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FIXED POINT RESULTS WITH RATE OF CONVERGENCE AND ERROR ESTIMATION

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Abstract. In this paper, we present some fixed point results metric spaces under certain admissibility conditions. A number of consequences and an illustration of the results are also discussed herein. Further, we present error estimation and rate of convergence of the fixed point iterations.

1. Introduction

The purpose of this paper is to establish new fixed point results using various ideas prevalent in the field of metric fixed point theory. Below we briefly mention the ideas that we have put together to obtain our results.

While going through the proofs of fixed point theorems of several contractive mappings, it is an interesting observation that the contraction condition is not used for every pair of points from the metric space. Thus, the contractive condition could be restricted to certain pairs without disturbing the proof. Such restrictions were made in two ways. One way is to introduce orderings such as partial orderings, graphs, etc. on the metric space and then assert that the contraction holds only for pairs related by the ordering, while the other way is to introduce admissibility conditions, which are some additional functional requirements. Admissibility conditions were introduced by Samet et al [11]. The same idea with many variations has been used in several papers [1, 12].

In another approach, the contractive conditions were relaxed by introducing weak inequalities. Originally this was done in metric spaces for Banach contractions in the work of Rhoades [10]. This led to the use of a type of inequalities known as weak

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inequalities, which could be used to establish new fixed point theorems in works such as [5, 9].

The use of rational terms in metric inequalities was initiated by Dass et al [4]. In later works the rational expressions were considered by many researchers. As a result, several new and important fixed point results could be established in metric fixed point theory [2, 8].

In this paper, we combine the above three trends in metric fixed point theory to obtain new results. In particular, we exhibit the following features in our theorems.

(a) We use certain admissibility conditions.

(b) We use three control functions to construct a weak contraction inequality.

(c) We use a conditional rational expression in the weak inequality we consider in our theorem.

We first prove the existence of the fixed point under certain conditions, and the uniqueness of the fixed point is proved by assuming an additional condition. There are several corollaries and illustrations. Our result extends some of the existing results. Finally, we give a discussion on the error and the rate of convergence of the fixed point iteration.

The conditional rational expression we use is a modification of the expression used by Fisher [6], which was proposed in a correction of the result of Khan [7]. The result of Khan [7] has many generalizations in works such as [8, 13].

2. Mathematical background

Definition 2.1. Let X be a nonempty set and $T : X \to X$. An element $x \in X$ is called a fixed point of T if $x = Tx$.

Definition 2.2. Let X be a nonempty set, $\alpha, \nu : X \times X \to [0, +\infty)$ be two mappings. A mapping $T : X \to X$ is said to be $(\alpha{-}\nu)$-dominated if $\alpha(x, Tx) \geq \nu(x, Tx)$ for all $x \in X$.

Definition 2.3. Let (X, d) be a metric space and $\alpha, \nu : X \times X \to [0, +\infty)$. Then X is said to have $(\alpha{-}\nu)$-regular property if for every sequence $\{x_n\}$ in X converging to $x \in X, \alpha(x_n, x_{n+1}) \geq \nu(x_n, x_{n+1}), \quad \text{for all } n$ implies $\alpha(x_n, x) \geq \nu(x_n, x), \quad \text{for all } n.$

Definition 2.4. Let (X, d) be a metric space and $\alpha, \nu : X \times X \to [0, +\infty)$. Then X is said to have $(\alpha{-}\nu)$-transitive property if $\alpha(x, y) \geq \nu(x, y)$ and $\alpha(y, z) \geq \nu(y, z)$ imply $\alpha(x, z) \geq \nu(x, z)$ for $x, y, z \in X$.

The above definitions are illustrated though the following example.

Example 2.5. Let $X = [0, 1]$ be equipped with the usual metric. Let $T : X \to X$ and $\alpha, \nu : X \times X \to [0, +\infty)$ be respectively defined as follows:

$$T(x) = \frac{\cos x}{16}, \quad \alpha(x, y) = e^{x+y}, \quad \text{for all } x, y \in X,$$

$$\nu(x,y) = \begin{cases} x+y, & \text{if} \quad 0 \le x \le 1, 0 \le y \le \frac{1}{8}, \\ e^{2(x+y)}, & \text{otherwise.} \end{cases}$$

As $y = Tx \in [0, \frac{1}{16}]$ for all $x \in [0,1]$, it follows that $\alpha(x,y) = e^{x+y} \ge x+y = \nu(x,y)$ for all $x \in [0,1]$, that is, T is a $(\alpha - \nu)$- dominated mapping.

Let $\{x_n\}$ be a sequence in X converging to $x \in X$, such that $\alpha(x_n, x_{n+1}) \ge \nu(x_n, x_{n+1})$, for all n. Then $x_1 \in [0,1]$ and $x_n \in [0, \frac{1}{8}]$ for all $n \ge 2$, which implies that $x \in [0, \frac{1}{8}]$. Therefore $\alpha(x_n, x) = e^{x_n + x} \ge x_n + x = \nu(x_n, x)$, for all n. So, X has $(\alpha{-}\nu)$- regular property.

Let $x, y, z \in X$ be such that $\alpha(x,y) \ge \nu(x,y)$ and $\alpha(y,z) \ge \nu(y,z)$. Then $x \in [0,1]$ and $y, z \in [0, \frac{1}{8}]$ which imply that $\alpha(x,z) = e^{x+z} \ge x+z = \nu(x,z)$. Therefore, X has $(\alpha{-}\nu)$-transitive property.

Next we describe following classes of functions which are used in our main findings.

•. Let Ψ denote the family of all functions $\psi : [0,+\infty) \to [0,+\infty)$ such that:

(i) ψ is non-decreasing; (ii) $\psi(t) < t$ for each $t > 0$.

•. Let Φ denote the family of all functions $\phi : [0,+\infty) \to [0,+\infty)$ such that

(i) ϕ is nondecreasing and continuous; (ii) for any sequence $\{t_n\} \subseteq [0,+\infty)$, $\lim_{n\to+\infty} \phi(t_n) = 0$ if and only if $\lim\limits_{n\to+\infty} t_n = 0$.

It is clear that $\phi(t) = 0$ if and only if $t = 0$.

•. Let Ω be the collection of all functions $\omega : [0,+\infty) \to [0,+\infty)$ such that ω is lower semi-continuous and $\omega(t) = 0$ if and only if $t = 0$.

Example 2.6. The following functions $\psi_i : [0,+\infty) \to [0,+\infty), (i = 1,2)$ belong to the class Ψ.

(i) $\psi_1(t) = k\ t$ for all $t \in [0,+\infty)$, where $0 < k < 1$.

(ii) $\psi_2(t) = \dfrac{t}{1+t}$ for all $t \in [0,+\infty)$.

Example 2.7. The following functions $\phi_i : [0,+\infty) \to [0,+\infty), (i = 1,2,3)$ belong to the class Φ.

(i) $\phi_1(t) = t$ for all $t \in [0,+\infty)$.

(ii) $\phi_2(t) = \ln \theta(t)$ for all $t \in [0,+\infty)$, where $\theta : (0,+\infty) \to (1,+\infty)$ is non decreasing function satisfying the property : each sequence $\{t_n\}$ in $(0,+\infty)$, $\lim_{n\to+\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to+\infty} t_n = 0$.

(iii) $\phi_3(t) = e^{F(t)}$, for all $t \in [0,+\infty)$, where $F : (0,+\infty) \to \mathbb{R}$ strictly increasing function satisfying the following property : for any sequence $\{a_n\}$ in $\mathbb{R}^+$, $\lim_{n\to+\infty} a_n = 0$ and $\lim_{n\to+\infty} F(a_n) = \pm\infty$ are equivalent.

Definition 2.8. Let (X,d) be a metric space and $\alpha, \nu : X \times X \to [0,+\infty)$. A mapping $T : X \to X$ is said to be a $(\phi - \psi - \omega)$- generalized rational weak contraction if there exist $\phi \in \Phi, \psi \in \Psi$ and $\omega \in \Omega$ such that for all $x, y \in X$ with $\alpha(x,y) \ge \nu(x,y)$,

$\phi(d(Tx,Ty)) \leq \psi(\phi(M(x,y))) - \omega(M(x,y))$, where $M(x,y)$ is a conditional rational expression defined as

$$M(x,y) = \begin{cases} \dfrac{d(x,Tx)\ d(x,Ty) + d(y,Ty)\ d(y,Tx)}{\max\{d(x,Ty), d(y,Tx)\}}, & \text{if } \max\{d(x,Ty), d(y,Tx)\} \neq 0, \\ 0, & \text{if } \max\{d(x,Ty), d(y,Tx)\} = 0. \end{cases}$$

3. Main results

First we prove the existence of fixed point of a self mapping T of a metric space. Then under addition assumptions we establish the uniqueness of the fixed point. We deduce some corollaries of the main result and illustrate it with an example.

Let (X,d) be a metric space and $\alpha,\ \nu : X \times X \to [0,\ +\infty)$ be two mappings. We designate the following properties by (A1), (A2) and (A3).

(A1) X has $(\alpha{-}\nu)$- regular property;

(A2) X has $(\alpha{-}\nu)$-transitive property;

(A3) for every $x, x^* \in X$, there exists a $u \in X$ such that $\alpha(x,u) \geq \nu(x,u)$ and $\alpha(x^*,u) \geq \nu(x^*,u)$.

Theorem 3.1. *Let (X,d) be a complete metric space and $\alpha, \nu : X \times X \to [0,+\infty)$ be two mappings such that the properties (A1) and (A2) hold. Suppose that $T : X \to X$ be a $(\alpha{-}\nu)$-dominated mapping and there exist $\phi \in \Phi$, $\psi \in \Psi$ and $\omega \in \Omega$ such that T is a $(\phi{-}\psi{-}\omega)$-generalized rational weak contraction. Then T has a fixed point in X.*

Proof. Let $x_0 \in X$. We construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \quad \text{for all } n \geq 0. \tag{1}$$

As T is $(\alpha{-}\nu)$-dominated, we have

$$\alpha(x_n, x_{n+1}) \geq \nu(x_n, x_{n+1}), \quad \text{for all } n \geq 0. \tag{2}$$

If there exists n_0 such that $d(x_{n_0}, x_{n_0+1}) = 0$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, that is, x_{n_0} is a fixed point of T. Hence we assume that $d(x_n, x_{n+1}) \neq 0$ for all $n \geq 0$. Let $r_n = d(x_n, x_{n+1})$ for all $n \geq 0$. Then $r_n = d(x_n, x_{n+1}) > 0$ for all $n \geq 0$. As T is a $(\phi{-}\psi{-}\omega)$-generalized rational weak contraction, we have for all $n \geq 0$ that

$$\phi(d(x_{n+1}, x_{n+2})) = \phi(d(Tx_n, Tx_{n+1})) \leq \psi(\phi(M(x_n, x_{n+1}))) - \omega(M(x_n, x_{n+1})). \tag{3}$$

If $\max\{d(x_{n_0}, Tx_{n_0+1}), d(x_{n_0+1}, Tx_{n_0})\} = 0$ for some n_0, then $M(x_{n_0}, x_{n_0+1}) = 0$. Using the properties of ϕ, ψ and ω, we have from the above inequality that $d(Tx_{n_0}, Tx_{n_0+1}) = 0$, that is, $d(x_{n_0+1}, x_{n_0+2}) = 0$, that is, $r_{n_0+1} = 0$, which contradicts our assumption that $r_n = d(x_n, x_{n+1}) > 0$ for all $n \geq 0$. Therefore, we assume that $\max\{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\} \neq 0$ for all n, that is, $d(x_n, x_{n+2}) \neq 0$ for all n. Now,

$$\begin{aligned} M(x_n, x_{n+1}) &= \frac{d(x_n, Tx_n)\ d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1})\ d(x_{n+1}, Tx_n)}{\max\{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\}} \\ &= \frac{d(x_n, x_{n+1})\ d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2})\ d(x_{n+1}, x_{n+1})}{\max\{d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\}} \end{aligned}$$

$$= \frac{d(x_n, x_{n+1})\ d(x_n, x_{n+2})}{d(x_n, x_{n+2})} = d(x_n, x_{n+1}) = r_n > 0. \quad (4)$$

Using (4) and the properties of ϕ, ψ and ω, we have from (3) that

$$\phi(r_{n+1}) = \psi(\phi(r_n)) - \omega(r_n) \leq \phi(r_n) - \omega(r_n) < \phi(r_n). \quad (5)$$

Therefore, $\{\phi(r_n)\}$ is monotone decreasing sequence of nonnegative real numbers. By a property of ϕ, $\{r_n\}$ is a monotone decreasing sequence of nonnegative real numbers. So, there exist $L \geq 0$ and $r \geq 0$ such that $\lim_{n\to+\infty} \phi(r_n) = L$ and $\lim_{n\to+\infty} r_n = r$. If possible, suppose that $r > 0$. By a property of ω,we have $\omega(r) > 0$. Taking limit superior on both sides of (5) and using the lower semi-continuity of ω, we have

$$L \leq L - \liminf_{n\to+\infty} \omega(r_n) \leq L - \omega(r) < L,$$

which is a contradiction. Therefore,

$$r = \lim_{n\to+\infty} r_n = \lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \quad (6)$$

We show that $\{x_n\}$ is a Cauchy sequence. Suppose, on contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we have two sequences $\{p(n)\}$ and $\{q(n)\}$ of natural numbers such that for every $n \in \mathbb{N}$, $p(n)$ is the smallest such positive integer for which

$$p(n) > q(n) > n, \quad d(x_{q(n)}, x_{p(n)}) \geq \epsilon \qquad \text{and} \qquad d(x_{q(n)}, x_{p(n)-1}) < \epsilon. \quad (7)$$

For every $n \in \mathbb{N}$, we have

$$\epsilon \leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < d(x_{p(n)}, x_{p(n)-1}) + \epsilon.$$

Taking limit as $n \to +\infty$ and using (6), we have

$$\lim_{n\to+\infty} d(x_{p(n)}, x_{q(n)}) = \lim_{n\to+\infty} d(x_{q(n)}, x_{p(n)}) = \epsilon.$$

Again for every $n \in \mathbb{N}$, we have $\epsilon \leq d(x_{q(n)}, x_{p(n)}) \leq d(x_{q(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{p(n)})$. Taking limit infimum on both sides of the above inequality and using (6), we have $\liminf_{n\to+\infty} d(x_{q(n)}, x_{p(n)+1}) \geq \epsilon$. Then there exists $n_2 \in \mathbb{N}$ such that $d(x_{q(n)}, x_{p(n)+1}) > \frac{\epsilon}{2}$, for all $n > n_2$. This implies that for all $n > n_2$,

$$\max\{d(x_{q(n)}, x_{p(n)+1}), d(x_{p(n)}, x_{q(n)+1})\} = \max\{d(x_{q(n)}, Tx_{p(n)}), d(x_{p(n)}, Tx_{q(n)})\} > \frac{\epsilon}{2}. \quad (8)$$

From (7), we have

$$\epsilon \leq d(x_{q(n)}, x_{p(n)}) \leq d(x_{q(n)}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{p(n)}).$$

Taking limit infimum on both sides of the above inequality and using (6), we have $\liminf_{n\to+\infty} d(x_{q(n)+1}, x_{p(n)+1}) \geq \epsilon$. Then there exists $n_3 \in \mathbb{N}$ such that, for all $n > n_3$, $d(x_{q(n)+1}, x_{p(n)+1}) > \frac{\epsilon}{2}$. By assumption (A2), we have $\alpha(x_{q(n)}, x_{p(n)}) \geq \nu(x_{q(n)}, x_{p(n)})$. By (8), $\max\{d(x_{q(n)}, Tx_{p(n)}), d(x_{p(n)}, Tx_{q(n)})\} > \frac{\epsilon}{2} \neq 0$ for all $n > n_2$. As ϕ is nondecreasing and T is a $(\phi - \psi - \omega)$- generalized rational weak contraction, we have for every $n > \max\{n_2, n_3\}$,

$$\phi(\frac{\epsilon}{2}) \leq \phi(d(x_{q(n)+1}, x_{p(n)+1})) = \phi(d(Tx_{q(n)}, Tx_{p(n)}))$$

$$\leq \psi(\phi(M(x_{q(n)}, x_{p(n)}))) - \omega(M(x_{q(n)}, x_{p(n)})) \leq \psi(\phi(M(x_{q(n)}, x_{p(n)})), \quad (9)$$

where,

$$0 \leq M(x_{q(n)}, x_{p(n)}) = \frac{d(x_{q(n)}, Tx_{q(n)})\ d(x_{q(n)}, Tx_{p(n)}) + d(x_{p(n)}, Tx_{p(n)})\ d(x_{p(n)}, Tx_{q(n)})}{\max\{d(x_{q(n)}, Tx_{p(n)}), d(x_{p(n)}, Tx_{q(n)})\}}$$

$$= \frac{d(x_{q(n)}, x_{q(n)+1})\ d(x_{q(n)}, x_{p(n)+1}) + d(x_{p(n)}, x_{p(n)+1})\ d(x_{p(n)}, x_{q(n)+1})}{\max\{d(x_{q(n)}, x_{p(n)+1}), d(x_{p(n)}, x_{q(n)+1})\}}$$

$$= \frac{d(x_{q(n)}, x_{q(n)+1}) d(x_{q(n)}, x_{p(n)+1})}{\max\{d(x_{q(n)}, x_{p(n)+1}), d(x_{p(n)}, x_{q(n)+1})\}} + \frac{d(x_{p(n)}, x_{p(n)+1}) d(x_{p(n)}, x_{q(n)+1})}{\max\{d(x_{q(n)}, x_{p(n)+1}), d(x_{p(n)}, x_{q(n)+1})\}}$$

$$\leq d(x_{q(n)}, x_{q(n)+1}) + d(x_{p(n)}, x_{p(n)+1}).$$

Taking limit as $n \to +\infty$ and using (6), we have $\lim_{n\to+\infty} M(x_{q(n)}, x_{p(n)}) = 0$. Then there exists $n_4 \in \mathbb{N}$ such that $M(x_{q(n)}, x_{p(n)}) < \frac{\epsilon}{2}$ for all $n > n_4$. Using the properties of ϕ and ψ, we have from (9) that for all $n > \max\{n_2, n_3, n_4\}$,

$$\phi(\frac{\epsilon}{2}) \leq \psi(\phi(M(x_{q(n)}, x_{p(n)}))) \leq \psi(\phi(\frac{\epsilon}{2})) < \phi(\frac{\epsilon}{2}),$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in X. As X is complete, there exists $x \in X$ such that

$$\lim_{n\to+\infty} x_n = x. \quad (10)$$

Next we show that x is a fixed point of T. If possible, let x is not a fixed point of T. Then $d(x, Tx) > 0$. By (2), (10) and the $(\alpha-\nu)$- regularity assumption of the space, we have $\alpha(x_n, x) \geq \nu(x_n, x)$ for all $n \geq 0$. As T is a $(\phi-\psi-\omega)$- generalized rational weak contraction, we have

$$\phi(d(x_{n+1}, Tx)) = \phi(d(Tx_n, Tx)) \leq \psi(\phi(M(x_n, x))) - \omega(M(x_n, x)), \quad \text{for all } n. \quad (11)$$

Suppose that for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k > k$ and $x_{n_k} = Tx$. Then $x = \lim_{k\to+\infty} x_{n_k} = Tx$, which contradicts the assumption $d(x, Tx) > 0$. Thus, there exists $m \in \mathbb{N}$ such that $x_n \neq Tx$ for each $n \geq m$. Hence, for each $n \geq m$, $\max\{d(x_n, Tx), d(x, Tx_n)\} > 0$. Therefore,

$$M(x_n, x) = \frac{d(x_n, Tx_n)\ d(x_n, Tx) + d(x, Tx)\ d(x, Tx_n)}{\max\{d(x_n, Tx), d(x, Tx_n)\}}$$

$$= \frac{d(x_n, x_{n+1})\ d(x_n, Tx) + d(x, Tx)\ d(x, x_{n+1})}{\max\{d(x_n, Tx), d(x, x_{n+1})\}}.$$

Taking limit as $n \to +\infty$ and using (10), we have $\lim_{n\to+\infty} M(x_n, x) = 0$. As $d(x, Tx) > 0$, there exists $n_5 \in \mathbb{N}$ such that $M(x_n, x) < \frac{d(x,Tx)}{2}$ for all $n > n_5$. Using the properties of ϕ and ψ, we get from (11) that for all $n > n_5$,

$$\phi(d(x_{n+1}, Tx)) \leq \psi(\phi(M(x_n, x))) - \omega(M(x_n, x))$$

$$\leq \psi(\phi(\frac{d(x, Tx)}{2})) - \omega(M(x_n, x)) < \phi(\frac{d(x, Tx)}{2}) - \omega(M(x_n, x)).$$

Take limit superior and use the continuity of ϕ and the lower semi-continuity of ω:

$$\phi(d(x, Tx)) \leq \phi(\frac{d(x, Tx)}{2}) - \liminf \omega(M(x_n, x))) \leq \phi(\frac{d(x, Tx)}{2}) - \omega(0) = \phi(\frac{d(x, Tx)}{2}).$$

By the nondecreasing property of ϕ, we have $d(x,Tx) \leq \dfrac{d(x,Tx)}{2}$, which is a contradiction. Therefore, $d(x,Tx)=0$, that is, $x=Tx$, that is, x is a fixed point of T. □

THEOREM 3.2. In addition to the hypothesis of Theorem 3.1, if (A3) holds then T has a unique fixed point.

Proof. By Theorem 3.1, the set of fixed points of T is nonempty. If possible, let x and x^* be two fixed points of T. Then $x = Tx$ and $x^* = Tx^*$. Our aim is to show that $x = x^*$. By the assumption (A3), there exists a $u \in X$ such that $\alpha(x,u) \geq \nu(x,u)$ and $\alpha(x^*,u) \geq \nu(x^*,u)$. Put $u_0 = u$. Then $\alpha(x,u_0) \geq \nu(x,u_0)$. Let $u_1 = Tu_0$. Similarly, as in the proof of Theorem 3.1, we define a sequence $\{u_n\}$ such that $u_{n+1} = Tu_n$, for all $n \geq 0$. As T is $(\alpha-\nu)$-dominated, we have

$$\alpha(u_n,u_{n+1}) \geq \nu(u_n,u_{n+1}), \quad \text{for all} \quad n \geq 0. \tag{12}$$

Arguing similarly as in proof of Theorem 3.1, we prove that $\{u_n\}$ is a Cauchy sequence in X and there exists a $p \in X$ such that $\lim_{n\to+\infty} u_n = p$. We claim that

$$\alpha(x,u_n) \geq \nu(x,u_n), \quad \text{for all} \quad n \geq 0. \tag{13}$$

As $\alpha(x,u_0) \geq \nu(x,u_0)$ and $\alpha(u_0,u_1) \geq \nu(u_0,u_1)$, by the assumption (A2), we have $\alpha(x,u_1) \geq \nu(x,u_1)$. Therefore, our claim is true for $n = 1$. We assume that $\alpha(x,u_m) \geq \nu(x,u_m)$ holds for some $m > 1$. By (12), $\alpha(u_m,u_{m+1}) \geq \nu(u_m,u_{m+1})$. Applying the assumption (A2), we have $\alpha(x,u_{m+1}) \geq \nu(x,u_{m+1})$ and this proves our claim.

If possible, suppose $x \neq p$. Then $d(x,p) > 0$. As T is a $(\phi-\psi-\omega)$- generalized rational weak contraction, using (13) we have

$$\phi(d(x,u_{n+1}))\leq\phi(d(Tx,Tu_n))\leq\psi(\phi(M(x,u_n)))-\omega(M(x,u_n)), \text{ for all } n \geq 0. \tag{14}$$

Suppose that for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k > k$ and $u_{n_k} = Tx$. Then $p = \lim_{k\to+\infty} u_{n_k} = Tx = x$, which implies that $d(x,p) = 0$, which contradicts that $d(x,p) > 0$. Therefore, there exists $m \in \mathbb{N}$ such that $u_n \neq Tx$ for each $n \geq m$. Hence, in this case $\max\{d(u_n,Tx), d(x,Tu_n)\} \neq 0$ for each $n \geq m$. Therefore,

$$M(x,u_n) = \frac{d(x,Tx)\ d(x,Tu_n) + d(u_n,Tu_n)d(u_n,Tx)}{\max\{d(x,Tu_n), d(u_n,Tx)\}} = \frac{d(u_n,u_{n+1})d(u_n,x)}{\max\{d(x,u_{n+1}), d(u_n,x)\}}.$$

Taking limit as $n \to +\infty$ on both sides, we have $\lim_{n\to+\infty} M(u_n,x) = 0$. As $d(x,p) > 0$, there exists $n_6 \in \mathbb{N}$ such that $M(x,u_n) < \frac{d(x,p)}{2}$ for all $n > n_6$. Using the properties of ϕ and ψ, we have from (14) that for all $n > n_6$,

$$\phi(d(x,u_{n+1})) \leq \psi(\phi(M(x,u_n))) - \omega(M(x,u_n)) < \phi(\frac{d(x,p)}{2}) - \omega(M(x,u_n)).$$

Taking limit superior in above inequality and using the continuity of ϕ and the lower semi-continuity of ω, we have

$$\phi(d(x,p)) \leq \phi(\frac{d(x,p)}{2})) - \liminf \omega(M(x,u_n)) = \phi(\frac{d(x,p)}{2}) - \omega(0) = \phi(\frac{d(x,p)}{2}).$$

By a property of ϕ, we have $d(x,p) \leq \dfrac{d(x,p)}{2}$, which is a contradiction. So, we have $d(x,p) = 0$, that is, $x = p$. Similarly, we can show that $x^* = p$. Then $x = x^*$ and hence T has a unique fixed point. $\square$

COROLLARY 3.3. *Let (X,d) be a complete metric space and $\psi \in \Psi, \phi \in \Phi$ and $\omega \in \Omega$. Then a mapping $T : X \to X$ has a unique fixed point if for all $x, y \in X$ the following inequality holds: $\phi(d(Tx,Ty)) \leq \psi(\phi(M(x,y))) - \omega(M(x,y))$, where $M(x,y)$ is same as in Definition 2.8.*

Proof. Taking two functions $\alpha, \nu : X \times X \to [0, +\infty)$ such that $\alpha(x,y) = \nu(x,y)$ for all $x, y \in X$, we have the required proof from that of Theorem 3.2. $\square$

COROLLARY 3.4. *Let (X,d) be a complete metric space and $\alpha, \nu : X \times X \to [0, +\infty)$ be two mappings such that the properties (A1), (A2) and (A3) hold. Suppose that $T : X \to X$ be a $(\alpha - \nu)$-dominated mapping and there exists a $\omega \in \Omega$ such that for all $x, y \in X$ with $\alpha(x,y) \geq \nu(x,y)$,*

$$d(Tx,Ty) \leq \frac{M(x,y)}{1 + M(x,y)} - \omega(M(x,y)),$$

where $M(x,y)$ is given in Definition 2.8. Then T has a unique fixed point in X.

Proof. Define two mappings $\psi, \phi : [0, +\infty) \to [0, +\infty)$ as $\psi(t) = \frac{t}{1+t}$, for all $t \in [0, +\infty)$ and $\phi(t) = t$, for all $t \in [0, +\infty)$. Then $\psi \in \Psi$ and $\phi \in \Phi$. Then the inequality of the theorem takes the form $\phi(d(Tx,Ty)) \leq \psi(\phi(M(x,y))) - \omega(M(x,y))$, where $M(x,y)$ is same as in Definition 2.8. Then we have the required proof from that of Theorem 3.2. $\square$

REMARK 3.5. Our results generalize the results in [6–8].

EXAMPLE 3.6. Take the complete metric space $X = \{0, 1, 2, 3, \ldots, n, \ldots\}$ with the metric "d" defined as $d(x,y) = \begin{cases} x + y, & \text{if } \ x \neq y, \\ 0, & \text{if } \ x = y. \end{cases}$.

Let $T : X \to X$ and $\alpha, \nu : X \times X \to [0, +\infty)$ be respectively defined as follows:

$$T(x) = \begin{cases} x - 1, & \text{if } x \neq 0, \\ 0, & \text{if } x \doteq 0, \end{cases}, \quad \alpha(x,y) = e^{x+y} \quad \text{and} \quad \nu(x,y) = x + y, \text{ for } x, y \in X.$$

Let $\psi, \phi, \omega \colon [0, +\infty) \to [0, +\infty)$ be given respectively by the formulas

$$\psi(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t - 1, & t \geq 1, \end{cases} \quad \phi(t) = \begin{cases} t^2, & 0 \leq t < 1, \\ t, & t \geq 1, \end{cases} \quad \omega(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ 1, & t > 1. \end{cases}$$

Let $x, y \in X$. Without loss of generality we take $x \geq y$ and discuss following cases.

Case 1. $y \neq 0$ and $x > y$.

$$\phi(d(Tx,Ty)) = \phi(d(x-1, y-1)) = x+y-2,$$
$$d(x,Tx) = 2x-1, \quad d(x,Ty) = x+y-1, \quad d(y,Ty) = 2y-1$$
$$d(y,Tx) = x+y-1 \quad (\text{if } y \neq Tx) \quad \text{or} \quad d(y,Tx) = 0 \quad (\text{if } y = Tx).$$

Now, $\max\{d(x,Ty), d(y,Tx)\} \neq 0$, $M(x,y) = 2(x+y-1)$ or $M(x,y) = 2x-1$

and $\psi(\phi(M(x,y))) - \omega(M(x,y)) = 2(x+y-1)-1-1 = 2(x+y-2)$

or $\psi(\phi(M(x,y))) - \omega(M(x,y)) = (2x-1)-1-1 = 2x-3 = x+x-3 = x+y+k-3$, where $k \geq 1$ is an integer (as $x > y$). Then we have $\phi(d(Tx,Ty)) \leq \psi(\phi(M(x,y))) - \omega(M(x,y))$.

Case 2. $y = 0$ and $x > y$.

$$\phi(d(Tx,Ty)) = \phi(d(x-1,0)) = x-1,$$
$$d(x,Tx) = 2x-1, \quad d(x,Ty) = x, \quad d(y,Ty) = 0, \quad d(y,Tx) = x-1.$$

Now, $\max\{d(x,Ty), d(y,Tx)\} \neq 0, \quad M(x,y) = 2x-1$

and $\psi(\phi(M(x,y))) - \omega(M(x,y)) = (2x-1)-1-1 = (x-1)+x-2 \quad \text{if } x = 1$

or $\psi(\phi(M(x,y))) - \omega(M(x,y)) = (2x-1)-1-0 = 2(x-1) \quad \text{if } x > 1.$

Then we have $\phi(d(Tx,Ty)) \leq \psi(\phi(M(x,y))) - \omega(M(x,y))$.

Case 3. $x = y \neq 0$.

$$\phi(d(Tx,Ty)) = 0, \quad d(x,Tx) = d(x,Ty) = d(y,Ty) = d(y,Tx) = 2x-1.$$

Now, $\max\{d(x,Ty), d(y,Tx)\} \neq 0, \quad M(x,y) = 2x-1$

and $\psi(\phi(M(x,y))) - \omega(M(x,y)) = (2x-1)-1-1 = (x-1)+x-2, \quad \text{if } x = 1$

or $\psi(\phi(M(x,y))) - \omega(M(x,y)) = (2x-1)-1-0 = 2(x-1) = 0, \quad \text{if } x > 1.$

Then we have $\phi(d(Tx,Ty)) \leq \psi(\phi(M(x,y))) - \omega(M(x,y))$.

Case 4. $x = y = 0$.

$$\phi(d(Tx,Ty)) = 0, \quad d(x,Tx) = d(x,Ty) = d(y,Ty) = d(y,Tx) = 0.$$

Now, $\max\{d(x,Ty), d(y,Tx)\}=0, \quad M(x,y)=0, \quad \psi(\phi(M(x,y)))-\omega(M(x,y))=0.$

Then we have $\phi(d(Tx,Ty)) \leq \psi(\phi(M(x,y))) - \omega(M(x,y))$.
All the conditions of Theorem 3.1 and Theorem 3.2 are satisfied and here 0 is the unique fixed point of T.

4. Error estimation and rate of convergence

For the purpose of the present section we formally state the following fixed point problem.
Problem A Let (X,d) be a metric space and $T : X \to X$ be a mapping. We consider the problem of finding a fixed point of T, that is, the problem of finding $x \in X$ such that $x = Tx$.

We now study the rate at which the iteration method of finding the fixed point converges if the initial approximation to the fixed point is sufficiently close to the desired fixed point. The same idea with many variations were utilized in several a recent work [3]. For this purpose we first define the order of convergence of the fixed point **Problem A**.

Definition 4.1. **Problem A** is said to be of order r or has the rate of convergence r with respect to $\{x_n\}$ defined in (1) if (i) T has a unique fixed point x, (ii) r

is a positive real number for which there exists a finite constant $C > 0$ such that $R_{k+1} \leq C\ [R_k]^r$, where $R_k = d(x,\ x_k)$ is the error in k-th iterate. The constant C is called the asymptotic error. If $r = 1$, we say that the iteration process has linear rate of convergence.

Theorem 4.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Let $\alpha, \nu : X \times X \to [0, +\infty)$ be two mappings and $\phi \in \Phi, \psi \in \Psi$, $\omega \in \Omega$ such that all the assumptions of Theorem 3.2 are satisfied. Then $R_{n+1} \leq 2R_n$, if $R_n \neq 0$.*

Proof. By Theorem 3.2, T has unique fixed point. Suppose x is the unique fixed point of T. Let $x_0 \in X$ be the initial approximation of x. We define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \geq 0$. Following Theorem 3.1, we have $\{x_n\}$ is a Cauchy sequence in X and $\{x_n\}$ converges to the fixed point of T in X. As we consider that x is the unique fixed point of T, we have $\lim_{n\to+\infty} x_n = x$.

If the error at n-th stage is zero, that is, if $R_n = 0$ then $d(x_n, x) = 0$, that is, $x_n = x$. Then $x_{n+1} = Tx_n = Tx = x$, which implies that $R_{n+1} = d(x_{n+1}, x) = 0$. Similarly, we can show that $R_{n+k} = 0$ for all $k = 1, 2, 3, \ldots$. Hence we assume that $R_n = d(x_n, x) \neq 0$ for all $n > 0$. If there exists n_0 such that $d(x_{n_0}, x_{n_0+1}) = 0$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, that is, x_{n_0} is a fixed point of T. As we consider that x is the unique fixed point of T, we have $x_{n_0} = x$ Therefor, $R_{n_0} = d(x_{n_0}, x) = 0$, which contradicts our assumption that $R_n \neq 0$ for all $n > 0$. Therefore, we assume that $r_n = d(x_n, x_{n+1}) > 0$ for all n. Following Theorem 3.1, we have $\alpha(x_n, x) \geq \nu(x_n, x)$ for all $n \geq 0$. As $\max\{d(x_n, Tx), d(x, Tx_n)\} = \max\{d(x_n, x), d(x, x_{n+1})\} = \max\{R_n, R_{n+1}\} > 0$ for all n, we have

$$\begin{aligned}\phi(R_{n+1}) &= \phi(d(x_{n+1}, x)) = \phi(d(x_{n+1}, Tx)) = \phi(d(Tx_n, Tx)) \\ &\leq \psi(\phi(M(x_n, x))) - \omega(M(x_n, x))) \leq \psi(\phi(M(x_n, x))),\end{aligned}$$

where

$$M(x_n, x) = \frac{d(x_n, Tx_n)d(x_n, Tx) + d(x, Tx)d(x, Tx_n)}{\max\{d(x_n, Tx), d(x, Tx_n)\}} = \frac{d(x_n, x_{n+1})R_n}{\max\{R_n, R_{n+1}\}} > 0.$$

Using the properties of ϕ and ψ, we have

$$\begin{aligned}\phi(R_{n+1}) &\leq \psi(\phi(M(x_n, x))) < \phi(M(x_n, x)) = \phi\Big(\frac{d(x_n, x_{n+1})\ R_n}{\max\{R_n, R_{n+1}\}}\Big) \\ &\leq \phi\Big(\frac{[d(x_n, x) + d(x, x_{n+1})]\ R_n}{\max\{R_n, R_{n+1}\}}\Big) = \phi\Big(\frac{[R_n + R_{n+1})]\ R_n}{\max\{R_n, R_{n+1}\}}\Big) \\ &\leq \phi\Big(\frac{2\ [\max\{R_n, R_{n+1}\}]\ R_n}{\max\{R_n, R_{n+1}\}}\Big) = \phi(2\ R_n),\end{aligned}$$

which, by a property of ϕ, implies that $R_{n+1} \leq 2\ R_n$. □

Remark 4.3. The rate of convergence of the iteration method of finding the fixed point is here linear.

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