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GENERALIZATION OF TWO THEOREMS OF STEINHAUS IN CATEGORY BASES

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Abstract. Here we unify two results of Steinhaus and their corresponding category analogues by extending them in the settings of category bases. We further show that in any perfect translation base, every abundant Baire set contains a full subset for which the thesis of the second theorem fails.

1. Introduction

In [10], Steinhaus proved that the distance set $D(A, B)$ of two sets A and B of positive Lebesgue measure contains an interval, where by distance set, we mean the set of all mutual distances between points of A and B. In the same paper, while dealing with subsets of the real line $\mathbb{R}$, he proved even stronger theorems:

THEOREM 1.1. *If $A_n n = 1^\infty$ is any infinite sequence of Lebesgue measurable sets with positive measures, then there exists an infinite sequence $a_n n = 1^\infty$ of distinct points such that $a_n \in A_n$ for all $n = 1, 2, 3, \dots$ and their mutual distances are all rational.*

THEOREM 1.2. *If E is an infinite Lebesgue measurable set, then there exists an enumerable set P composed of points whose mutual distances are all rational numbers, and a set Z of measure zero such that $P \subseteq E \subseteq P' \cup Z$, where P' represents the derived set of P.*

A set A has the Baire property [7] if it can be expressed as the symmetric difference $G \Delta P$ of an open set G and a set P of first category. Equivalently, $A = (G \backslash P) \cup Q$, where G is open and P, Q are sets of first category. Steinhaus's theorem on the distance set has an exact category analogue in the realm of Baire category, which is due to Picard [8]. He showed that if A and B are second category subsets of $\mathbb{R}$ with the Baire property, then their distance set contains an interval. Picard's theorem has

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been generalized by K. P. S. and M. Bhaskara Rao [1] for sets in topological groups, by Kominek [3] in topological vector spaces, and by Sander [9] in topological spaces with respect to the class of globally solvable mappings. The above two theorems (Theorem 1.1 and Theorem 1.2) were also given more general forms. Alongside the measure-theoretic results, these were set forth by Miller, Xenikakis, and Polychronis [4] for sets with the Baire property in the real line, in the light of certain specified classes of mappings $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, each of which has first-order continuous partial derivatives f_x and f_y non-vanishing on an open set containing $A \times A$, where A is the union of sets in the sequence $A_{n}{}_{n=1}^{\infty}$.

In this paper, we prove two results, each of which unifies Theorem 1.1 and Theorem 1.2 with their corresponding category analogues in point-meager Baire category bases having a countable pseudo-base. Similar types of unifications were earlier established by Morgan [5] for perfect, translation category bases in the real line, but here the results are based on a different approach.

2. Preliminaries and results

All the definitions stated below are taken from [5].

Let X be a nonempty set and $\mathcal{C} \subseteq P(X)$ a nonempty family of sets. Let us call the nonempty members of $\mathcal{C}$ regions.

Definition 2.1. We call a pair $(X, \mathcal{C})$ a category base, if
(i) Every point of X belongs to some region; i.e., $X = \bigcup \mathcal{C}$.

(ii) Let A be a region and $\mathcal{D}$ be any non-empty family of disjont regions having cardinality less than the cardinality of $\mathcal{C}$. Then

(a) There exists a region $D \in \mathcal{D}$ such that $A \cap D$ contains a region provided $A \cap (\bigcup \mathcal{D})$ contains a region.

(b) There exists a region $B \subseteq A$ which is disjoint from every region in $\mathcal{D}$ provided $A \cap (\bigcup \mathcal{D})$ contains no region.

Definition 2.2. In a category base $(X, \mathcal{C})$, A set A is called
(i) *singular*, if every region contains a region disjoint from A.

(ii) *meager*, if A can be expressed as countable union of singular sets and co-meager, if the complement of A is meager.

(iii) *abundant*, if A is not meager.

(iv) *abundant everywhere*, if A is abundant in every region. In particular, A is abundant everywhere in a region if A is abundant in every subregion of that region.

(v) *Baire*, if every region contains a subregion in which either A or its complement (in X) is meager.

A category base $(X,\mathcal{C})$ is called
(i) *point-meager*, if every singleton set in it is a singular set.

(ii) *a Baire base*, if every region in it is abundant.

(iii) *separable*, if there is a countable subfamily of regions such that every region is abundant in at least one region from the subfamily.

(iv) *a pseudobase*, if there is a subfamily of regions such that every region contains at least one region from the subfamily.

However, having a countable pseudobase is a condition stronger than separability.

DEFINITION 2.3. Any pair $(X,\mathcal{J})$ (or, briefly $\mathcal{J}$) is called a complete family if there exists a sequence $\{\Psi_n\}_{n=1}^{\infty}$, $\Psi_n : \mathcal{J} \mapsto \mathcal{J}$ satisfying
(i) $\Psi_n[A] \subseteq A$ for every $A \in \mathcal{J}$ and $n = 1, 2, 3, \ldots$.

(ii) $\bigcap\limits_{n=1}^{\infty} A_n \neq \phi$ for every sequence $\{A_n\}_{n=1}^{\infty}$ $(A_n \in \mathcal{J})$ for which the corresponding sequence $\{\Psi_n[A_n]\}_{n=1}^{\infty}$ is descending.

For any mapping $f : X \times X \mapsto X$, Let $f_x(y) = f(x, y)$ $(y \in X)$ and $f^y(x) = f(x, y)$ $(x \in X)$ represent the x-section and y-section of f. In the following theorem, the notion of a dense set has been used in the following manner. In a category base $(X,\mathcal{C})$, we call a set dense if it has nonempty intersection with every region.

We now state our results.

THEOREM 2.4. *Let $(X,\mathcal{C})$ be a point-meager, Baire base in which the intersection of no two regions (in case it is nonempty) is a singular set, Moreover, let $\mathcal{C}$ be a complete family and $f : X \times X \mapsto X$ be a mapping such that:*
(i) f_x is one-to-one and $\{f^y[C] : C \in \mathcal{C}\} = \mathcal{C}$ for every $x, y \in X$.

(ii) Given D as a dense subset of X, there exists $\eta \in D$ such that $A \cap f^{\eta}[B] \neq \phi$ whenever $A, B \in \mathcal{C}\backslash\{\phi\}$.
Then, given any sequence $\{A_n\}_{n=1}^{\infty}$ of abundant Baire sets in $(X,\mathcal{C})$, there exist sequences $\{a_n\}_{n=1}^{\infty}(a_n \in A_n)$ and $\{\eta_n\}_{n=1}^{\infty}(\eta_n \in D)$ of distinct points satisfying the identity $f(a_1, \eta_1) = f(a_2, \eta_2) = \ldots = f(a_n, \eta_n) = \ldots$.

Proof. In the given category base $(X,\mathcal{C})$, we choose regions $C_0, C_1, C_2, \ldots, C_n, \ldots$ such that C_0 is arbitrary but fixed and $C_n\backslash A_n$ is meager for $n = 1, 2, 3, \ldots$. As every abundant Baire set is abundant everywhere in some region and each A_n is an abundant Baire set, this choice is justified by [5, Chapter 1, Section III, Theorem 2]. Let $C_n\backslash A_n = \bigcup\limits_{i=1}^{\infty} F_i^n$ where F_i^n are singular sets $(n, i = 1, 2, \ldots)$.

We now choose the region $D_1^1 \subseteq C_1\backslash F_1^1$ and $\eta_1 \in D$ (by (ii)) such that $C_0 \cap f^{\eta_1}[D_1^1] \neq \emptyset$. Since, by assumption, the intersection of no two regions is singular, and because $\mathcal{C}$ is a complete family, there exists a region H_1 such that $\Psi_1[H_1] \subseteq H_1 \subseteq C_0 \cap f^{\eta_1}[D_1^1]$. Using conditions (i) and (ii) once again, and also because $\mathcal{C}$ is a complete family, we may further choose regions D_1^2, D_2^2, H_2, and a point $\eta_2 \in D$ such that the inclusion relations hold:

$$D_1^2 \subseteq (f^{\eta_1})^{-1}[\Psi_1[H_1]]\backslash F_2^1,$$
$$D_2^2 \subseteq C_2\backslash(F_1^2 \cup F_2^2).$$

and
$$\Psi_2[H_2] \subseteq H_2 \subseteq f^{\eta_1}[D_1^2] \cap f^{\eta_2}[D_2^2].$$

In a similar manner in general, in the $(n+1)$th step, we find regions D_k^{n+1} ($k = 1, 2, \ldots, n+1$), H_{n+1} and a point $\eta_{n+1} \in D$ for which the following relations are satisfied:

$$D_1^{n+1} \subseteq (f^{\eta_1})^{-1}[\Psi_1[H_n]]\backslash F_{n+1}^1,$$
$$D_k^{n+1} \subseteq (f^{\eta_k})^{-1}[f^{\eta_{k-1}}[D_{k-1}^{n+1}]]\backslash F_{n+1}^k,$$
$$\vdots$$
$$D_{n+1}^{n+1} \subseteq C_{n+1}\backslash \bigcup_{k=1}^{n+1} F_k^{n+1}.$$

and
$$\Psi_{n+1}[H_{n+1}] \subseteq H_{n+1} \subseteq f^{\eta_n}[D_n^{n+1}] \cap f^{\eta_{n+1}}[D_{n+1}^{n+1}].$$

As our category base is a point-meager Baire base, we may choose the sequence $\{\eta_n\}_{n=1}^{\infty}$ in a manner that all its members are distinct. It is easy to check that the sequence $\{\Psi_n[H_n]\}_{n=1}^{\infty}$ is decreasing and so $\bigcap\limits_{n=1}^{\infty} H_n \neq \phi$ because $\mathcal{C}$ is complete. But

$$\bigcap_{n=1}^{\infty} H_n \subseteq f^{\eta_1}(A_1) \cap f^{\eta_2}(A_2) \cap \ldots \cap f^{\eta_n}(A_n) \cap \ldots$$

Hence there exists a sequence $\{a_n\}_{n=1}^{\infty}$ ($a_n \in A_n$) made up of distinct points (because f_x is one-to-one) such that $f(a_1, \eta_1) = f(a_2, \eta_2) = \ldots = f(a_n, \eta_n) = \ldots$. □

In the following theorem, the notion of cluster point of a sequence has been used in the following sense. In a category base $(X, \mathcal{C})$, a point a is a cluster point of $E = \{a_n\}_{n=1}^{\infty}$ if for every region C containg a and for every natural number m, there exists a natural number $p > m$ such that $a_p \in C$.

Theorem 2.5. *Let $(X, \mathcal{C})$ be a point meager, Baire base in which the conditions stated in the above theorem are satisfied. Moreover, let $(X, \mathcal{C})$ has a countable pseudobase. Then given any abundant Baire set A, there exists*
(i) a countably infinite subset $P = \{a_1, a_2, \ldots, a_n, \ldots\}$ of A and an infinite sequence $\{\eta_n\}_{n=1}^{\infty}$ of distinct terms satisfying $f(a_1, \eta_1) = f(a_2, \eta_2) = \ldots = f(a_n, \eta_n) = \ldots$ and
(ii) a meager set H such that $P \subseteq A \subseteq P' \cup H$, where P' denotes the set of all cluster points of P.

Proof. Let $\mathcal{B}$ be a countable pseudobase. Since any category base possessing a countable pseudobase is separable, it is easy to check that $X\backslash \bigcup \mathcal{B}$ is meager. We set $A^* = A \cap (\bigcup \mathcal{B})$ and $A_n = A^* \cap B_n$ (n=1,2,3,...) where $\mathcal{B} = \{B_1, B_2, \ldots, B_n, \ldots\}$. We may assume that no A_n is meager for otherwise, we may remove it without affecting the conclusions. By Theorem 2.4, there exist infinite sequences $\{a_n\}_{n=1}^{\infty}$ ($a_n \in A_n$) and $\{\eta_n\}_{n=1}^{\infty}$ ($\eta_n \in D$) of distinct terms satisfying $f(a_1, \eta_1) = f(a_2, \eta_2) = \ldots = f(a_n, \eta_n) = \ldots$. This proves (i).

We set $P = \{a_1, a_2, \ldots, a_n, \ldots\}$ and claim that every point of A^* is a cluster point of P. For any $x \in A^*$, we choose arbitrarily a region C containing x and a natural number m. Let k_1 be the least natural number such that $B_{k_1} \subseteq C$. If $k_1 > m$, we put $p = k_1$. Then $p > m$ and $a_p \in A^* \cap B_p \subseteq C$. Otherwise, if $k_1 \leq m$, we carry out the following procedure: choose $B_{k_2} \subseteq B_{k_1} \backslash \{a_{k_1}\}$ and select $a_{k_2} \in A^* \cap B_{k_2}$. Again choose $B_{k_3} \subseteq B_{k_2} \backslash \{a_{k_2}\}$ and select $a_{k_3} \in A^* \cap B_{k_3}$. Since $(X, \mathcal{C})$ is point-meager Baire base, we may continue in this manner to obtain by Archimedean property a natural number k_j such that $k_j > m$. We put $k_j = p$ so that $a_p \in A^* \cap B_p \subseteq C$. This serves our purpose. Finally, setting $H = A \cap (X \backslash \bigcup \mathcal{B})$ we get $P \subseteq A \subseteq P' \cup H$ which proves (ii). □

Earlier, Morgan [5, Chapter 6, Section II, Theorem 3, Theorem 5] unified the two theorems of Steinhaus (stated in the introduction) and their category analogues in perfect translation bases. From the fact that f in Theorem 2.4 and Theorem 2.5 may be chosen, in particular, as the function $(x, y) \mapsto x + y$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, one may observe that these theorems resemble those of Morgan in the sense that they also unify the two theorems under a different approach.

Following Grzegorek and Labuda [2], in any category base, we say that a set F is a full subset of a set E if $F \subseteq E$ and, for any Baire set B, $F \cap B$ is nonmeager whenever $E \cap B$ is nonmeager. If E is a Baire set, this is equivalent to stating that $E \backslash F$ cannot contain any abundant Baire set. It is interesting to note that in a perfect translation base, given any abundant Baire set B, there exists a full subset of B for which the thesis of Theorem 2.5 fails. To prove this, we proceed as follows.

First, we construct a Vitali set V in $\mathbb{R}$, which is also Bernstein in nature [6]. Since $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} (V + r)$ (where $\mathbb{Q}$ is the set of rationals), for some $r_0 \in \mathbb{Q}$, $(V + r_0) \cap B$ is abundant. Moreover, $(V + r_0) \cap B$ is a full subset of B, because, if not, it would be possible to fit an abundant Baire set in B that is disjoint from $(V + r_0) \cap B$. However, this is impossible because every abundant Baire set in any perfect base contains a perfect set [6, Chapter 5, Section II, Theorem 8], and V is Bernstein. Furthermore, Theorem 2.5 fails for the set $(V + r_0) \cap B$ because it is a subset of a Vitali set, and therefore no two distinct points in it can be at a rational distance from one another.

Now, $(V + r_0) \cap B$ is also a non-Baire set. Otherwise, by [5, Chapter 6, Section II, Theorem 3, Theorem 5], $(V + r_0) \cap B$ would not be an ARD set [6]. But the non-Baireness of $(V + r_0) \cap B$ can also be derived without using Morgan's theorems. In fact, we can prove more: no abundant subset of $(V + r_0) \cap B$ can be Baire. Suppose, for the sake of contradiction, that E is an abundant Baire set contained in $(V + r_0) \cap B$. Choose a fixed $r' \in \mathbb{Q}$. Since $\mathbb{Q} \backslash r'$ is dense, by [5, Chapter 6, Section I, Theorem 3], $\bigcup_{r \in \mathbb{Q} \backslash r'} (E + r)$ is co-meager. Consequently, the set $E + r'$ – which is disjoint from $\bigcup_{r \in \mathbb{Q} \backslash r'} (E + r)$ – is meager, which is a contradiction.

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