) \neq 2$, and if $\mathscr{R}$ admits a nonzero generalized derivation F associated with a derivation d such that $[F(t), \sigma(t)] \in \mathscr{J}_Z$ for all $t \in \mathscr{R}$, then $\mathscr{R}$ is commutative." In this direction, a lot of work has been done in recent years (see [3, 6, 7] and the references therein).

The main goal of our work is to investigate the commutativity of σ-prime rings that satisfy some central identities involving pairs of derivations. Our motivation for this manuscript comes from the types of identities studied by Mamouni et al. in [10], and motivated by these types of identities, we generalized some results from [10]. To prove our main results, we need some lemmas as well as some facts.

2. The main results

LEMMA 2.1. *Let $\mathscr{R}$ be a σ-prime ring. If $a \in \mathscr{R}$ and $z \in \mathscr{J}_Z$ such that $az \in \mathscr{J}_Z$ and $a\sigma(z) \in \mathscr{J}_Z$, then either $a \in \mathscr{J}_Z$ or $z = 0$.*

Proof. Since, $az \in \mathscr{J}_Z$ and $a\sigma(z) \in \mathscr{J}_Z$, we have $(0) = [az, r] = [a\sigma(z), r]$ for all $r \in \mathscr{R}$, which implies that $(0) = z[a, r] = \sigma(z)[a, r]$. Therefore $(0) = z\mathscr{R}[a, r] = \sigma(z)\mathscr{R}[a, r]$, by the definition of σ-prime rings we have either $z = 0$ or $a \in \mathscr{J}_Z$. □

Lemma 2.2. *Let $\mathscr{R}$ be a σ-prime ring. If $a \in \mathscr{R}$ and $z \in \mathscr{J}_Z$ such that $az \in \mathscr{J}_Z$ and $\sigma(a)z \in \mathscr{J}_Z$, then either $a \in \mathscr{J}_Z$ or $z = 0$.*

Proof. Since, $az \in \mathscr{J}_Z$ and $\sigma(a)z \in \mathscr{J}_Z$, we have $(0) = [az, r] = [\sigma(a)z, r]$ for all $r \in \mathscr{R}$, which implies that $0 = z[a, r] = z[\sigma(a), r]$. Therefore $(0) = z\mathscr{R}[a, r] = z\mathscr{R}[\sigma(a), r]$, by the definition of σ-prime rings we have either $z = 0$ or $a \in \mathscr{J}_Z$. □

Lemma 2.3. *Let $\mathscr{R}$ be a σ-prime ring of* $\mathrm{char}(\mathscr{R}) \neq 2$*, then $\mathscr{R}$ is 2-torsion free.*

Proof. Let $u \in \mathscr{R}$ and $2u = 0$ suggests, $2u(vw) = 0$ for all $v, w \in \mathscr{R}$ and $u\mathscr{R}(2w) = 0$ for all $w \in \mathscr{R}$. Since $\mathrm{char}(\mathscr{R}) \neq 2$ and $\mathscr{R} \neq (0)$ then there exist $0 \neq p \in \mathscr{R}$ such that $2p \neq 0$, forces $u\mathscr{R}(2p) = (0) = u\mathscr{R}\sigma(2p)$, by the definition of σ-prime rings we have, either $u = 0$ or $2p = 0$. The second case is not possible by the assumption and first case implies that $\mathscr{R}$ is 2-torsion free. □

Lemma 2.4. *In σ-prime ring, $\mathscr{J}_Z \cap \mathscr{J}_H$ and $\mathscr{J}_Z \cap \mathscr{J}_S$ are free from zero-divisor.*

Proof. Let $a \in \mathscr{R}$ and $b \in \mathscr{J}_Z \cap \mathscr{J}_H$, such that $ab = 0$, implies $abu = 0$ for all $u \in \mathscr{R}$ provide us $a\mathscr{R}b = (0) = a\mathscr{R}\sigma(b)$, by the definition of σ-prime ring, we have either $a = 0$ or $b = 0$. □

Lemma 2.5. *Let $\mathscr{R}$ be a 2-torsion free σ-prime ring with an involution σ which is of the second kind. If $t_1^2 \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$, then $\mathscr{R}$ is commutative.*

Proof. We are given that $t_1^2 \in! \mathscr{J}_Z$ for all $t_1 \in! \mathscr{R}$. By linearizing the last relation and using it, we obtain $t_1t_2 + t_2t_1 \in! \mathscr{J}_Z$ for all $t_1, t_2 \in! \mathscr{R}$. Since σ is of the second kind, there exists $0 \neq c \in! \mathscr{J}_Z \cap \mathscr{J}_S$. Now, replacing t_2 by c, we have $t_1c \in! \mathscr{J}_Z$ for all $t_1 \in! \mathscr{R}$, which implies that $[t_1, r]c = 0$ for all $r \in! \mathscr{R}$. Applying Lemma 2.4, we get that $[t_1, r] = 0$ for all $t_1, r \in! \mathscr{R}$, which implies that $\mathscr{R}$ is commutative. □

Lemma 2.6. *Let $\mathscr{R}$ be a 2-torsion free σ-prime ring and ψ be the derivation on $\mathscr{R}$. If σ is of the second kind and $\psi(h) = 0$ for all $h \in \mathscr{J}_H \cap \mathscr{J}_Z$, then $\psi(z) = 0$ for all $z \in \mathscr{J}_Z$.*

Proof. By the given hypothesis, we have $\psi(h) = 0$, where $h \in \mathscr{J}_H \cap \mathscr{J}_Z$, then $\psi(k^2) = 0$ for all $k \in \mathscr{J}_S \cap \mathscr{J}_Z$. Hence $k\psi(k) = 0$, making use of Lemma 2.4, we have either $k = 0$ or $\psi(k) = 0$. The first case is not possible because σ is of the second kind. So we have $\psi(k) = 0$ for $k \in \mathscr{J}_S \cap \mathscr{J}_Z$. Now for any $z \in \mathscr{J}_Z$ we have $2z = h + k$, where $h = z + \sigma(z)$ and $k = z - \sigma(z)$. Therefore, we have $\psi(2z) = \psi(h) + \psi(k) = 0$. Consequently, $\psi(z) = 0$ for all $z \in \mathscr{J}_Z$. □

Fact 2.7. *Let $\mathscr{R}$ be a 2-torsion free σ-prime rings with an involution σ which is of the second kind, if $\mathscr{R}$ is normal, then $\mathscr{R}$ is commutative.*

Proof. Since $\mathscr{R}$ is normal, i.e., $hk = kh$ where $h \in \mathscr{J}_H$ *and* $k \in \mathscr{J}_S$ respectively. Take any $t_1 \in \mathscr{R}$, then $t_1 - \sigma(t_1) \in \mathscr{J}_S$ and

$$h(t_1 - \sigma(t_1)) = (t_1 - \sigma(t_1))h, \quad \text{for all } t_1 \in \mathscr{R} \text{ and } h \in \mathscr{J}_H. \tag{1}$$

Take $s \in \mathscr{J}_S \cap \mathscr{J}_Z$, then $s(t_1 + \sigma(t_1)) \in \mathscr{J}_S$ for all $t_1 \in \mathscr{R}$, using the normality condition of $\mathscr{R}$, we have $hs(t_1 + \sigma(t_1)) = s(t_1 + \sigma(t_1))h$ for all $t_1 \in \mathscr{R}$ and $h \in \mathscr{J}_H$. This implies that

$$s\{h(t_1 + \sigma(t_1)) - (t_1 + \sigma(t_1))h\} = 0, \text{ for all } \quad t_1 \in \mathscr{R} \text{ and for all } \quad h \in \mathscr{J}_H.$$

Invoking Lemma 2.4, we have either $s = 0$ or $h(t_1 + \sigma(t_1)) = (t_1 + \sigma(t_1))h$. The first case is not possible, because σ is of the second kind and the latter case together with (1), gives us $ht_1 = t_1 h$ for all $t_1 \in \mathscr{R}$ and $h \in \mathscr{J}_H$. Replacing t_1 by t_2, we obtain

$$ht_2 = t_2 h, \quad \text{for all } t_2 \in \mathscr{R} \text{ and } h \in \mathscr{J}_H. \tag{2}$$

Substituting h by $t_1 + \sigma(t_1)$ in (2), we get

$$\{t_1 + \sigma(t_1)\}t_2 = t_2\{t_1 + \sigma(t_1)\} \text{ for all } t_1, t_2 \in \mathscr{R}. \tag{3}$$

Now, we take $s \in \mathscr{J}_S \cap \mathscr{J}_Z$, then $s(t_1 - \sigma(t_1)) \in \mathscr{J}_H$ and using (2), we have $s\{(t_1 - \sigma(t_1))t_2 - t_2(t_1 - \sigma(t_1))\} = 0$ for all $t_1, t_2 \in \mathscr{R}$. Making use of Lemma 2.4, we have either $s = 0$ or $(t_1 - \sigma(t_1))t_2 = t_2(t_1 - \sigma(t_1))$ but the first case is not possible, due to σ is of the second kind and the latter case implies that

$$(t_1 - \sigma(t_1))t_2 = t_2(t_1 - \sigma(t_1)) \text{ for all } t_1, t_2 \in \mathscr{R}. \tag{4}$$

Using (3), together with (4), we get, $t_1 t_2 = t_2 t_1$ for all $t_1, t_2 \in \mathscr{R}$. □

Fact 2.8. *Let $\mathscr{R}$ be a 2-torsion free σ-prime rings with an involution σ which is of the second kind, then σ is centralizing iff $\mathscr{R}$ is commutative.*

Proof. Let

$$[t_1, \sigma(t_1)] \in \mathscr{J}_Z \text{ for all } \quad t_1 \in \mathscr{R}. \tag{5}$$

Linearizing (5), we get

$$[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)] \in \mathscr{J}_Z \text{ for all } \quad t_1, t_2 \in \mathscr{R}.$$

Replacing t_2 by $\sigma(t_2)$, we get

$$[[t_1, t_2], t_1] + [[\sigma(t_2), \sigma(t_1)], t_1] = 0 \text{ for all } t_1, t_2 \in \mathscr{R}. \tag{6}$$

Replacing t_2 by $t_2 t_1$ in (6), we get

$$[[t_1, t_2], t_1]t_1 + \sigma(t_1)[[\sigma(t_2), \sigma(t_1)], t_1] + [\sigma(t_1), t_1][\sigma(t_2), \sigma(t_1)] = 0 \text{ for all } t_1, t_2 \in \mathscr{R}. \tag{7}$$

Combining (6) in (7), we get

$$[[t_1, t_2], t_1]t_1 - \sigma(t_1)[[t_2, t_1], t_1] + [\sigma(t_1), t_1][\sigma(t_2), \sigma(t_1)] = 0 \text{ for all } t_1, t_2 \in \mathscr{R}. \tag{8}$$

Taking $t_2 t_1$ for t_2 in the above equation, we obtain

$$[[t_1, t_2], t_1]t_1^2 - \sigma(t_1)[[t_2, t_1], t_1]t_1 + [\sigma(t_1), t_1]\sigma(t_1)[\sigma(t_2), \sigma(t_1)] = 0 \text{ for all } t_1, t_2 \in \mathscr{R}. \tag{9}$$

Using (8) in (9), and replacing t_1 by $\sigma(t_1)$ and t_2 by $\sigma(t_2)$, we have

$$[t_1, \sigma(t_1)]\{t_1[t_2, t_1] - [t_2, t_1]\sigma(t_1)\} = 0 \text{ for all } t_1, t_2 \in \mathscr{R}. \tag{10}$$

Exchanging t_2 by $t_2 t_1$ in (10), we capture

$$[t_1, \sigma(t_1)]\{t_1[t_2, t_1]t_1 - [t_2, t_1]t_1\sigma(t_1)\} = 0 \text{ for all } t_1, t_2 \in \mathscr{R}. \tag{11}$$

Invoking (10) in (11), we obtain

$$[t_1, \sigma(t_1)][t_2, t_1]\{-t_1\sigma(t_1) + \sigma(t_1)t_1\} = 0 \text{ for all } t_1, t_2 \in \mathscr{R}.$$

The last relation further implies

$$[t_1, \sigma(t_1)]^2 \mathscr{R}[t_2, t_1] = 0 \text{ for all } t_1, t_2 \in \mathscr{R}. \tag{12}$$

Replacing t_1 by $\sigma(t_1)$ and t_2 by $\sigma(t_2)$ in (12), we find

$$[t_1, \sigma(t_1)]^2 \mathscr{R}[t_2, t_1] = 0 = [t_1, \sigma(t_1)]^2 \mathscr{R}\ \sigma\{[t_2, t_1]\}, \text{ for all } t_1, t_2 \in \mathscr{R}.$$

By the definition of σ-prime ring, we get

$$[t_1, \sigma(t_1)]^2 = 0 \text{ or } [t_1, t_2] = 0 \text{for all } t_1, t_2 \in \mathscr{R}.$$

The later case suggests that $\mathscr{R}$ is commutative. The first case implies that

$$[t_1, \sigma(t_1)]^2 = 0 \text{ for all } t_1 \in \mathscr{R}.$$

Since $[t_1, \sigma(t_1)] \in \mathscr{J}_Z \cap \mathscr{J}_H$ and making use of Lemma 2.4, we get

$$[t_1, \sigma(t_1)] = 0 \text{ for all } t_1 \in \mathscr{R}.$$

Using Fact 2.7, $\mathscr{R}$ is commutative. □

FACT 2.9. *Let $\mathscr{R}$ be a 2-torsion free σ-prime ring with an involution σ, which is of the second kind. Then $t_1 \circ \sigma(t_1) \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$ iff $\mathscr{R}$ is commutative.*

Proof. By the given condition, we have

$$t_1 \circ \sigma(t_1) \in \mathscr{J}_Z \quad \text{for all } t_1 \in \mathscr{R}.$$

Linearizing the above relation, we get

$$t_1 \circ \sigma(t_2) + t_2 \circ \sigma(t_1) \in \mathscr{J}_Z \text{ for all } t_1, t_2 \in \mathscr{R}.$$

The last relation further implies that

$$[t_1 \circ \sigma(t_2), r] + [t_2 \circ \sigma(t_1), r] = 0 \quad \text{for all } t_1, t_2, r \in \mathscr{R}. \tag{13}$$

Replacing t_2 by $\sigma(t_2)$ in (13), we find that

$$[t_1 \circ t_2, r] + [\sigma(t_2) \circ \sigma(t_1), r] = 0 \quad \text{for all } t_1, t_2, r \in \mathscr{R}. \tag{14}$$

Taking t_1 in place of t_2 in (14), we grasp

$$[t_1^2, r] + [\sigma(t_1)^2, r] = 0 \text{ for all } t_1, r \in \mathscr{R}. \tag{15}$$

Assuming $t_2 \in \mathscr{J}_Z \setminus \{0\}$ and $t_1 = t_1^2$ in (13), we have

$$[t_1^2, r]t_2 + [\sigma(t_1)^2, r]\sigma(t_2) = 0 \text{ for all } t_1, r \in \mathscr{R}. \tag{16}$$

Making use of (15) and (16), we obtain

$$[t_1^2, r]\{t_2 - \sigma(t_2)\} = 0 \qquad \text{for all } t_1, t_2, r \in \mathscr{R}.$$

Now, $\{t_2 - \sigma(t_2)\} \in \mathscr{J}_S \cap \mathscr{J}_Z$, by using Lemma 3, we have either $[t_1^2, r] = 0$ or $\{t_2 - \sigma(t_2)\} = 0$, the latter case is not possible, because σ is of the second kind, the first case implies that $[t_1^2, r] = 0$ for all $t_1, r \in \mathscr{R}$. So, $t_1^2 \in \mathscr{Z}(\mathscr{R})$ for all $t_1 \in \mathscr{R}$. Invoking Lemma 2.5, $\mathscr{R}$ is commutative. □

FACT 2.10. *Let $\mathscr{R}$ be a 2-torsion free σ-prime ring. If σ is of the second kind involution and $\psi \neq 0$ be a σ-centralizing derivation on $\mathscr{R}$, then $\mathscr{R}$ is commutative.*

Proof. By the given condition

$$[\psi(t_1), \sigma(t_1)] \in \mathscr{J}_Z \quad \text{for all } t_1 \in \mathscr{R}.$$

Replacing t_1 by $t_1 + t_2$, we get

$$[\psi(t_1), \sigma(t_2)] + [\psi(t_2), \sigma(t_1)] \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}. \tag{17}$$

Substituting t_2h in place of t_2 in (17) and using it, where $0 \neq h \in \mathscr{J}_Z \cap \mathscr{J}_H$, we obtain

$$[t_2, \sigma(t_1)]\psi(h) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}.$$

Replacing t_1 with $\sigma(t_1)$ and t_2 by $\sigma(t_2)$, in the above relation we get

$$\sigma([t_2, \sigma(t_1)])\psi(h) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}.$$

By using Lemma 2.2, we have either $[t_2, \sigma(t_1)] \in \mathscr{J}_Z$ or $\psi(h) = 0$, the first case implies the commutativity of $\mathscr{R}$ and the later case implies that $\psi(z) = 0$ for all $z \in \mathscr{J}_Z$. Replacing t_2 by t_2z in (17), where $z \in \mathscr{J}_Z$, we obtain

$$[\psi(t_1), \sigma(t_2)]\sigma(z) + [\psi(t_2), \sigma(t_1)]z \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}. \tag{18}$$

Combining (18) and (17), we obtain

$$[[\psi(t_1)], t_2], r](\sigma(z) - z) = 0 \quad \text{for all } t_1, t_2, r \in \mathscr{R}.$$

Since, $\sigma(z) - z \in \mathscr{J}_Z \cap \mathscr{J}_H$, by Lemma 2.4 we obtain

$$[\psi(t_1)], t_1] \in \mathscr{J}_Z \quad \text{for all } t_1 \in \mathscr{R}.$$

By [13, Theorem 1], $\mathscr{R}$ is commutative. □

Theorem 2.11. *Let $\mathscr{R}$ be a noncommutative σ-prime ring with* $\operatorname{char}(\mathscr{R}) \neq 2$. *If σ is of the second kind and ψ_1, ψ_2 are derivations on $\mathscr{R}$ satisfying $\psi_1(t_1)\sigma(t_1) - \sigma(t_1)\psi_2(t_1) \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$, then $\psi_1 = \psi_2 = 0$.*

Proof. Given that

$$\psi_1(t_1)\sigma(t_1) - \sigma(t_1)\psi_2(t_1) \in \mathscr{J}_Z \quad for \quad all \quad t_1 \in \mathscr{R}. \tag{19}$$

Linearizing the above, we achieve

$$\psi_1(t_1)\sigma(t_2) + \psi_1(t_2)\sigma(t_1) - \sigma(t_1)\psi_2(t_2) - \sigma(t_2)\psi_2(t_1) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}. \tag{20}$$

Replacing t_2 by $\sigma(t_2)$ in (20), we have

$$\psi_1(t_1)t_2 + \psi_1(\sigma(t_2))\sigma(t_1) - \sigma(t_1)\psi_2(\sigma(t_2)) - t_2\psi_2(t_1) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}. \tag{21}$$

Replacing t_2 by t_2h, where $0 \neq h \in \mathscr{J}_Z \cap \mathscr{J}_H$, we receive

$$\begin{aligned}\psi_1(t_1)t_2h + \psi_1(\sigma(t_2))\sigma(t_1)h + \sigma(t_2)\sigma(t_1)\psi_1(h) - \sigma(t_1)\psi_2(\sigma(t_2))h \\ - \sigma(t_1)\sigma(t_2)\psi_2(h) - t_2\psi_2(t_1)h \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}.\end{aligned} \tag{22}$$

Invoking (21) in (22) and using Lemma 2.4, we have

$$\sigma(t_2)\sigma(t_1)\psi_1(h) - \sigma(t_1)\sigma(t_2)\psi_2(h) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}. \tag{23}$$

Substituting t_2 by h, where $0 \neq h \in \mathscr{J}_Z \cap \mathscr{J}_H$, we get

$$h\sigma(t_1)\psi_1(h) - \sigma(t_1)h\psi_2(h) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}.$$

The last relation further implies that

$$h\{\sigma(t_1)\psi_1(h) - \sigma(t_1)\psi_2(h)\} \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}. \tag{24}$$

Replacing t_1 by $\sigma(t_1)$ in (24) and using Lemma 2.4, we obtain

$$t_1\{\psi_1(h) - \psi_2(h)\} \in \mathscr{J}_Z \quad \text{for all } t_1 \in \mathscr{R}.$$

The last relation further implies that

$$[t_1, r]\{\psi_1(h) - \psi_2(h)\} = 0 \quad \text{for all } t_1, r \in \mathscr{R}. \tag{25}$$

Replacing r by ru, where $u \in \mathscr{R}$ and using (25), we have

$$[t_1, r]u\{\psi_1(h) - \psi_2(h)\} = 0 \quad \text{for all } t_1, r, u \in \mathscr{R}.$$

The last relation further implies that

$$[t_1, r] \ \mathscr{R} \ \{\psi_1(h) - \psi_2(h)\} = (0) = \sigma\{[t_1, r]\} \ \mathscr{R} \ \{\psi_1(h) - \psi_2(h)\} \quad \text{for all } t_1, r \in \mathscr{R}.$$

By the definition of σ-prime ring we have either $[t_1, r] = 0$ or $\{\psi_1(h) - \psi_2(h)\} = 0$, the first case implies that the commutative of $\mathscr{R}$ which is not possible by our assumption. The latter case implies that $\psi_1(h) = \psi_2(h)$ and by (23), we have

$$\{\sigma(t_2)\sigma(t_1) - \sigma(t_1)\sigma(t_2)\}\psi_2(h) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}.$$

The last relation further implies

$$[\sigma(t_2), \sigma(t_1)] \quad \psi_2(h) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}.$$

On manipulating the last relation, we obtain

$$[t_2, t_1]\psi_2(h) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}. \tag{26}$$

The above relation further implies that

$$\sigma([t_2, t_1])\psi_2(h) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}.$$

The last relation together with (26) and Lemma 2.2, we have either $\psi_2(h) = 0$ or $[t_2, t_1] \in \mathscr{J}_Z$, for all $t_1, t_2 \in \mathscr{R}$. Replacing t_1 by $\sigma(t_2)$, then $\mathscr{R}$ is commutative by the Fact 2.8, which is not possible by our assumption. The first case implies that $\psi_2(h) = 0 = \psi_1(h)$ for all $h \in \mathscr{J}_Z \cap \mathscr{J}_H$. Replacing t_2 by h in (22), where $h \in \mathscr{J}_Z \cap \mathscr{J}_H$, we obtain

$$\psi_1(t_1) - \psi_2(t_1) \in \mathscr{J}_Z \quad \text{for all } t_1 \in \mathscr{R}.$$

Let assume $\phi(t_1) = \psi_1(t_1) - \psi_2(t_1)$, so $\phi(t_1) \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$, if $\psi_1 \neq \psi_2$ then ϕ is centralizing derivation so, by [13, Theorem 1], $\mathscr{R}$ is commutative, which is not possible by our assumption. Now, if $\psi_1 = \psi_2$, then (19), gives $[\psi_1(t_1), \sigma(t_1)] \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$. Fact 2.10, implies $\psi_1 = 0$ □

Corollary 2.12 ([10, Theorem 1]). *Let $\mathscr{R}$ be a noncommutative prime ring with an involution σ which is of the second kind, with* char$(\mathscr{R}) \neq 2$*, if ψ_1, ψ_2 are derivations of $\mathscr{R}$ satisfying $\psi_1(t_1)\sigma(t_1) - \sigma(t_1)\psi_2(t_1) \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$, then $\psi_1 = \psi_2 = 0$.*

Theorem 2.13. *Let $\mathscr{R}$ be a noncommutative σ-prime rings with an involution σ which is of the second kind with* char$(\mathscr{R}) \neq 2$*, if ψ_1 and ψ_2 are derivations of $\mathscr{R}$ satisfying $\psi_1(\sigma(t_1))t_1 - \sigma(t_1)\psi_2(t_1) \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$, then $\psi_1 = \psi_2 = 0$.*

Proof. Given that

$$\psi_1(\sigma(t_1))t_1 - \sigma(t_1)\psi_2(t_1) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}.$$

Linearizing the above relation, we achieve

$$\psi_1(\sigma(t_1))t_2 + \psi_1(\sigma(t_2))t_1 - \sigma(t_1)\psi_2(t_2) - \sigma(t_2)\psi_2(t_1) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}. \tag{27}$$

Replacing t_2 by t_2h, where $0 \neq h \in \mathscr{J}_Z \cap \mathscr{J}_H$, we receive

$$\psi_1(\sigma(t_1))t_2h - \sigma(t_1)\psi_2(t_2)h - \sigma(t_1)t_2\psi_2(h) + \psi_1(\sigma(t_2))t_1h$$

$$+\,\sigma(t_2)t_1\psi_1(h)+\sigma(t_2)\psi_2(t_1)h\in\mathscr{J}_Z \quad \text{for all } t_1,t_2\in\mathscr{R}. \tag{28}$$

Using (27) in (28), we obtain

$$\sigma(t_2)t_1\psi_1(h)-\sigma(t_1)t_2\psi_2(h)\in\mathscr{J}_Z \quad \text{for all } t_1,t_2\in\mathscr{R}. \tag{29}$$

Taking $t_2=t_1$ in (29), we gain

$$\sigma(t_1)t_1\{\psi_1(h)-\psi_2(h)\}\in\mathscr{J}_Z \quad \text{for all } t_1\in\mathscr{R}.$$

The last relation further implies that

$$[\sigma(t_1)t_1,r]\ \{\psi_1(h)-\psi_2(h)\}=0 \quad \text{for all } t_1\in\mathscr{R}. \tag{30}$$

Replacing r by ru, where $u\in\mathscr{R}$ and using (30), we have

$$[\sigma(t_1)t_1,r]\mathscr{R}\{\psi_1(h)-\psi_2(h)\}=(0)=\sigma([\sigma(t_1)t_1,r])\mathscr{R}\{\psi_1(h)-\psi_2(h)\} \quad \text{for all } t_1\in\mathscr{R}.$$

By the definition of σ-prime rings we have, either $\sigma(t_1)t_1\in\mathscr{J}_Z$ or $\psi_1(h)=\psi_2(h)$, the first case implies that the commutativity of $\mathscr{R}$, which is not possible by our assumption. The later case together with (29), gives us

$$\{\sigma(t_2)t_1-\sigma(t_1)t_2\}\psi_1(h)\in\mathscr{J}_Z \quad \text{for all } t_1,t_2\in\mathscr{R}. \tag{31}$$

The formal relation further implies that

$$\sigma(\sigma(t_2)t_1-\sigma(t_1)t_2)\psi_1(h)\in\mathscr{J}_Z \quad \text{for all } t_1,t_2\in\mathscr{R}.$$

The previous relation together with (31) and Lemma 2.2, we have either $\sigma(t_2)t_1-\sigma(t_1)t_2\in\mathscr{J}_Z$ for all $t_1,t_2\in\mathscr{R}$ or $\psi_1(h)=0$ for all $h\in\mathscr{J}_Z\cap\mathscr{J}_H$. The initial case implies that

$$\sigma(t_2)t_1-\sigma(t_1)t_2\in\mathscr{J}_Z \quad \text{for all } t_1,t_2\in\mathscr{R}. \tag{32}$$

Taking t_1s in place of t_1, where $0\neq s\in\mathscr{J}_Z\cap\mathscr{J}_S$, making use of Lemma 2.4, we obtain

$$\sigma(t_2)t_1+\sigma(t_1)t_2\in\mathscr{J}_Z \quad \text{for all } t_1,t_2\in\mathscr{R}. \tag{33}$$

Combining (32) and (33) and using $\text{char}(\mathscr{R})\neq 2$, we obtain

$$\sigma(t_2)t_1\in\mathscr{J}_Z \quad \text{for all } t_1,t_2\in\mathscr{R}.$$

Replacing t_2 by t_1, we achieve

$$\sigma(t_1)t_1\in\mathscr{J}_Z \quad \text{for all } t_1\in\mathscr{R}. \tag{34}$$

Replacing t_1 by $\sigma(t_1)$, we get

$$t_1\sigma(t_1)\in\mathscr{J}_Z \quad \text{for all } t_1\in\mathscr{R}. \tag{35}$$

Combining (35) and (34), we obtain

$$[\sigma(t_1),t_1]\in\mathscr{J}_Z \quad \text{for all } t_1\in\mathscr{R}.$$

By Fact 2.8, $\mathscr{R}$ is commutative, which is not true by our assumption. Now, if $\psi_1(h)=0$ for all $h\in\mathscr{J}_Z\cap\mathscr{J}_H$, then $\psi_2(h)=0$, Lemma 2.6 implies $\psi_1(z)=\psi_2(z)=0$, for all $z\in\mathscr{J}_Z$, replacing t_2 by h in (27), where $h\in\mathscr{J}_Z\cap\mathscr{J}_H$, we obtain

$$\psi_1(\sigma(t_1))-\psi_2(t_1)\in\mathscr{J}_Z \quad \text{for all } t_1\in\mathscr{R}. \tag{36}$$

Replacing t_1 by s in (27), where $s\in\mathscr{J}_Z\cap\mathscr{J}_S$, we obtain

$$\psi_1(\sigma(t_1))+\psi_2(t_1)\in\mathscr{J}_Z \quad \text{for all} \quad t_1\in\mathscr{R}. \tag{37}$$

The last relation together with (36), implies that $\psi_1(t_1)\in\mathscr{J}_Z$ for all $t_1\in\mathscr{R}$. By [13, Theorem 1], $\psi_1 = 0$, so (37), implies $\psi_2(t_1) \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$, so by same result $\psi_2 = 0$. □

Corollary 2.14 ([10, Theorem 2]). *Let $\mathscr{R}$ be a noncommutative prime rings with an involution σ which is of the second kind with* $\mathrm{char}(\mathscr{R}) \neq 2$, *if ψ_1 and ψ_2 are derivations of $\mathscr{R}$ satisfying $\psi_1(\sigma(t_1))t_1 - \sigma(t_1)\psi_2(t_1) \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$, then $\psi_1 = \psi_2 = 0$.*

Theorem 2.15. *Let $\mathscr{R}$ be a σ-prime rings with an involution σ which is of the second kind with* $\mathrm{char}(\mathscr{R}) \neq 2$, *if ψ_1 and ψ_2 are derivations on $\mathscr{R}$ such that $\psi_1\sigma = \sigma\psi_1$, or ($\psi_2\sigma = \sigma\psi_2$), then following assertions are equivalent:*

(i) $\psi_1(t_1) \circ \psi_2(\sigma(t_1)) - t_1 \circ \sigma(t_1) \in \mathscr{J}_Z$ *for all* $t_1 \in \mathscr{R}$.

(ii) $\psi_1(t_1) \circ \psi_2(\sigma(t_1)) + t_1 \circ \sigma(t_1) \in \mathscr{J}_Z$ *for all* $t_1 \in \mathscr{R}$.

(iii) $[\psi_1(t_1), \psi_2(\sigma(t_1))] - t_1 \circ \sigma(t_1) \in \mathscr{J}_Z$ *for all* $t_1 \in \mathscr{R}$.

(iv) $[\psi_1(t_1), \psi_2(\sigma(t_1))] + t_1 \circ \sigma(t_1) \in \mathscr{J}_Z$ *for all* $t_1 \in \mathscr{R}$.

(v) $\mathscr{R}$ *is commutative.*

Proof. Clearly (v) $\Longrightarrow$ (i)–(iv).

If $\psi_1 = 0$ or $\psi_2 = 0$, then the above relation reduces to $t_1 \circ \sigma(t_1) \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$. Then $\mathscr{R}$ is commutative by Fact 2.9.

Now, we assume $\psi_1 \neq 0$ and $\psi_2 \neq 0$.

(i) $\Longrightarrow$ (v) Given that

$$\psi_1(t_1) \circ \psi_2(\sigma(t_1)) - t_1 \circ \sigma(t_1) \in \mathscr{J}_Z \quad \text{for all } t_1 \in \mathscr{R}.$$

Linearizing the above equation, we receive

$$\psi_1(t_1) \circ \psi_2(\sigma(t_2)) + \psi_1(t_2) \circ \psi_2(\sigma(t_1)) - t_1 \circ \sigma(t_2) - t_2 \circ \sigma(t_1) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}.$$

Replacing t_2 by t_2h, where $0 \neq h \in \mathscr{J}_Z \cap \mathscr{J}_H$, in the above equation and using it, we get

$$\{\psi_1(t_1) \circ \sigma(t_2)\}\psi_2(h) + \{t_2 \circ \psi_2(\sigma(t_1))\}\psi_1(h) \in \mathscr{J}_Z \quad \text{for all } t_1, t_2 \in \mathscr{R}. \tag{38}$$

Putting h in the place of t_2 where; $0 \neq h \in \mathscr{J}_Z \cap \mathscr{J}_H$, in the above relation and using Lemma 2.4, we get

$$\psi_1(t_1)\psi_2(h) + \psi_2(\sigma(t_1))\psi_1(h) \in \mathscr{J}_Z \quad \text{for all } t_1 \in \mathscr{R}. \tag{39}$$

Putting s in the place of t_2 in (38), where, $0 \neq s \in \mathscr{J}_Z \cap \mathscr{J}_S$, and using Lemma 2.4, we get

$$-\psi_1(t_1)\psi_2(h) + \psi_2(\sigma(t_1))\psi_1(h) \in \mathscr{J}_Z \quad \text{for all } t_1 \in \mathscr{R}. \tag{40}$$

Combining (39) and (40) and using $\mathrm{char}(\mathscr{R}) \neq 2$, we obtain

$$\psi_2(t_1)\psi_1(h) \in \mathscr{J}_Z \quad \text{for all } t_1 \in \mathscr{R}.$$

The previous relation further implies that

$$\{[\psi_2(t_1), r]\}\psi_1(h) = 0 \quad \text{for all } t_1, r \in \mathscr{R}.$$

Since, σ commutes with ψ_1, then $\psi_1(h) \in \mathscr{J}_Z \cap \mathscr{J}_H$, so by Lemma 2.4, we have either $\psi_1(h) = 0$ or $[\psi_2(t_1), r] = 0$. The first case is not possible because σ is of the second kind, the later case implies that $[\psi_2(t_1), r] = 0$ for all $t_1, r \in \mathscr{R}$. In particular, taking $r = t_1$, we have $[\psi_2(t_1), t_1] = 0$ for all $t_1 \in \mathscr{R}$. By Fact 2.10, $\mathscr{R}$ is commutative.

(ii) $\Longrightarrow$ (v) Given that

$$\psi_1(t_1)\circ\psi_2(\sigma(t_1))+t_1\circ\sigma(t_1)\in\mathscr{J}_Z\quad\text{for all } t_1\in\mathscr{R}.$$

Linearizing the above equation, we obtain

$$\psi_1(t_1)\circ\psi_2(\sigma(t_2))+\psi_1(t_2)\circ\psi_2(\sigma(t_1))+t_1\circ\sigma(t_2)+t_2\circ\sigma(t_1)\in\mathscr{J}_Z\quad\text{for all } t_1,t_2\in\mathscr{R}.$$

Replacing t_2 by t_2h, where $0\neq h\in\mathscr{J}_Z\cap\mathscr{J}_H$, in the above equation, we get

$$\{\psi_1(t_1)\circ\sigma(t_2)\}\psi_2(h)+\{t_2\circ\psi_2(\sigma(t_1))\}\psi_1(h)\in\mathscr{J}_Z\quad\text{for all } t_1,t_2\in\mathscr{R}.$$

The above equation is same as (38), so by the same argument $\mathscr{R}$ is commutative.

(iii) $\Longrightarrow$ (v) Given that

$$[\psi_1(t_1),\ \psi_2(\sigma(t_1))]-t_1\circ\sigma(t_1)\in\mathscr{J}_Z\quad\text{for all } t_1\in\mathscr{R}.$$

Taking $t_1=t_1+t_2$ in the above relation, we obtain

$$[\psi_1(t_1),\psi_2(\sigma(t_2))]+[\psi_1(t_2),\psi_2(\sigma(t_1))]$$
$$-t_1\circ\sigma(t_2)-t_2\circ\sigma(t_1)\in\mathscr{J}_Z\quad\text{for all } t_1,t_2\in\mathscr{R}.\tag{41}$$

Replacing t_2 by t_2h in (41) and using it, where $0\neq h\in\mathscr{J}_Z\cap\mathscr{J}_H$, we gain

$$[\psi_1(t_1),\ \sigma(t_2)]\ \psi_2(h)+[t_2,\psi_2(\sigma(t_1))]\psi_1(h)\in\mathscr{J}_Z\quad\text{for all } t_1,t_2\in\mathscr{R}.\tag{42}$$

Substituting t_2 by t_2s in (42), where $0\neq s\in\mathscr{J}_Z\cap\mathscr{J}_S$, we obtain

$$-[\psi_1(t_1),\sigma(t_2)]\ \psi_2(h)+[t_2,\psi_2(\sigma(t_1))]\psi_1(h)\in\mathscr{J}_Z\quad\text{for all } t_1,t_2\in\mathscr{R}.\tag{43}$$

By combining (42) and (43), we achieve

$$[\psi_1(t_1),\sigma(t_2)]\psi_2(h)\in\mathscr{J}_Z\quad\text{for all } t_1,t_2\in\mathscr{R}.$$

The previous relation further implies that

$$[[\psi_1(t_1),\sigma(t_2)],r]\psi_2(h)=0\quad\text{for all } t_1,t_2,r\in\mathscr{R}.$$

Since, σ commutes with ψ_2, then $\psi_2(h)\in\mathscr{J}_Z\cap\mathscr{J}_H$, so by Lemma 2.4, we have either $\psi_2(h)=0$ or $[[\psi_1(t_1),\ \sigma(t_2)],r]=0$. The first case is not possible because σ is of the second kind, the latter case implies that $[\psi_1(t_1),\sigma(t_2)]\in\mathscr{J}_Z$ for all $t_1,t_2\in\mathscr{R}$. In particular, taking $t_2=\sigma(t_1)$, we have $[\psi_1(t_1),\ t_1]\in\mathscr{J}_Z$ for all $t_1\in\mathscr{R}$. By Fact 2.10, $\mathscr{R}$ is commutative.

(iv) $\Longrightarrow$ (v) Given that

$$[\psi_1(t_1),\psi_2(\sigma(t_1))]+t_1\circ\sigma(t_1)\in\mathscr{J}_Z\quad\text{for all } t_1\in\mathscr{R}.$$

Taking $t_1=t_1+t_2$ in the above relation, we obtain

$$[\psi_1(t_1),\psi_2(\sigma(t_2))]+[\psi_1(t_2),\psi_2(\sigma(t_1))]+t_1\circ\sigma(t_2)+t_2\circ\sigma(t_1)\in\mathscr{J}_Z\quad\text{for all } t_1,t_2\in\mathscr{R}.$$

Replacing t_2 by t_2h in the above equation, where $0\neq h\in\mathscr{J}_Z\cap\mathscr{J}_H$, we get

$$\{\psi_1(t_1)\circ\sigma(t_2)\}\psi_2(h)+\{t_2\circ\psi_2(\sigma(t_1))\}\psi_1(h)\in\mathscr{J}_Z\quad\text{for all } t_1,t_2\in\mathscr{R}.$$

The above equation is same as (42), so by the same argument $\mathscr{R}$ is commutative. □

Corollary 2.16. *Let $\mathscr{R}$ be a σ-prime rings with an involution σ which is of the second kind, with $char(\mathscr{R})\neq 2$, if ψ_1 and ψ_2 are derivations on $\mathscr{R}$ such that $\psi_1\sigma=\sigma\psi_1$, or ($\psi_2\sigma=\sigma\psi_2$), then following assertions are equivalent:*
(i) $\psi_1(t_1)\circ\psi_2(t_2)-t_1\circ t_2\in\mathscr{J}_Z$ *for all* $t_1,t_2\in\mathscr{R}$.

(ii) $\psi_1(t_1)\circ\psi_2(t_1)+t_1\circ t_2\in\mathscr{J}_Z$ *for all* $t_1,t_2\in\mathscr{R}$.

(iii) $[\psi_1(t_1), \psi_2(t_2)] - t_1 \circ t_2 \in \mathscr{J}_Z$ *for all* $t_1, t_2 \in \mathscr{R}$.

(iv) $[\psi_1(t_1), \psi_2(t_2)] + t_1 \circ (t_2) \in \mathscr{J}_Z$ *for all* $t_1, t_2 \in \mathscr{R}$.

(v) $\mathscr{R}$ *is commutative.*

As it is well-known that the zero-divisor is impossible in the center of a prime ring, but in σ-prime rings center is not free from zero divisor. The follwing example explain that the above fact.

Example 2.17. Consider $\mathscr{R} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Big| a, b \in \mathbb{Z} \right\}$, define σ in such away, $\sigma\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$. It is easy to verify that $\mathscr{R}$ is a σ-prime ring with an involution σ. For any non zero a, $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathscr{J}_Z$, and for any nonzero b, $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in \mathscr{R}$ and $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This shows the fact.

The following examples show that the second kind is necessary in Theorem 2.15.

Example 2.18. Consider $\mathscr{R} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big| a, b, c, d \in \mathbb{Z} \right\}$. Define σ in such a way that $\sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. It is easy to verify that $\mathscr{R}$ is a σ-prime ring with an involution σ of the first kind.

Moreover, define ψ_1 and ψ_2 by $\psi_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$ and $\psi_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$. Here, ψ_1 and ψ_2 satisfy the condition $\psi_1(t_1) \circ \psi_2(\sigma(t_1)) - t_1 \circ \sigma(t_1) \in \mathscr{J}_Z$ for all $t_1 \in \mathscr{R}$. However, $\mathscr{R}$ is noncommutative.

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