MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
77, 3 (2025), 201–209
September 2025

research paper
оригинални научни рад
DOI: 10.57016/MV-I390kbnd

SOME REVERSE INEQUALITIES INVOLVING TSALLIS OPERATOR ENTROPY

Nasrollah Goudarzi

Abstract. This paper establishes some upper and lower bounds for Tsallis operator entropy. Some applications are also given when A and B are bounded above and below by positive constants.

We also corrected the error in the article by H. R. Moradi et al., published in 2017.

1. Introduction and preliminaries

A capital letter means an operator on a Hilbert space $\mathscr{H}$. The $\mathbf{1}_{\mathscr{H}}$ symbol will denote the identity operator. An operator, X, is said to be strictly positive (denoted by $X > 0$) if X is positive and invertible. For two strictly positive operators A, B and $p \in [0,1]$, the operator geometric mean $A \sharp_p B$ is specified by

$$A \sharp_p B := A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^p A^{\frac{1}{2}},$$

and we note that $A \sharp_p B = A^{1-p} B^p$ if A commutes with B. The weighted operator arithmetic mean asserts that $A \nabla_p B := (1-p) A + pB$, for any $p \in [0,1]$.

Let f be a continuous function defined on the interval I of real numbers, let B be a self-adjoint operator on the Hilbert space $\mathscr{H}$, and let A be a positive and invertible operator on $\mathscr{H}$. Assume that the spectrum $sp\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \subset \overset{o}{I}$ (the interior of I). Then, by employing the continuous functional calculus, we can define the perspective $\mathscr{P}_f\,(B|A)$ by appointing

$$\mathscr{P}_f\,(B|A) := A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}.$$

If A and B are commutative, then $\mathscr{P}_f\,(B|A) := Af\left(BA^{-1}\right)$ provided that $sp\left(BA^{-1}\right) \subset \overset{o}{I}$. For related works, the reader can refer to [10].

2020 Mathematics Subject Classification: 47A12, 47A30, 15A60

Keywords and phrases: Tsallis relative operator entropy; positive operator; relative operator entropy; geometric mean.

Tsallis relative operator entropy $T_p(A|B)$ in the paper by Yanagi-Kuriyama-Furuichi [11], for $p \in (0,1]$, is defined by

$$T_p(A|B) := \frac{A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p A^{\frac{1}{2}} - A}{p}.$$

Notice that $T_p(A|B)$ can be written by using the notation of $A\sharp_p B$ as follows:

$$T_p(A|B) := \frac{A\sharp_p B - A}{p},$$

for $p \in (0,1]$. The relative operator entropy $S(A|B)$ in [3] is defined by

$$S(A|B) = A^{\frac{1}{2}}\left(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

as an extension of [11]. The relation between $S(A|B)$ and $T_p(A|B)$ and $T_{-p}(A|B)$ was considered in [5] and the following inequalities was proved

$$\begin{gathered} T_{-p}(A|B) \leq S(A|B) \leq T_p(A|B) \\ A - AB^{-1}A \leq T_p(A|B) \leq B - A. \end{gathered} \tag{1}$$

For proofs and more facts about the Tsallis relative operator entropy, we refer the reader to [4, 6–8].

It has been shown in [2] that the following inequality for weighted arithmetic and harmonic operator means

$$\begin{aligned} &\left[p(1-p)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2+1\right]H_p(a,b) \leq A_p(a,b) \\ &\leq \left[p(1-p)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2+1\right]H_p(a,b), \end{aligned} \tag{2}$$

for any $a, b > 0$ and $p \in [0,1]$, where $A_p(a,b)$ and $H_p(a,b)$ are the scalar weighted arithmetic mean and harmonic mean, respectively, namely $A_p(a,b) = (1-p)a + pb$, $H_p(a,b) = \frac{ab}{(1-p)b+pa}$.

Motivated by the above points, we establish in this paper some inequalities for Tsallis relative operator entropy, relative operator entropy, and perspective of two operators under various assumptions for the positive invertible operators A and B.

2. Main results

Although the following result has been shown in [9, Theorem 2.1], we give proof for the reader's convenience. Notice that we aim to demonstrate that this theorem is also true under a different condition, shown in the subsequent remark.

THEOREM 2.1. *Let A, B be positive invertible operators and let $M > m > 0$ be such that $m^{-\frac{1}{p}}A \leq B \leq M^{-\frac{1}{p}}A$, where $0 < p \leq 1$. Then*

$$\frac{1}{-p^2}\left[\left(p\,(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left(A^{\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)^{-1}A^{\frac{1}{2}}\right)-A\right]\geq T_{-p}\left(A|B\right)$$
$$\geq\frac{1}{-p^2}\left[\left(p\,(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left(A^{\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)^{-1}A^{\frac{1}{2}}\right)-A\right]. \quad (3)$$

Proof. If we write the inequality (2) for $a=1$ and $b=x\in(0,\infty)$ then we have

$$\left[p\,(1-p)\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2+1\right]\times\left((1-p)+px^{-1}\right)^{-1}\leq(1-p)+px$$
$$\leq\left[p\,(1-p)\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2+1\right]\times\left((1-p)+px^{-1}\right)^{-1}, \quad (4)$$

for any $0<p\leq 1$. If $x\in[m,M]\subset(,\infty)$, then $\max\{m,1\}\leq\max\{x,1\}\leq\max\{M,1\}$ and $\min\{m,1\}\leq\min\{x,1\}\leq\min\{M,1\}$. We have

$$\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2\leq\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2$$

and

$$\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2\leq\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2$$

for any $x\in[m,M]\subset(0,\infty)$. Therefore by (4) we get

$$\left[p\,(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right]\left((1-p)+px^{-1}\right)^{-1}\leq(1-p)+px$$
$$\leq\left[p\,(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right]\left((1-p)+px^{-1}\right)^{-1}$$

or, equivalently, we have

$$\frac{1}{-p^2}\left[\left(p\,(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left((1-p)+px^{-1}\right)^{-1}-1\right]\geq\frac{(x-1)}{-p}$$
$$\geq\frac{1}{-p^2}\left[\left(p\,(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left((1-p)+px^{-1}\right)^{-1}-1\right] \quad (5)$$

for any $x\in[m,M]$ and any $0<p\leq 1$. If we use the continuous functional calculus for the positive invertible operator X with $m\mathbf{1}_{\mathscr{H}}\leq X\leq M\mathbf{1}_{\mathscr{H}}$, then we have from (5) that

$$\frac{1}{-p^2}\left[\left(p\,(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left((1-p)\,\mathbf{1}_{\mathscr{H}}+pX^{-1}\right)^{-1}-\mathbf{1}_{\mathscr{H}}\right]\geq\frac{(X-\mathbf{1}_{\mathscr{H}})}{-p}$$
$$\geq\frac{1}{-p^2}\left[\left(p\,(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left((1-p)\,\mathbf{1}_{\mathscr{H}}+pX^{-1}\right)^{-1}-\mathbf{1}_{\mathscr{H}}\right].$$

By writing $X = \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-p}$, for any $0 < p \leq 1$, we obtain

$$\frac{1}{-p^2}\left[\left(p\,(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left((1-p)\,\mathbf{1}_{\mathscr{H}}+p\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p\right)^{-1}-\mathbf{1}_{\mathscr{H}}\right]$$

$$\geq \frac{\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-p}-\mathbf{1}_{\mathscr{H}}}{-p} \tag{6}$$

$$\geq \frac{1}{-p^2}\left[\left(p\,(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left((1-p)\,\mathbf{1}_{\mathscr{H}}+p\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p\right)^{-1}-\mathbf{1}_{\mathscr{H}}\right].$$

If we multiply the inequality (6) both sides with $A^{\frac{1}{2}}$, then we get

$$\frac{1}{-p^2}\left[\left(p\,(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left(A^{\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)^{-1}A^{\frac{1}{2}}\right)-A\right]\geq T_{-p}\left(A|B\right)$$

$$\geq \frac{1}{-p^2}\left[\left(p\,(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left(A^{\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)^{-1}A^{\frac{1}{2}}\right)-A\right].$$

This completes the proof of Theorem 2.1, since

$$A^{-\frac{1}{2}}\left((1-p)\,A+pA^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^pA^{\frac{1}{2}}\right)A^{-\frac{1}{2}}$$
$$= A^{-\frac{1}{2}}\left((1-p)\,A+p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}} = A^{-\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}}. \qquad \square$$

A particular case of Theorem 2.1 can be seen as follows.

Remark 2.2. Let A, B positive invertible operators and let m, m', M, M' be such that the condition $0 < m\mathbf{1}_{\mathscr{H}} \leq A \leq m'\mathbf{1}_{\mathscr{H}} < M'\mathbf{1}_{\mathscr{H}} \leq B \leq M\mathbf{1}_{\mathscr{H}}$ holds. Put $h{:=}\frac{M}{m}$ and $h'{:=}\frac{M'}{m'}$, then we have $A < h'A = \frac{M'}{m}A \leq B \leq \frac{M}{m}A = hA$. By (3) we get

$$\frac{1}{-p^2}\left(p\,(1-p)\left(\frac{h'-1}{h'}\right)^2+1\right)\left(\left(A^{-\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}}\right)-A\right)\geq T_{-p}\left(A|B\right)$$

$$\geq \frac{1}{-p^2}\left(p\,(1-p)\,(h-1)^2+1\right)\left(\left(A^{-\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}}\right)-A\right) \tag{7}$$

for any $0{<}p{\leq}1$. Notice that if $0{<}m\mathbf{1}_{\mathscr{H}}{\leq}B{\leq}m'\mathbf{1}_{\mathscr{H}}{<}M'\mathbf{1}_{\mathscr{H}}{\leq}A{\leq}M\mathbf{1}_{\mathscr{H}}$ holds, then for $h{:=}\frac{M}{m}$ and $h'{:=}\frac{M'}{m'}$ we also $\frac{1}{h}A \leq B \leq \frac{1}{h'}A < A$. Finally, from (7) we get (3).

The following result is similar to the previous theorem but for the positive parameter.

Theorem 2.3. *Let A, B be positive invertible operators and let $M > m > 0$ such that $m^{\frac{1}{p}}A \leq B \leq M^{\frac{1}{p}}A$. Then, for $0 < p \leq 1$*

$$\frac{1}{p^2}\left[\left(p\,(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left(A^{-\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}}\right)-A\right]\leq T_p\left(A|B\right)$$

$$\leq \frac{1}{p^2}\left[\left(p\,(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left(A^{-\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}}\right)-A\right]. \tag{8}$$

Proof. If we write the inequality (2) for $a = 1$ and $b = x \in (0, \infty)$ then we have

$$\left[p(1-p)\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2+1\right]\times\left((1-p)+px^{-1}\right)^{-1}\leq(1-p)+px$$

$$\leq\left[p(1-p)\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2+1\right]\times\left((1-p)+px^{-1}\right)^{-1}. \tag{9}$$

for any $0 < p \leq 1$. If $x \in [m, M] \subset (,\infty)$, then $\max\{m,1\} \leq \max\{x,1\} \leq \max\{M,1\}$ and $\min\{m,1\} \leq \min\{x,1\} \leq \min\{M,1\}$. We have

$$\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2\leq\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2$$

and

$$\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2\leq\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2$$

for any $x \in [m, M] \subset (0, \infty)$. Therefore by (9), we get

$$\left[p(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right]\left((1-p)+px^{-1}\right)^{-1}\leq(1-p)+px$$

$$\leq\left[p(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right]\left((1-p)+px^{-1}\right)^{-1}$$

or, equivalently, we have

$$\frac{1}{p^2}\left[\left(p(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left((1-p)+px^{-1}\right)^{-1}-1\right]\leq\frac{(x-1)}{p}$$

$$\leq\frac{1}{p^2}\left[\left(p(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left((1-p)+px^{-1}\right)^{-1}-1\right] \tag{10}$$

for any $x \in [m, M]$ and any $0 < p \leq 1$. If we use the continuous functional calculus for the positive invertible operator X with $m\mathbf{1}_{\mathscr{H}} \leq X \leq M\mathbf{1}_{\mathscr{H}}$, then we have from (10) that for any $0 < p \leq 1$

$$\frac{1}{p^2}\left[\left(p(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left((1-p)\mathbf{1}_{\mathscr{H}}+pX^{-1}\right)^{-1}-\mathbf{1}_{\mathscr{H}}\right]\leq\frac{(X-\mathbf{1}_{\mathscr{H}})}{p}$$

$$\leq\frac{1}{p^2}\left[\left(p(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left((1-p)\mathbf{1}_{\mathscr{H}}+pX^{-1}\right)^{-1}-\mathbf{1}_{\mathscr{H}}\right].$$

By writing $X = \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p$, we obtain

$$\frac{1}{p^2}\left[\left(p(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left((1-p)\mathbf{1}_{\mathscr{H}}+p\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-p}\right)^{-1}-\mathbf{1}_{\mathscr{H}}\right]$$

$$\leq\frac{\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p-\mathbf{1}_{\mathscr{H}}}{p} \tag{11}$$

$$\leq \frac{1}{p^2}\left[\left(p\,(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left((1-p)\,\mathbf{1}_{\mathscr{H}}+p\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-p}\right)^{-1}-\mathbf{1}_{\mathscr{H}}\right]$$

for any $0 < p \leq 1$.

If we multiply the inequality (11) both sides with $A^{\frac{1}{2}}$, then we get

$$\frac{1}{p^2}\left[\left(p\,(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left(A^{-\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}}\right)-A\right]\leq T_p\left(A|B\right)$$

$$\leq \frac{1}{p^2}\left[\left(p\,(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left(A^{-\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}}\right)-A\right].$$

Since

$$A^{-\frac{1}{2}}\left((1-p)\,A+pA^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{p}A^{\frac{1}{2}}\right)A^{-\frac{1}{2}}$$

$$=A^{-\frac{1}{2}}\left((1-p)\,A+p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}}=A^{-\frac{1}{2}}\left(A\nabla_p\left(A\sharp_p B\right)\right)A^{-\frac{1}{2}}.$$

This completes the proof of Theorem 2.3. □

A particular case of Theorem 2.3 can be seen as follows.

REMARK 2.4. Let A, B positive invertible operators and positive real numbers m, m', M, M' such that the condition $0 < m\mathbf{1}_{\mathscr{H}} \leq A \leq m'\mathbf{1}_{\mathscr{H}} < M'\mathbf{1}_{\mathscr{H}} \leq B \leq M\mathbf{1}_{\mathscr{H}}$ holds. Put $h{:=}\frac{M}{m}$ and $h'{:=}\frac{M'}{m'}$, then we have $A < h'A = \frac{M'}{m}A \leq B \leq \frac{M}{m}A = hA$. By (8), we get

$$\frac{1}{p^2}\left[\left(p\,(1-p)\right)\left(\frac{h'-1}{h'}\right)^2+1\left(B^{\frac{1-p}{2}}\left(A^p !_p B^p\right)B^{\frac{1-p}{2}}-B\right)\right]\leq T_p\left(B|A\right)$$

$$\leq \frac{1}{p^2}\left[\left(p\,(1-p)\right)(h-1)^2+1\left(B^{\frac{1-p}{2}}\left(A^p !_p B^p\right)B^{\frac{1-p}{2}}-B\right)\right] \tag{12}$$

for any $0 < p \leq 1$. Notice that if $0 < m\mathbf{1}_{\mathscr{H}} \leq B \leq m'\mathbf{1}_{\mathscr{H}} < M'\mathbf{1}_{\mathscr{H}} \leq A \leq M\mathbf{1}_{\mathscr{H}}$ holds, then for $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$ we also $\frac{1}{h}A \leq B \leq \frac{1}{h'}A < A$. Finally, from (12), we get (8).

A new upper bound for the relative operator entropy is given in the following theorem. This estimate covers the previous bound (1) by letting $t = 1$.

THEOREM 2.5. *Let A, B be positive invertible operators. Then for any $t > 0$,*

$$S\left(A|B\right)\leq \ln tA - A + \frac{1}{t}B. \tag{13}$$

Proof. We use some ideas from [9, Theorem 2.4]. If f is convex and differentiable, then

$$f(x) \geq f(t) + (x-t)\,f'(t) \tag{14}$$

for any x on the domain of f and $t > 0$. Using the continuous functional calculus for a positive operator X, we have from (14) in the operator order that

$$f(X) \geq f(t)\,\mathbf{1}_{\mathscr{H}} + \left(X - t\mathbf{1}_{\mathscr{H}}\right) f'(t) \tag{15}$$

Now, if we take $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (15), then we get

$$f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\geq f(t)\,\mathbf{1}_{\mathscr{H}}+\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-t\mathbf{1}_{\mathscr{H}}\right)f'(t)$$

If we take $f(x) = -\ln x$ in (13), and then multiplying both sides with $A^{\frac{1}{2}}$ we deduce the desired result. $\square$

Note that the sign "$\geq$" should be reversed to "$\leq$" in [9, Corollary 2.2] because $f(t) = \frac{t^p-1}{p}$, $(0 < p \leq 1)$ is a concave function. Therefore, [9, Remark 2.5] isn't correct.

A new lower bound for Tsallis relative operator entropy is provided in the following theorem.

Theorem 2.6. *For any invertible positive operator A and B such that $A \leq B$, and $0 < p \leq 1$ we have*

$$A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + \mathbf{1}_{\mathscr{H}}}{2}\right)^{p-1} A^{-\frac{1}{2}}(B-A) \leq T_p(B|A). \tag{16}$$

Proof. Consider the function $f(t) = t^{p-1}$, $0 < p \leq 1$. It is readily to check that $f(t)$ is convex on $[1, \infty)$. We also have $\int_1^x t^{p-1}dt = \frac{x^p-1}{p}$. On the other hand, by utilizing the left-hand side of Hermite-Hadamard inequality, one can see that

$$\left(\frac{x+1}{2}\right)^{p-1}(x-1) \leq \frac{x^p-1}{p}, \tag{17}$$

where $x \in [1, \infty)$ and $p \in (0, 1]$.

By taking $x = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and then multiplying both sides by $A^{\frac{1}{2}}$ in (17) we deduce desired inequality (16). $\square$

Remark 2.7. Easy computations reveal that $1 - \frac{1}{x} \leq \left(\frac{x+1}{2}\right)^{p-1}(x-1)$, i.e. Theorem 2.6 provides an improvement for the first inequality in (1).

Theorem 2.8. *For any invertible positive operator A and B such that $A \leq B$, and $0 < p \leq 1$ we have $A\sharp_p B - A\sharp_{p-1}B \leq T_p(B|A)$.*

Proof. Assume that f is convex and differentiable. Whence

$$f(t) - f(s) + f'(t)s \leq f'(t)t. \tag{18}$$

By taking into account that $f(t) = -t^p$, $p \in [0, 1]$ is convex and $s = 1$ from (18) we infer $t^p - t^{p-1} \leq \frac{t^p-1}{p}$. We obtain the desired result using arguments similar to those in the proof of Theorem 2.6. $\square$

Remark 2.9. Define $g_p(t) = t^p - t^{p-1} - 1 + \frac{1}{t}$. This function is always positive, i.e. $1 - \frac{1}{t} \leq t^p - t^{p-1}$. Thus, we can conclude that Equation 19 improves the first inequality in (1).

We close this paper by presenting the lower and upper bound for $\mathscr{P}_f$.

Theorem 2.10. *Let A, B be two positive and invertible operators and let $f : [m, M] \to \mathbb{R}$ be a convex function. If $mA \leq B \leq MB$ for some scalars $0 < m < M$, then*

$$\mathscr{P}_f(A|B) \leq \frac{MA-B}{M-m}f(m) + \frac{B-mA}{M-m}f(M)$$

$$-\left(\frac{1}{2}A-\frac{1}{M-m}A^{\frac{1}{2}}\left|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-\frac{m+M}{2}\mathbf{1}_{\mathscr{H}}\right|A^{\frac{1}{2}}\right)\left(f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right)$$

and

$$\frac{MA-B}{M-m}f(m)+\frac{B-mA}{M-m}f(M)\leq\mathscr{P}_f(A|B)$$
$$+\left(\frac{1}{2}A+\frac{1}{M-m}A^{\frac{1}{2}}\left|\frac{M+m}{2}\mathbf{1}_{\mathscr{H}}-A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right|A^{\frac{1}{2}}\right)\left(f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right).$$

Proof. In [1] it has been proved that

$$\min\{p_1,p_2\}\left(f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right)\leq p_1f(x)+p_2f(y)-f(p_1x+p_2y), \quad (19)$$

and

$$p_1f(x)+p_2f(y)-f(p_1x+p_2y)\leq\max\{p_1,p_2\}\left(f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right), \quad (20)$$

for every convex function f on an interval I and $x,y\in I$, $p_1,p_2\in[0,1]$ such that $p_1+p_2=1$. Putting $p_1=\frac{t-m}{M-m}$, $p_2=\frac{M-t}{M-m}$, $x=M$ and $y=m$, in (19), it follows that

$$f(t)\leq\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)$$
$$-\left(\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{m+M}{2}\right|\right)\left(f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right) \quad (21)$$

due to $\min\left\{\frac{M-t}{M-m},\frac{t-m}{M-m}\right\}=\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{m+M}{2}\right|$. By (21), we infer that

$$f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\leq\frac{M\mathbf{1}_{\mathscr{H}}-A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{M-m}f(m)+\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-m\mathbf{1}_{\mathscr{H}}}{M-m}f(M)$$
$$-\left(\frac{1}{2}\mathbf{1}_{\mathscr{H}}-\frac{1}{M-m}\left|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-\frac{m+M}{2}\mathbf{1}_{\mathscr{H}}\right|\right)\left(f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right).$$

Multiplying both sides of the above inequality by $A^{\frac{1}{2}}$, gives

$$\mathscr{P}_f(A|B)\leq\frac{MA-B}{M-m}f(m)+\frac{B-mA}{M-m}f(M)$$
$$-\left(\frac{1}{2}A-\frac{1}{M-m}A^{\frac{1}{2}}\left|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-\frac{m+M}{2}\mathbf{1}_{\mathscr{H}}\right|A^{\frac{1}{2}}\right)\left(f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right).$$

On the other hand, it follows from (20) that

$$\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)$$
$$\leq f(t)+\left(\frac{1}{2}+\frac{1}{M-m}\left|\frac{M+m}{2}-t\right|\right)\left(f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right),$$

since $\max\left\{\frac{M-t}{M-m},\frac{t-m}{M-m}\right\}=\frac{1}{2}+\frac{1}{M-m}\left|\frac{M+m}{2}-t\right|$. Applying the same method as in the above, we get

$$\frac{MA-B}{M-m}f(m)+\frac{B-mA}{M-m}f(M) \le \mathscr{P}_f(A|B)$$
$$+\left(\frac{1}{2}A+\frac{1}{M-m}A^{\frac{1}{2}}\left|\frac{M+m}{2}\mathbf{1}_{\mathscr{H}}-A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right|A^{\frac{1}{2}}\right)\left(f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right),$$

as desired. □

References

[1] S. S. Dragomir, *Bounds for the normalised Jensen functional*, Bull. Austral. Math. Soc., **74** (2006), 471–478.

[2] S. S. Dragomir, *Multiplicative inequalities for weighted arithmetic and harmonic operator means*, Acta Univ Sapientiae Matem., **11(1)** (2019), 54–65.

[3] J. I. Fujii, E. Kamei, *Relative operator entropy in non-commutative information theory*, Math. Japon., **34** (1989), 341–348.

[4] S. Furuichi, H. R. Moradi, *Some remarks on Tsallis relative operator entropy*. Rev. Real Acad. Cienc. Exactas Fis. Nat. - A: Mat., **114(69)** (2020).

[5] S. Furuichi, K. Yanagi, K. Kuriyama, *A note on operator inequalities of Tsallis relative operator entropy*, Linear Algebra Appl., **407** (2005), 19–31.

[6] H. R. Moradi, S. Furuichi, A complementary inequality to the information monotonicity for Tsallis relative operator entropy, Linear Multilinear Algebra, **68(8)** (2020), 1557–1567.

[7] H. R. Moradi, S. Furuichi, N. Minculete, Estimates for Tsallis relative operator entropy, Math. Inequal. Appl., **20(4)** (2017), 1079–1088.

[8] H. R. Moradi, S. Furuichi, M. Sababheh, *Some operator inequalities via convexity*. Linear Multilinear Algebra., **70** (2022), 7740–7752.

[9] H. R. Moradi, M. Shah Hosseini, M. E. Omidvar, S. S. Dragomir, *Some lower and upper bounds for relative operator entropy*, U.P.B. Sci. Bull., Series A., **79(3)** (2017), 97–106.

[10] I. Nikoufar, *On operator inequalities of some relative operator entropies*, Adv. Math., **259** (2014), 376–383.

[11] K. Yanagi, K. Kuriyama, S. Furuichi, *Generalized Shannon inequalities based on Tsallis relative operator entropy*, Linear Algebra Appl., **394** (2005), 109–118.

(received 01.02.2023; in revised form 18.09.2023; available online 28.03.2025)

Department of Nuclear Engineering, Abadeh Branch, Islamic Azad University, Abadeh, Iran
E-mail: na.goudarzi@iau.ac.ir
ORCID iD: https://orcid.org/0009-0003-8733-1856