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ON LOCAL FRACTAL FUNCTIONS OF HIGHER ORDER

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Abstract. In this short note we prove the existence of local fractal functions of the Orlicz-Sobolev class of order $m \geq 0$. The graph of a local fractal function coincides with the attractor of an appropriate iterated function system, whose construction is fairly standard. Local fractal functions appear naturally as the fixed points of the Read-Bajraktarević operator when restricted to a suitable Orlicz-Sobolev space. Our results extend some of the outcomes obtained by Massopust on Lebesgue and Sobolev spaces to higher order, dimension and function spaces (where the role of the norm is now played by a Young function).

1. Introduction

An iterated function system (IFS) consists of a finite set of contracting functions defined on a complete metric space, with the images in the same space. Hutchinson and Mandelbrot introduced IFSs in the literature in the early 1980s, and their applications were later widely popularized by Barnsley in the 1990s. For example, the *Collage Theorem* asserts that there always exists a unique nonempty compact set which is equal to the union of its images under the collection of contractions. Such a set is called an *attractor* of the IFS. Conversely, Barnsley demonstrated that it is possible to begin with a determined figure, and then build an IFS whose attractor converges to that figure. The method is known as the Random Iteration Algorithm, dubbed the *Chaos Game* by Barnsley himself [5]. During the last decades, this classic framework has been extended to more general spaces [23] and contracting functions (see [20] for a detailed exam of generalized IFSs).

Iterated function systems are, moreover, tightly connected with the *Read-Bajraktarević equation,* introduced by M. Bajraktarević in the 1950s [4]. In its elementary form the Read-Bajraktarević equation is given by $u(x) = \nu(x, u(b(x)))$, where $b : I \to I, \nu : I \times \mathbb{R} \to \mathbb{R}$ and the unknown $u : I \to \mathbb{R}$ are functions on a closed real interval I. Later in the 1980s, the associated Read-Bajraktarević operator

$$\mathbf{T}u(x) = \nu(x, u(b(x))) \tag{1}$$

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appeared in the works of C. J. Read in the context of the invariant subspace problem [22]. Initially defined on the space $C^\infty(I)$ of infinitely differentiable functions on I, the operator (1) is closely related to Bajraktarević's functional equation, as follows. Assume that b, ν are continuous and b is surjective. Suppose that there exists a constant $c \in (0, 1)$ such that for $x, y \in I$ and $y_1, y_2 \in \mathbb{R}$, $|\nu(x, y_1) - \nu(x, y_2)| \leq c|y_1 - y_2|$. Then the operator (1) is contractive on $C^\infty(I)$ [18]. The unique fixed point u^* is obtained as a limit of the iterations $u_k(x) = \nu(x, u_{k-1}(b(x)))$, $k \in \mathbb{N}$. The initial condition u_0 is any function in $C^\infty(I)$, and the sequence of iterations converges to u^* in the sense that $\sup_{x \in I} |u_k(x) - u^*(x)| \to 0$ as $k \to \infty$ [19]. The fixed point u^* is called a smooth local fractal function [17], or a local fractal function of the smooth class. The graph of a local fractal function is the local attractor of an associated local IFS, whose construction is fairly standard in one dimension. Characterizations of local fractal functions of the Hölder, Lebesgue and Sobolev classes are well known. For example, Massopust et al. proved that the set of discontinuities of a bounded local fractal function is at most countably infinite [6]. The existence of local fractal functions of the Orlicz and of the Orlicz-Sobolev classes of order one was proved later in [2] in $N \geq 1$ dimensions. More specifically, we considered those IFSs whose attractors are the graphs of local fractal functions either of the Orlicz class, or of the Orlicz-Sobolev class on a nonempty connected and bounded subset of $\mathbb{R}^N$, which is partitioned into nonempty connected and convex subsets. (The hypothesis of convexity was required to tackle the converse problem: given a fixed point of the Read-Bajraktarević operator, how do we construct a contractive local IFS with the attractor being the graph of the fixed point, in N dimensions?) We proved that a local fractal function is the fixed point of the induced Read-Bajraktarević operator. A hint about the existence problem in higher order appeared in [2] as well. This short note is a follow-up in that direction. More specifically, in this article we prove that local fractal functions of an Orlicz-Sobolev class of order $m \geq 2$ appear naturally as the fixed points of the restriction of the Read-Bajraktarević operator. Our results extend some of the outcomes obtained by Massopust on Lebesgue and Sobolev spaces to higher order, dimension and function spaces, where the role of the norm is now played by a Young function.

2. Iterated function systems

Let (X, d) denote a complete metric space and $\{w_i : X \to X\}_{i=1}^n$ a set of n continuous maps. Then $\mathcal{F} = (X, w_1, \ldots, w_n)$ is called an n-map iterated function system, or IFS. (The letter N is commonly used in the literature to denote the number of maps in the definition of the IFS. We will use n instead, and we will rather employ N to denote the dimension of the domain that appears later). We say that the functions w_i belong in the IFS $\mathcal{F}$, and we write $w_i \in \mathcal{F}$. These structures were introduced in the works of Hutchinson (1981), Mandelbrot (1982) and Barnsley (1993), as follows. Let $\mathcal{H}(X)$ denote the set of nonempty, compact subsets of X. Associated with an IFS is the set-valued map $\mathbf{w} : \mathcal{H}(X) \to \mathcal{H}(X)$,

$$\mathbf{w}(S) = \bigcup_{i=1}^{n} w_i(S), \quad S \subset X. \tag{2}$$

The IFS $\mathcal{F}$ is called contractive if there is a metric $\tilde{d}$, which is equivalent to d, such that each $w_i \in \mathcal{F}$ is a contraction with respect to $\tilde{d}$. That is, for each $1 \leq i \leq n$ there exists a $c_i \in [0, 1)$ such that $\tilde{d}(w_i(x), w_i(y)) \leq c_i \tilde{d}(x, y)$ for all $x, y \in X$. Hutchinson demonstrated [12] that if this is the case, then $\mathbf{w}$ is itself a contraction on $\mathcal{H}(X)$, $d_{\mathcal{H}(X)}(\mathbf{w}(A), \mathbf{w}(B)) \leq c\, d_{\mathcal{H}(X)}(A, B)$, where $A, B \in \mathcal{H}$ and $c = \max_{1 \leq i \leq n}\{c_i\}$. The expression

$$d_{\mathcal{H}(X)}(A, B) = \max\left\{\max_{a\in A}\min_{b\in B} d(a, b), \max_{b\in B}\min_{a\in A} d(a, b)\right\}$$

is the Hausdorff distance between compact sets. In this case, the Banach theorem implies the existence of a unique fixed point $A \in \mathcal{H}(X)$, called the attractor of the IFS, which satisfies $A = \mathbf{w}(A)$. From (2), the attractor A is *self-similar* since it may be expressed as a union of contracted copies of itself. It is also known that non-contractive IFSs (i.e., such that the maps w_i's are not contractions with respect to any topologically equivalent metric in X) can yield attractors. For more details and examples we refer the reader to [5, 15].

The following notion is due to Barnsley and Hurd [7]. Let $\{X_i\}_{i=1}^n \subseteq X$ be a family of nonempty subsets, equipped with a family of continuous maps $\{w_i : X_i \to X\}_{i=1}^n$. Then $\mathcal{F}_{\text{loc}} = \{(X_i, w_i)\}_{i=1}^n$ is called a local iterated function system (or local IFS). A local IFS is called contractive if there exists a metric, equivalent to d, for which every $w \in \mathcal{F}_{\text{loc}}$ is contractive. Let 2^X denote the power set of X. Associated with any local IFS $\mathcal{F}_{\text{loc}} = \{(X_i, w_i)\}_{i=1}^n$ is the operator $\mathbf{w}_{\text{loc}} : 2^X \to 2^X$ defined by $\mathbf{w}_{\text{loc}}(S) = \bigcup_{i=1}^n w_i(S \cap X_i)$.

Definition 2.1. An element $A \in 2^X$ is a local attractor of the local IFS, if $A = \mathbf{w}_{\text{loc}}(A)$.

Suppose that $\mathcal{F}$ and $\mathcal{F}_{\text{loc}}$ are both contractive. It is well known [17, Proposition 1] that if X is compact, and for every $i = 1, \ldots, n$ the set X_i is closed, then the attractor A of $\mathcal{F}_{\text{loc}}$ is a subset of the attractor of $\mathcal{F}$.

3. Orlicz and Orlicz-Sobolev spaces

This section is brief summary on Orlicz and Orlicz-Sobolev spaces. For further details we refer the reader to [10, 21]. A function $\Phi : [0, \infty) \to [0, \infty)$ which is increasing, continuous, unbounded and such that $\Phi(0) = 0$ is called a φ-function [16, p. 11]. If any such a Φ is, moreover, convex then it has the integral representation

$$\Phi(t) = \int_0^t \phi(s)\, ds \tag{3}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing, right-continuous function satisfying $\phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. This function ϕ is called the right derivative of Φ. A convex φ-function Φ satisfying

$$\frac{\Phi(t)}{t} \to 0 \quad \text{as} \quad t \to 0 \quad \text{and} \quad \frac{\Phi(t)}{t} \to \infty \quad \text{as} \quad t \to \infty,$$

is denominated a Young function [16, pp. 47]. Young functions are sometimes called N-functions; however, to avoid confusion we will not employ that denomination. (This definition is ambiguous. Some authors call a convex function $\Phi : [0,\infty) \to [0,\infty]$ a Young function if Φ is not identically zero and $\lim_{t\to 0^+} \Phi(t) = \Phi(0) = 0$). For example, if $\phi(t) = pt^{p-1}$ for $t \geq 0$, $1 \leq p < \infty$, then $\Phi(t) = t^p$ and

$$\|u\|_p = \Phi^{-1}\Big(\int_X \Phi(|u(x)|)dx\Big)$$

for $u \in L^p(X)$, and where $\Phi^{-1}(t) = t^{1/p}$ is the inverse function.

Given a Young function Φ with the integral representation (3), the right-inverse function of ϕ is defined for $s \geq 0$ by $\psi(s) = \sup\{t : \phi(t) \leq s\}$. If ϕ is continuous and increasing then ψ is the ordinary inverse function ϕ^{-1}. The function ψ has the same properties as ϕ: it is positive for $s > 0$, right-continuous for $s \geq 0$ and satisfies $\psi(0) = 0$ and $\psi(s) \to \infty$ as $s \to \infty$. Hence the integral $\overline{\Phi}(t) = \int_0^t \psi(s)\, ds$ is a Young function as well, called the conjugate (or complementary) of Φ.

In what follows, we will fix an integer $N \geq 1$. Let $X \subset \mathbb{R}^N$ be a bounded subset and let Φ be a Young function. The Orlicz class $\mathcal{L}_\Phi(X)$ is the set of (equivalence classes of) real-valued measurable functions u such that $\Phi(u) \in L^1(X)$. In general, $\mathcal{L}_\Phi(X)$ is not a vector space [14]. However, the linear hull $L_\Phi(X)$ of $\mathcal{L}_\Phi(X)$ is a vector space which is Banach with respect to the Luxemburg norm

$$\|u\|_\Phi = \inf\Big\{\tau > 0 : \int_X \Phi\Big(\frac{u}{\tau}\Big)dx \leq 1\Big\}.$$

We denote by $E_\Phi(X)$ the closure (for the norm-topology) of $L^\infty(X)$ in $L_\Phi(X)$. The space $E_\Phi(X)$ is separable and Banach for the inherited norm. In general, $E_\Phi(X) \subseteq \mathcal{L}_\Phi(X) \subseteq L_\Phi(X)$, and $E_\Phi(X) = L_\Phi(X)$ if and only if Φ satisfies a Δ_2-condition (at infinity). This means that, for $r > 1$, there exist a positive constant $\gamma(r)$ such that

$$\Phi(rt) \leq \gamma(r)\, \Phi(t), \quad \text{for} \quad t \geq T \tag{4}$$

where T is also positive. (The Δ_2-condition is *global* if $T = 0$). It is known that if Φ and $\overline{\Phi}$ satisfy a Δ_2-condition at infinity then the spaces $L_\Phi(X)$ and $L_{\overline{\Phi}}(X)$ are reflexive and separable. It follows that one can identify the dual space of $E_\Phi(X)$ with $L_{\overline{\Phi}}(X)$ and the dual space of $E_{\overline{\Phi}}(X)$ with $L_\Phi(X)$, see [1,10] for details.

3.1 Multivariate Stirling numbers

A multi-index of length N is an N-tuple $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$ such that every component β_j is a nonnegative integer. The order of the multi-index is the number $|\boldsymbol{\beta}| = \beta_1 + \cdots + \beta_N$, and we set $\boldsymbol{\beta}! = \beta_1! \cdots \beta_N!$ As well, if $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^N$ then the multivariate exponentiation is defined by $\vec{v}^{\boldsymbol{\beta}} = v_1^{\beta_1} \cdots v_N^{\beta_N}$.

The set of multi-indices of length N is denoted by $\mathbb{N}^N$. (This set includes the zero multi-index $\mathbf{0}$, for which all components are null). For $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ in $\mathbb{N}^N$, we write $\boldsymbol{\beta} \leq \boldsymbol{\sigma}$ if $\beta_j \leq \sigma_j$ for all $j = 1, \ldots, N$.

Let k be a nonnegative integer. A standard result in combinatorics says that the number of nonnegative integer solutions $(\sigma_1, \ldots, \sigma_N)$ of the linear equation $\sigma_1 + \cdots + \sigma_N = k$ is equal to $\binom{k+N-1}{N-1}$. Hence, if m is a positive integer, the number of multi-indices $\boldsymbol{\sigma}$ satisfying $1 \le |\boldsymbol{\sigma}| \le m$ is

$$N_m = \sum_{k=1}^{m} \binom{k+N-1}{N-1} \in \mathbb{N}. \tag{5}$$

Following [8], we introduce a linear order on $\mathbb{N}^N$, as follows. We write $\boldsymbol{\sigma} \prec \boldsymbol{\beta}$ provided one of the following conditions holds true:
a) $|\boldsymbol{\sigma}| < |\boldsymbol{\beta}|$, b) $|\boldsymbol{\sigma}| = |\boldsymbol{\beta}|$ and $\sigma_1 < \beta_1$,

c) $|\boldsymbol{\sigma}| = |\boldsymbol{\beta}|$ with $\sigma_1 = \beta_1, \ldots, \sigma_j = \beta_j$ and $\sigma_{j+1} < \beta_{j+1}$ for some $1 \le j < N$.
For $\boldsymbol{\sigma}, \boldsymbol{\beta}$ multi-indices and k a nonnegative integer, we define the set

$$E_k(\boldsymbol{\beta},\boldsymbol{\sigma}) = \Big\{(\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_k;\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_k) : |\boldsymbol{\mu}_j| > 0, \mathbf{0} \prec \boldsymbol{\theta}_1 \prec \ldots \prec \boldsymbol{\theta}_k, \sum_{j=1}^{k} \boldsymbol{\mu}_j = \boldsymbol{\sigma}, \sum_{j=1}^{k} |\boldsymbol{\mu}_j| \boldsymbol{\theta}_j = \boldsymbol{\beta}\Big\},$$

where $\boldsymbol{\mu}_j, \boldsymbol{\theta}_j \in \mathbb{N}^N$. If $k = 0$ then we let $E_0(\boldsymbol{\beta},\boldsymbol{\sigma}) = \emptyset$. Observe that if either $|\boldsymbol{\sigma}| > |\boldsymbol{\beta}|$, or if $\boldsymbol{\sigma} = \mathbf{0}$ but $\boldsymbol{\beta} \ne \mathbf{0}$, then by definition $E_k(\boldsymbol{\beta},\boldsymbol{\sigma}) = \emptyset$ for any $k \in \mathbb{N}$. In addition, note that $E_k(\boldsymbol{\beta},\boldsymbol{\sigma}) = \emptyset$ as well whenever $|\boldsymbol{\sigma}| < k$ or $|\boldsymbol{\beta}| < k$. For $1 \le |\boldsymbol{\sigma}| \le |\boldsymbol{\beta}|$, we define nonnegative integers

$$\boldsymbol{S}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}} = \sum_{k=1}^{|\boldsymbol{\beta}|} \sum_{E_k(\boldsymbol{\beta},\boldsymbol{\sigma})} \boldsymbol{\beta}! \prod_{j=1}^{k} \frac{1}{\boldsymbol{\mu}_j!(\boldsymbol{\theta}_j!)^{|\boldsymbol{\mu}_j|}}, \tag{6}$$

which are called the multivariate Stirling numbers [8]. Following the univariate case, let

$$\boldsymbol{S}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}} = \begin{cases} 1, & \boldsymbol{\sigma} = \boldsymbol{\beta} = \mathbf{0} \\ 0, & \boldsymbol{\sigma} = \mathbf{0}, |\boldsymbol{\beta}| \ge 1. \end{cases} \tag{7}$$

A close relationship between the univariate Stirling numbers of the second kind S_n^k and the multivariate Stirling numbers is established in [8, Corollary 2.9] through the formula $\sum_{|\boldsymbol{\sigma}|=k} \boldsymbol{S}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}} = N^k S_{|\boldsymbol{\beta}|}^k$, where $1 \le k \le |\boldsymbol{\beta}|$. Our convention (7) extends it to all $k \ge 0$. Further, if $m \ge 0$ is an integer, define $S_m = \max_{0 \le |\boldsymbol{\sigma}| \le |\boldsymbol{\beta}| \le m} \boldsymbol{S}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}}$. For example, let $\boldsymbol{\beta} = \hat{\boldsymbol{e}}_s$ and $\boldsymbol{\sigma} = \hat{\boldsymbol{e}}_r$ for some $s, r \in \{1, \ldots, N\}$, where $\{\hat{\boldsymbol{e}}_1, \ldots, \hat{\boldsymbol{e}}_N\}$ is the canonical basis of $\mathbb{R}^N$. Then $E_1(\boldsymbol{\beta},\boldsymbol{\sigma}) = \{(\hat{\boldsymbol{e}}_r; \hat{\boldsymbol{e}}_s)\}$ and $E_k(\boldsymbol{\beta},\boldsymbol{\sigma}) = \emptyset$ for the other indices k. Thus, in this case $\boldsymbol{S}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}} = 1/\hat{\boldsymbol{e}}_r!(\hat{\boldsymbol{e}}_s!)^{|\hat{\boldsymbol{e}}_r|} = 1$ and (7) yields $S_0 = S_1 = 1$.

Nota

Even in lower dimensions the calculation of the multivariate Stirling numbers is quite tedious. For example, for $\boldsymbol{\beta} = (3,2)$ and $\boldsymbol{\sigma} = (2,1)$ (and dropping the $\boldsymbol{\beta}, \boldsymbol{\sigma}$-dependence) we have,

$$E_2 = \Big\{ \left(\binom{1}{1}, \binom{1}{0}; \binom{0}{1}, \binom{3}{0}\right), \left(\binom{1}{1}, \binom{1}{0}; \binom{1}{0}, \binom{1}{2}\right), \left(\binom{0}{1}, \binom{2}{0}; \binom{1}{0}, \binom{1}{1}\right),$$
$$\left(\binom{1}{0}, \binom{1}{1}; \binom{1}{0}, \binom{1}{1}\right), \left(\binom{2}{0}, \binom{0}{1}; \binom{0}{1}, \binom{3}{0}\right), \left(\binom{2}{0}, \binom{0}{1}; \binom{1}{0}, \binom{1}{2}\right) \Big\},$$

where each multi-index is represented by a vector. Likewise, the set E_3 is given by

$$\Big\{\Big(\binom{1}{0},\binom{1}{0},\binom{0}{1};\binom{1}{0},\binom{0}{2},\binom{2}{0}\Big),\Big(\binom{1}{0},\binom{1}{0},\binom{0}{1};\binom{0}{1},\binom{1}{1},\binom{2}{0}\Big),$$
$$\Big(\binom{1}{0},\binom{1}{0},\binom{0}{1};\binom{0}{1},\binom{1}{0},\binom{2}{1}\Big),\Big(\binom{1}{0},\binom{0}{1},\binom{1}{0};\binom{1}{0},\binom{0}{2},\binom{2}{0}\Big),$$
$$\Big(\binom{1}{0},\binom{0}{1},\binom{1}{0};\binom{0}{1},\binom{1}{1},\binom{2}{0}\Big),\Big(\binom{1}{0},\binom{0}{1},\binom{1}{0};\binom{0}{1},\binom{1}{0},\binom{2}{1}\Big),$$
$$\Big(\binom{0}{1},\binom{1}{0},\binom{1}{0};\binom{1}{0},\binom{0}{2},\binom{2}{0}\Big),\Big(\binom{0}{1},\binom{1}{0},\binom{1}{0};\binom{0}{1},\binom{1}{1},\binom{2}{0}\Big),$$
$$\Big(\binom{0}{1},\binom{1}{0},\binom{1}{0};\binom{0}{1},\binom{1}{0},\binom{2}{1}\Big)\Big\},$$

whereas $E_1 = E_4 = E_5 = \emptyset$. Hence $\boldsymbol{S}^{\boldsymbol{\sigma}}_{\boldsymbol{\beta}} = 12\big(\frac{1}{6} + 8(\frac{1}{2}) + 4(\frac{1}{4}) + \frac{1}{12} + 1\big) = 75$.

3.2 Orlicz-Sobolev spaces of higher order

Let $u : X \subset \mathbb{R}^N \to \mathbb{R}$ be a real-valued measurable function on an open set. The weak (distributional) derivative of order $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N) \in \mathbb{N}^N$ of u is given by

$$D^{\boldsymbol{\beta}}_x u = \frac{\partial^{|\boldsymbol{\beta}|} u}{\partial x^{\boldsymbol{\beta}}} = \frac{\partial^{|\boldsymbol{\beta}|} u}{\partial x_1^{\beta_1} \cdots \partial x_N^{\beta_N}},$$

with the usual convention that the derivative of order zero is the identity operator: $\partial^0 u/\partial x_j^0 = u$, for any index $j = 1, \ldots, N$. If $u_1, \ldots, u_N$ are real-valued and measurable in a common domain, then we write

$$\mathbf{D}^{\boldsymbol{\beta}}_x(u_1, \ldots, u_N)(x) = (D^{\boldsymbol{\beta}}_x u_1(x), \ldots, D^{\boldsymbol{\beta}}_x u_N(x)). \tag{8}$$

Let $m \geq 0$ be an integer. The collection

$$W^m L_\Phi(X) = \{u \in L_\Phi(X) : D^{\boldsymbol{\beta}}_x u \in L_\Phi(X), |\boldsymbol{\beta}| \leq m, \boldsymbol{\beta} \in \mathbb{N}^N\} \tag{9}$$

is the Orlicz-Sobolev space of order m. This space is Banach when equipped with the norm

$$\|u\|_{m,\Phi} = \sum_{|\boldsymbol{\beta}| \leq m} \|D^{\boldsymbol{\beta}}_x u\|_\Phi. \tag{10}$$

Evidently, $W^0 L_\Phi(X) = L_\Phi(X)$. Likewise, the set of functions

$$W^m E_\Phi(X) = \{u \in L_\Phi(X) : D^{\boldsymbol{\beta}}_x u \in E_\Phi(X), |\boldsymbol{\beta}| \leq m, \boldsymbol{\beta} \in \mathbb{N}^N\}$$

is a closed subspace of $W^m L_\Phi(X)$ and hence, is also a Banach space with the same norm (10). The space $W^m E_\Phi(X)$ is separable, while $W^m L_\Phi(X)$ is not separable in general. Obviously, in this case as well, $W^0 E_\Phi(X) = E_\Phi(X)$. If Φ satisfies a Δ_2-condition then $W^m E_\Phi(X) = W^m L_\Phi(X)$ is reflexive. The space (9) is always identified with a subspace of the product $\Pi_{|\alpha| \leq m} L_\Phi(X)$. It is not our aim to examine the structure of Orlicz-Sobolev spaces in this note. For further details we refer the reader to [1, 10, 16].

4. Local fractal functions

Let $X \subset \mathbb{R}^N$ (with $N \geq 1$) be a nonempty connected and bounded subset and let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function with the integral representation (3). We will assume that there exist $p_\Phi, q_\Phi > 0$ such that

$$p_\Phi \leq \frac{t\,\phi(t)}{\Phi(t)} \leq q_\Phi < \infty, \quad \text{for} \quad t \neq 0. \tag{11}$$

These numbers play a role in the characterization of Orlicz and Orliz-Sobolev spaces [3, 13]. For example, estimates (11) ensure that Φ satisfies a global Δ_2-condition [1]. By [9, Lemma 2.5], the complementary $\overline{\Phi}$ satisfies a global Δ_2-condition if and only if $p_\Phi > 1$, etc.

Lemma 4.1 ([9]). *Let ρ, t be nonnegative real numbers. Then*

$$\min\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\,\Phi(t) \leq \Phi(\rho t) \leq \max\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\,\Phi(t).$$

In the sequel, we fix an integer $m \geq 0$. Let $\{X_i\}_{i=1}^n$ be a family of nonempty and connected subsets of X and $\{\alpha_i : X_i \to X\}_{i=1}^n$ be a collection of diffeomorphisms fulfilling the condition $\alpha_i(X_i) \cap \alpha_{i'}(X_{i'}) = \emptyset$ for $i \neq i'$, and $X = \dot{\bigcup}_{i=1}^n \alpha_i(X_i)$. Such a family is called a local parametrization of X and each element α_i is called a local coordinate. We will assume that the local coordinates are uniformly bounded up to the order m :

$$b_i(m) := \max\left\{1, \max_{\substack{1\leq j\leq N \\ |\boldsymbol{\theta}|\leq m}} \sup_{x\in X_i} |D_x^{\boldsymbol{\theta}}(\alpha_i^{-1})_j(x)|\right\} < \infty,$$

where y parametrizes the source space of α_i and $(\alpha_i^{-1})_j = y_j \circ \alpha_i^{-1}$ is the j^{th} component. (This would happen, e.g. if $\alpha_i^{-1}(x) = Ax$, with A nonsingular:

$$b_i(m) = \max\left\{1, \max_{1\leq j\leq N} \sup_{x\in X_i} |A_{j\bullet}x|\right\} < \infty,$$

where $A_{j\bullet}$ is the j^{th} row). Finally, let $\{\lambda_i : X_i \to \mathbb{R}\}_{i=1}^n \subset W^m L_\Phi(X)$, and let $\{R_i\}_{i=1}^n$ be a collection of functions with $R_i \in (L^\infty(X_i), \|\cdot\|_{i,\infty})$. Define Read-Bajraktarević operator $\mathbf{T} : W^m L_\Phi(X) \to \mathbb{R}^X$,

$$\mathbf{T}u(x) = \sum_{i=1}^n (\lambda_i \circ \alpha_i^{-1})(x)\mathbb{1}_{\alpha_i(X_i)}(x) + \sum_{i=1}^n (R_i \circ \alpha_i^{-1})(x)(u|_{X_i} \circ \alpha_i^{-1})(x)\mathbb{1}_{\alpha_i(X_i)}(x), \tag{12}$$

where $\mathbb{1}_{\alpha_i(X_i)}$ is the characteristic function of $\alpha_i(X_i)$ (i.e., $\mathbb{1}_{\alpha_i(X_i)}(x) = 1$ if $x \in \alpha_i(X_i)$ and it is zero otherwise), and $u|_{X_i}$ is the restriction of $u \in W^m L_\Phi(X)$ to X_i. Note that for any fixed point u^* of $\mathbf{T}$, $u^* \circ \alpha_i = \lambda_i + R_i\, u^*|_{X_i}$, $i = 1, \ldots, n$.

A *local fractal function of the Orlicz-Sobolev class* $W^m L_\Phi(X)$ is a fixed point $u^* \in W^m L_\Phi(X)$ of the operator (12), whose graph $G(u^*) = \{(x, u^*(x)) : x \in X\}$ is the attractor of a contractive local IFS in N dimensions. The *realization* problem consists in the construction of the corresponding local IFS. This question was examined previously in [2] under a few additional mild hypotheses (convexity of X_i, etc.) In this note we treat the (in some sense) converse problem on the existence of local fractal functions of the Orlicz-Sobolev class $W^m L_\Phi(X)$.

4.1 Local fractal functions of higher order

We introduce positive constants

$$a_i = \sup_{x \in X_i} |\det \mathrm{J}_x \alpha_i|, \quad i = 1, \dots, n, \tag{13}$$

where $\mathrm{J}_x\alpha_i$ is the Jacobian matrix of the transformation α_i at the point $x \in X_i$.

LEMMA 4.2. *The Read-Bajraktarević operator is well defined on the Orlicz space $L_\Phi(X)$ and sends this space into itself. Moreover, write $r_i = \max\{1, \|R_i\|_{i,\infty}\}$, $i = 1, \dots, n$. If the sum $\sum_{i=1}^n a_i r_i^{q_\Phi} < 1$, then (12) is a contraction on $L_\Phi(X)$.*

Proof. Since the α_i's are diffeomorphisms and $\lambda_i \in L_\Phi(X)$ for $i = 1, \dots, n$, the operator (12) is well defined and $\mathbf{T}(L_\Phi(X)) \subseteq L_\Phi(X)$. Let $\tau > 0$. After changing the coordinate $y = \alpha_i^{-1}(x)$ and subsequent re-labeling $x \mapsto y$, Lemma 4.1 implies

$$\begin{aligned}
&\int_X \Phi\Big(\frac{1}{\tau}|\mathbf{T}u(x) - \mathbf{T}v(x)|\Big)dx \\
&\le \sum_{i=1}^n \int_{\alpha_i(X_i)} \Phi\Big(\frac{1}{\tau}|R_i \circ \alpha_i^{-1}(x)|\, \big|u|_{X_i} \circ \alpha_i^{-1}(x) - v|_{X_i} \circ \alpha_i^{-1}(x)\big|\Big)dx \\
&\le \sum_{i=1}^n \int_{X_i} \Phi\Big(\frac{r_i}{\tau}\big|u|_{X_i}(x) - v|_{X_i}(x)\big|\Big)|\det \mathrm{J}_x\alpha_i|dx \\
&\le \Big(\sum_{i=1}^n a_i r_i^{q_\Phi}\Big) \int_X \Phi\Big(\frac{1}{\tau}|u(x) - v(x)|\Big)dx.
\end{aligned}$$

The definition of the Luxemburg norm yields the desired conclusion. □

THEOREM 4.3. *Let $R_i(x) = c_i \in \mathbb{R}$ for any $x \in X_i$ and $i = 1, \dots, n$. Then the Read-Bajraktarević operator is well defined on the Orlicz-Sobolev space $W^m L_\Phi(X)$, and sends this space into itself. In addition, assume that $q_\Phi > 1$ and*

$$M := \sum_{i=1}^n a_i r_i^{q_\Phi} (b_i(m))^{m q_\Phi} < \frac{1}{(N_m S_m)^{q_\Phi}}, \tag{14}$$

where $r_i = \max\{1, |c_i|\}$, and N_m is the number defined in (5). Then the operator (12) is a contraction on $W^m L_\Phi(X)$.

Proof. Let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_N)$ be a multi-index with $|\boldsymbol{\beta}| \le m$. The case $m = 0$ is covered in Lemma 4.2. We thus assume that $1 \le |\boldsymbol{\beta}| \le m$. Since $\lambda_i \in W^m L_\Phi(X)$ for all $i = 1, \dots, n$, the operator (12) is well defined and sends this Orlicz-Sobolev space into itself. Choose $u, v \in W^m L_\Phi(X)$ and, for every index i, form the difference $g_i = u|_{X_i} - v|_{X_i}$. Then $g_i \circ \alpha_i^{-1}$ is $D_x^{\boldsymbol{\beta}}$-differentiable and the (weak) derivative of the composite is obtained via the generalized Faà di Bruno formula [8, 11],

$$D_x^{\boldsymbol{\beta}}(g_i \circ \alpha_i^{-1})(x) = \sum_{|\boldsymbol{\sigma}|=1}^{|\boldsymbol{\beta}|} D_y^{\boldsymbol{\sigma}} g_i(y) \sum_{k=1}^{|\boldsymbol{\beta}|} \sum_{E_k(\boldsymbol{\beta},\boldsymbol{\sigma})} \boldsymbol{\beta}! \prod_{j=1}^k \frac{\big(\mathbf{D}_x^{\boldsymbol{\theta}_j}(\alpha_i^{-1})(x)\big)^{\boldsymbol{\mu}_j}}{\boldsymbol{\mu}_j!(\boldsymbol{\theta}_j!)^{|\boldsymbol{\mu}_j|}},$$

where $\mathbf{D}_x^{\boldsymbol{\theta}_j}(\alpha_i^{-1})(x) = \big(D_x^{\boldsymbol{\theta}_j}(\alpha_i^{-1})_1(x), \ldots, D_x^{\boldsymbol{\theta}_j}(\alpha_i^{-1})_N(x)\big)$ is the vector derivative (8) and $y = \alpha_i^{-1}(x)$ parametrizes the source space of α_i. Therefore,

$$|D_x^{\boldsymbol{\beta}}(g_i \circ \alpha_i^{-1})(x)| \leq \sum_{|\boldsymbol{\sigma}|=1}^{|\boldsymbol{\beta}|} |D_y^{\boldsymbol{\sigma}} g_i(y)| \big(b_i(m)\big)^{|\boldsymbol{\sigma}|} \boldsymbol{S}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}} \leq \big(b_i(m)\big)^m S_m \sum_{|\boldsymbol{\sigma}|=1}^{m} |D_y^{\boldsymbol{\sigma}} g_i(y)|,$$

where $\boldsymbol{S}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}}$ is the multivariate Stirling number (6). Hence, if $\tau > 0$,

$$\begin{aligned}\int_X \Phi\Big(\frac{1}{\tau}\big|D_x^{\boldsymbol{\beta}}\mathbf{T}u(x) - D_x^{\boldsymbol{\beta}}\mathbf{T}v(x)\big|\Big)dx &\leq \sum_{i=1}^{n} \int_{\alpha_i(X_i)} \Phi\Big(\frac{r_i}{\tau}\big|D_x^{\boldsymbol{\beta}}(g_i \circ \alpha_i^{-1}(x))\big|\Big)dx \\ &\leq \sum_{i=1}^{n} \int_{\alpha_i(X_i)} \Phi\Big(\frac{r_i}{\tau}\big(b_i(m)\big)^m S_m \sum_{|\boldsymbol{\sigma}|=1}^{m} |D_y^{\boldsymbol{\sigma}} g_i(y)|\Big)dx,\end{aligned}$$

for every $i = 1, \ldots, n$. Since Φ is convex the right-hand side above is indeed bounded by

$$\frac{1}{N_m} \sum_{|\boldsymbol{\sigma}|=1}^{m} \sum_{i=1}^{n} \int_{\alpha_i(X_i)} \Phi\Big(\frac{r_i}{\tau} N_m \big(b_i(m)\big)^m S_m \big|(D_y^{\boldsymbol{\sigma}} g_i)\big(\alpha_i^{-1}(x)\big)\big|\Big)dx.$$

Inasmuch as $q_\Phi > 1$, the change of the coordinate $y = \alpha_i^{-1}(x)$ (and subsequent relabeling $x \mapsto y$) and an application of Lemma 4.1 yield

$$\int_X \Phi\Big(\frac{1}{\tau}|D_x^{\boldsymbol{\beta}}\mathbf{T}u(x) - D_x^{\boldsymbol{\beta}}\mathbf{T}v(x)|\Big)dx \leq M N_m^{q_\Phi - 1} S_m^{q_\Phi} \sum_{|\boldsymbol{\sigma}|=1}^{m} \int_X \Phi\Big(\frac{1}{\tau}\big|D_x^{\boldsymbol{\sigma}} u(x) - D_x^{\boldsymbol{\sigma}} v(x)\big|\Big)dx.$$

The definition of the Luxemburg norm thus entails

$$\|D_x^{\boldsymbol{\beta}}\mathbf{T}u - D_x^{\boldsymbol{\beta}}\mathbf{T}v\|_\Phi \leq M N_m^{q_\Phi - 1} S_m^{q_\Phi} \sum_{|\boldsymbol{\sigma}|=1}^{m} \|D_x^{\boldsymbol{\sigma}} u - D_x^{\boldsymbol{\sigma}} v\|_\Phi,$$

and hence $\|\mathbf{T}u - \mathbf{T}v\|_{m,\Phi} \leq M\big(N_m S_m)^{q_\Phi} \|u - v\|_{m,\Phi}$. The operator (12) is thus a contraction on $W^m L_\Phi(X)$. This concludes the proof. □

Nota

Observe that if $m = 1$ (and then $N_1 = N$ and $S_1 = 1$, as demonstrated in Section 3.1), the condition on the constant M reduces to $\sum_{i=1}^n a_i(b_i(1))^{q_\Phi} < 1/N^{q_\Phi}$, which corresponds to the case examined in [2]. Theorem 4.3 is thus a generalization of the results obtained by Massopust to higher orders, higher dimensions, and different function spaces.

4.2 Example

The case of the Sobolev space $W^{m,p}(0,1)$ with $1 < p \leq \infty$ is well documented in the literature. Let $\{X_i\}_{i=1}^n$ be a collection of nonempty open intervals of $X = (0,1)$ and let $\{x_1 < \cdots < x_{n-1}\}$ be a partition of the same interval. The local parametrization $\{\alpha_i\}_i$ of the domain X is so chosen that the local coordinate $\alpha_i : X_i \to [0,1]$ is a linear map with $\alpha_i(X_i) = (x_{i-1}, x_i)$, where $x_0 = 0$ and $x_n = 1$. The function R_i is a real constant r_i and $\lambda_i \in W^{m,p}(X_i)$, for every $i = 1, \ldots, n$. The induced Read-Bajraktarević operator $\mathbf{T} : W^{m,p}(0,1) \to \mathbb{R}^{(0,1)}$ is well defined and sends $W^{m,p}(0,1)$ into itself. If $d_i > 0$ denotes the ordinary derivative of α_i and if the following conditions are met,

$$\begin{cases} \max\limits_{k=0,1,\ldots,m} \sum\limits_{i=1}^{n} \dfrac{|r_i|^p}{d_i^{kp-1}} < 1, & 1 \leq p < \infty; \\ \max\limits_{k=0,1,\ldots,m} \sum\limits_{i=1}^{n} \dfrac{|r_i|}{d_i^{k}} < 1, & p = \infty, \end{cases} \tag{15}$$

then the Read-Bajraktarević operator is contractive on $W^{m,p}(0,1)$. The unique fixed point $u^* \in W^{m,p}(0,1)$ is called a local fractal function of class $W^{m,p}(0,1)$. Note that for values $1 < p < \infty$ this example itself results from Theorem 4.3 applied in one dimension with the Young function $\Phi(t) = t^p$, so that $p_\Phi = q_\Phi = p$ in (11), while $a_i = d_i$ in (13) and $b_i(m) = \max\{1, 1/d_i\}$. Note as well that our general requirement (14) is an improved (finer) version of (15) in the sense that, as per (5), its right-hand side reduces to $1/(mS_m)^p$ when $N = 1$.

4.3 Concluding remarks

In one dimension the conclusions in the example above hold as well under the assumption that each function α_i is either a smooth bounded diffeomorphism, or a bounded invertible real-analytic map, from X_i onto the semi-open interval $[x_{i-1}, x_i)$. It would be natural to extend our results to sets X that belong in sub-domains of differentiable and real analytic manifolds. The problem whether the Read-Bajraktarević operator may be extended to these categories of spaces is, to our knowledge, untreated. It seems reasonable that the operator (12) will exhibit particular symmetries in these cases, stemming from Schwartz reflections around the image of the real line, etc. These generalizations may be useful in other theoretical contexts. We look forward to addressing this problem in a future publication.

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