MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
77, 3 (2025), 189–200
September 2025

research paper
оригинални научни рад
DOI: 10.57016/MV-SBdK8647

WELL-POSEDNESS STUDY FOR SOLUTIONS TO NONLINEAR DEGENERATE PARABOLIC PROBLEMS WITH VARIABLE EXPONENT

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Abstract. The purpose of this article is to prove the existence and uniqueness of weak solutions for nonlinear parabolic problem whose model is

$$\begin{cases} \frac{\partial v}{\partial t} - \operatorname{div}\Big[|\nabla v - \Theta(v)|^{q(x)-2}(\nabla v - \Theta(v))\Big] + \beta(v) = f & \text{in} \quad Q_T := (0,T)\times\Omega, \\ v = 0 & \text{on} \quad \Sigma_T := (0,T)\times\partial\Omega, \\ v(\cdot,0) = v_0 & \text{in} \quad \Omega. \end{cases}$$

We transform the parabolic problem into the elliptic problem by using time discretization technique by Euler forward scheme and Rothe method combined with the theory of variable exponent Sobolev spaces.

1. Introduction

Our aim in this paper is to study the existence and uniqueness results of weak solutions for the following nonlinear parabolic problem

$$(P) \qquad \begin{cases} \frac{\partial v}{\partial t} - \operatorname{div}(\Phi(\nabla v - \Theta(v))) + \beta(v) = f & \text{in} \quad Q_T := (0,T)\times\Omega, \\ v = 0 & \text{on} \quad \Sigma_T := (0,T)\times\partial\Omega, \\ v(\cdot,0) = v_0 & \text{in} \quad \Omega. \end{cases}$$

$\Omega \subset \mathbb{R}^d (d \geq 3)$ is an open bounded domain with Lipschitz boundary $\partial\Omega; T$ is a fixed positive number; ∇v is the gradient of v and $\Phi(\xi) := |\xi|^{q(x)-2}\xi$, for all $\xi \in \mathbb{R}^d$ with $1 < q(x) < d$.

We consider the following hypotheses:

(H1) β is a non decreasing continuous real function on $\mathbb{R}$, surjective such that $\beta(0) = 0$ and $|\beta(x)| \leq M|x|$, where M is a positive constant.

2020 Mathematics Subject Classification: 35K55, 35A01, 35A02

Keywords and phrases: Nonlinear parabolic problem; existence; weak solution; variable exponent; semi-discretization; uniqueness; Rothe method.

(H2) $f \in L^{\infty}(Q_T)$ and $v_0 \in L^{\infty}(\Omega) \cap W^{1,q(x)}(\Omega)$.

(H3) Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^d$ and $\Theta(0) = 0$ such that $|\Theta(x) - \Theta(y)| \leq \lambda|x - y|$, for all $x, y \in \mathbb{R}$, and λ is a positive constant.

The study of variable exponents spaces appeared in the literature for the first time in 1931, in an article by Orlicz [21], but the field of variable exponent function spaces has witnessed an explosive growth in recent years. The developments in science lead to a period of intense study of variable exponent spaces. Also observed were problems related to modelling of so-called electrorheological fluids, the study of thermorheological fluids and image processing. For more general application of this kind of problem we refer the reader to [15, 18, 19, 24].

The problem (P) arises in various physical contexts like chemical heterogeneous catalysts, non-Newtonian fluids and as well as the theory of heat conduction in electrically conducting materials (see for example [6, 22, 24]). Here we shall mention one of them which are related to turbulent flows.

Model: Flow through a porous medium in a turbulent regime

This model is governed by the continuity equation

$$\frac{\partial \theta}{\partial t} + \operatorname{div} u = 0,$$

and Darcy's law

$$u = -K(\theta) \operatorname{grad} \phi(\theta),$$

where $\theta(x,t)$ is the volumetric moisture content, $k(\theta)$ is the hydraulic conductivity, and the total potential ϕ is given by $\phi(\theta) = \psi(\theta) + z$.

With $\psi(\theta)$ the hydrostatic potential and z the gravitational potential. In turbulent regimes, the flow rate is different from that which can be predicted by the Darcy law, and so several authors proposed a nonlinear relation between u and $K(\theta) \operatorname{grad} \phi$, $|u|^{p-2}u = -K(\theta) \operatorname{grad} \phi(\theta)$, $p > 2$.

If e denotes the unit vector in the vertical direction, we obtain

$$\frac{\partial \theta}{\partial t} - \operatorname{div}\left(|\nabla\varphi(\theta) - K(\theta)e|^{q-2}(\nabla\varphi(\theta) - K(\theta)e)\right) = 0,$$

where

$$\varphi(\theta) = \int_0^{\theta} K(s)\phi'(s)ds, q = \frac{p}{p-1}.$$

In the last years, the problem (P) or special cases of it has been extensively treated by many authors in elliptic or parabolic case, we invite the reader to see for example the works [1, 2, 4, 7, 10, 11, 16].

We recall that the Euler forward scheme has been used by several authors while studying time discretization of nonlinear parabolic problems, we refer for example to the works [9, 12, 18, 20, 23] for some details.

The advantage of our method is that we can not only obtain the existence and uniqueness of weak solutions to the problem (P), but also compute the numerical approximations. In the particular case when $\Theta = 0$, the author in [12] showed the existence and uniqueness of entropy solutions in Orlicz spaces by using our Rothe time-discretization method.

In [17], Ahmed Jamea proved the existence and uniqueness of weak solutions to nonlinear parabolic problems with variable exponent

$$\begin{cases} \frac{\partial u}{\partial t} - \triangle_{p(x)} u + \alpha(u) = f & \text{in } Q_T :=]0,T[\times\Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial \eta} = 0 & \text{on } \Sigma_T :=]0,T[\times\partial\Omega, \\ u(.,0) = u_0 & \text{in } \Omega. \end{cases}$$

His approach is based essentially on time discretization technique by Euler forward scheme and Rothe method.

In [3], the authors showed the existence, uniqueness and the stability questions of an entropy solution to nonlinear parabolic equations

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(\Phi(\nabla u - \Theta(u))) + |u|^{p(x)-2}u + \alpha(u) = \mu & \text{in } Q_T =]0,T[\times\Omega, \\ \Phi(\nabla u - \Theta(u)) \cdot \eta + \gamma(u) = g & \text{on } \Sigma_T =]0,T[\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases}$$

With diffuse Radon measure data that does not charge sets of zero $p(.)$-capacity by a time discretization technique and Rothe method.

This work is divided into five sections. In Section 1, we introduce the problem (P) and state the assumptions . In Section 2, we present some preliminary results and notations, also we state our main result. In Section 3 we discretize the problem (P) by the Euler forward scheme, we show the existence and uniqueness of weak solution for the discretized problems and we show some stabilty results. At the last section, we finish this work by treated the convergence and existence results for the problem (P), moreover we confirme the uniqueness of solution.

2. Preliminary results and notations

In this step, we recall some preliminary results and notations which will be used in the sequel of this work. As the problem (P) depends on the variable $q(x)$, we should use the Lebesgue and Sobolev spaces with variable exponents.

We consider the set

$$C^+(\bar\Omega) = \{q : \bar\Omega \to \mathbb{R}^+ : q \text{ is continuous and such that } 1 < q_- < q_+ < \infty\},$$

where $q_- = \min_{x\in\bar\Omega} q(x)$ and $q_+ = \max_{x\in\bar\Omega} q(x)$.

For $q(\cdot) \in C^+(\bar\Omega)$, we define the Lebesgue space with variable exponent $L^{q(\cdot)}(\Omega)$ by

$$L^{q(\cdot)}(\Omega) = \{v : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |v|^{q(x)} \mathrm{d}x < \infty\},$$

endowed with the Luxemburg norm

$$\|v\|_{q(x)} = \|v\|_{L^{q(\cdot)}(\Omega)} = \inf\{\nu > 0, \quad \int_\Omega \left|\frac{v(x)}{\nu}\right|^{q(\cdot)} dx \le 1\}.$$

The space $\left(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\right)$ is a reflexive, uniformly convex Banach space, and its dual space is isomorphic to $L^{q'(\cdot)}(\Omega)$, where $\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1$.

Proposition 2.1 ([13, Holder inequality]). *Let $q(\cdot)$, $q'(\cdot) \in C^+(\bar{\Omega})$ with $\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1$. Then, for any $v \in L^{q(\cdot)}(\Omega)$ and $w \in L^{q'(\cdot)}(\Omega)$ we have $\left|\int_\Omega v \cdot w dx\right| \leq \left(\frac{1}{q_-} + \frac{1}{q'_-}\right) \|v\|_{q(\cdot)} \|w\|_{q'(\cdot)}$.*

We also consider the function $\rho_{q(\cdot)} : L^{q(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{q(\cdot)}(v) = \rho_{L^{q(\cdot)}(\Omega)}(v) = \int_\Omega |v(x)|^{q(x)} dx.$$

The connection between $\rho_{q(\cdot)}$ and $\| \cdot \|_{q(\cdot)}$ is established by the next result.

Proposition 2.2 ([13]). *1. Let v an element of $L^{q(\cdot)}(\Omega)$ we have*

(i) $\|v\|_{q(\cdot)} < 1$ (respectively $>, = 1$) $\Leftrightarrow \rho_{q(\cdot)}(v) < 1$ (respectively $>, = 1$).

(ii) $\|v\|_{q(\cdot)} = a \Leftrightarrow \rho_{q(\cdot)}(v) = a$ (when $a \neq 0$).

(iii) If $\|v\|_{q(\cdot)} < 1$, then $\|v\|^{q+}_{q(\cdot)} \leq \rho_{q(\cdot)}(v) \leq \|v\|^{q-}_{q(\cdot)}$.

(iv) If $\|v\|_{q(\cdot)} > 1$, then $\|v\|^{q-}_{q(\cdot)} \leq \rho_{q(.)}(v) \leq \|v\|^{q+}_{q(\cdot)}$.

2. For a sequence $(v_n)_{n\in\mathbb{N}} \subset L^{q(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, the following statements are equivalent:

(i) $\lim_{n\to\infty} v_n = v$ in $L^{q(\cdot)}(\Omega)$.

(ii) $\lim_{n\to\infty} \rho_{q(\cdot)}(v_n - v) = 0$.

(iii) $v_n \to v$ in measure in Ω.

The variable exponent Sobolev space $W^{1,q(\cdot)}(\Omega)$ consists of all $v \in L^{q(\cdot)}(\Omega)$ such that the absolute value of gradient is in $L^{q(\cdot)}(\Omega)$. Let the norm $\|v\|_{1,p(\cdot)} = \|v\|_{q(\cdot)} + \|\nabla v\|_{q(\cdot)}$. Then $\left(W^{1,q(\cdot)}(\Omega), \| \cdot \|_{1,q(\cdot)}\right)$ is a separable and reflexive Banach space.

We assume it is a measurable function $q(\cdot) : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \exists C > 0 : |q(x) - q(y)| \leq \frac{C}{-\ln|x-y|}, & \text{for } |x-y| < \frac{1}{2} \\ 1 < \operatorname{ess}\inf_{x\in\Omega} q(x) \leq \operatorname{ess}\sup_{x\in\Omega} q(x) < N. \end{cases} \tag{1}$$

Proposition 2.3 ([14, $q(\cdot)$-Poincarè inequality]). *Let Ω be a bounded open set and let $q(\cdot) : \Omega \to [1, \infty)$ satisfy (1). Then, there exists a constant C depending only on $q(\cdot)$ and Ω, such that the inequality $\|v\|_{q(\cdot)} \leq C \|\nabla v\|_{q(\cdot)}$, holds for every $v \in W^{1,q(\cdot)}_0(\Omega)$.*

Proposition 2.4 ([8, Sobolev embedding]). . *Let Ω be a bounded open set, with a Lipschitz boundary and let $q(\cdot) : \Omega \to [1, \infty)$ satisfy (2.1). Then, we get the following continuous embedding: $W^{1,q(\cdot)}(\Omega) \hookrightarrow L^{q^*(\cdot)}(\Omega)$, where $q^*(\cdot) = \frac{Nq(\cdot)}{N-q(\cdot)}$.*

Lemma 2.5. *For $\xi, \eta \in \mathbb{R}^d$ and $1 < q < \infty$, we have $\frac{1}{q}|\xi|^q - \frac{1}{q}|\eta|^q \leq |\xi|^{q-2}\xi(\xi - \eta)$.*

Proof. We consider the function $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $x \mapsto x^q - qx + (q-1)$. We have $g(x) \geq \min_{y\in\mathbb{R}^+} g(y) = g(1) = 0$ for all $x \in \mathbb{R}^+$. Therefore, we take $x = \frac{|\eta|}{|\xi|}$(if $|\xi| = 0$, the result is obvious) in the inequality above to get the result of the lemma by using Cauchy-Schwarz inequality. □

Lemma 2.6 ([6]). *Let q, q' two reals numbers such that $q > 1, q' > 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$, we have* $\left||\xi|^{q-2}\xi - |\eta|^{q-2}\eta\right|^{q'} \leq C\{(\xi-\eta)\left(|\xi|^{q-2}\xi - |\eta|^{q-2}\eta\right)\}^{\frac{\alpha}{2}}\{|\xi|^q + |\eta|^q\}^{1-\frac{\alpha}{2}}$, $\forall \xi, \eta \in \mathbb{R}^d$, *where $\alpha = 2$ if $1 < q \leq 2$ and $\alpha = q'$ if $q \geq 2$.*

Remark 2.7. Hereinafter, $c_i, (i \in \mathbb{N})$ are positive constants independent of N.

Definition 2.8. A measurable function $v : Q_T \to \mathbb{R}$ is a weak solution to nonlinear parabolic problems (P) in Q_T if $v(.,0) = v_0$ in $\Omega, v \in C\left(0,T; L^2(\Omega)\right) \cap L^{q(x)}\left(0,T; W^{1,q(x)}(\Omega)\right)$, $\frac{\partial v}{\partial t} \in L^2(Q_T)$ and we have

$$\int_0^T\int_\Omega \frac{\partial v}{\partial t}\varphi dxdt + \int_0^T\int_\Omega \Phi(\nabla v - \Theta(v)).\nabla\varphi dxdt + \int_0^T\int_\Omega \beta(v)\varphi dxdt = \int_0^T\int_\Omega f\varphi dxdt, \quad \forall\varphi \in C^1(Q_T). \tag{2}$$

Here, we state our main result of this article.

Theorem 2.9. *Under the hypotheses (H1), (H2) and (H3) there exists a unique weak solution for the nonlinear parabolic problem (P).*

3. The semi-discrete problem and stability results

3.1 The semi-discrete problem

In this section, we discretize the problem (P) by Euler forward scheme and we study the questions of existence and uniqueness under the assumptions (H1), (H2) and (H3) to the following discretized problems

$$(P_n) \qquad \begin{cases} V_n - \tau \operatorname{div}\left(\Phi\left(\nabla V_n - \Theta\left(V_n\right)\right)\right) + \tau\beta\left(V_n\right) = \tau f_n + V_{n-1} & \text{in } \Omega, \\ V_n = 0 & \text{on } \partial\Omega, \\ V_0 = u_0 & \text{in } \Omega, \end{cases}$$

where $N\tau = T, 0 < \tau < 1$, $1 \leq n \leq N$, $t_n = n\tau$ and $f_n(\cdot) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(s,\cdot)ds$, in Ω.

A weak solution to the discretized problems (P_n) is a sequence $(V_n)_{0\leq n\leq N}$ such that $V_0 = v_0$ and V_n is defined by induction as a weak solution to the problem

$$\begin{cases} v - \tau \operatorname{div}\left(\Phi\left(\nabla v - \Theta\left(v\right)\right)\right) + \tau\beta\left(v\right) = \tau f_n + V_{n-1} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

i.e. for $V_n \in L^\infty(\Omega) \cap W^{1,q(x)}(\Omega)$ and $\forall\varphi \in W^{1,q(x)}(\Omega), \forall\tau > 0$, we have

$$\int_\Omega V_n\varphi dx + \tau\int_\Omega \Phi(\nabla V_n - \Theta(V_n)).\nabla\varphi dx + \tau\int_\Omega \beta\left(V_n\right)\varphi dx = \int_\Omega \left(\tau f_n + V_{n-1}\right)\varphi dx. \tag{4}$$

Theorem 3.1. *Under the hypotheses (H1), (H2) and (H3) the problem (P_n) has a unique weak solution $(V_n)_{0\leq n\leq N}$ and for all $n = 1, \ldots, N, V_n \in L^\infty(\Omega) \cap W^{1,q(x)}(\Omega)$.*

Proof. For $n = 1$, we denote by $V = V_1$, we rewrite the problem 3 as

$$\begin{cases} -\tau \operatorname{div}(\Phi(\nabla V - \Theta(V))) + \bar{\beta}(V) = F & \text{in} \quad \Omega, \\ V = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5}$$

By the hypothese (H2), the function $F = \tau f_1 + u_0$ is an element of $L^\infty(\Omega)$ and the function $\bar{\beta}(s) = \tau\beta(s) + s$ is a non decreasing continuous real function on $\mathbb{R}$ surjective such that $\bar{\beta}(0) = 0$. Therefore, thanks to [5], the problem (5) has a unique weak solution V_1 in $L^\infty(\Omega) \cap W^{1,q(x)}(\Omega)$.

By induction, we deduce by the same manner that the problem (P_n) has a unique weak solution $(V_n)_{0\le n\le N}$ such that $n = 1, \dots, N, V_n \in L^\infty(\Omega) \cap W^{1,q(x)}(\Omega)$. □

3.2 Stability results

In this section, we show some a priori estimates for the discrete weak solution $(V_n)_{1\le n\le N}$ which we use later to derive convergence results for the Euler forward scheme.

Theorem 3.2. *Under the hypotheses (H1), (H2) and (H3) there exists a positive constant $C(v_0, f, F)$ depending on the data but not on N such that for all $n = 1, \dots, N$, we have*

$$\|V_n\|_\infty \le C(v_0, f, F), \tag{6}$$

$$\sum_{i=1}^{n} \|V_i - V_{i-1}\|_2^2 \le C(v_0, f, F), \tag{7}$$

$$\tau \sum_{i=1}^{n} \int_\Omega \Phi(\nabla V_i - \Theta(V_i)).\nabla V_i dx \le C(v_0, f, F). \tag{8}$$

Proof. **For** (6). Let $k > 0$ and $1 \le n \le N$, we have $V_n \in L^\infty(\Omega)$. Then, multiplying (P_n) by $|V_n|^k V_n$ and integrating over Ω, we obtain that

$$\int_\Omega |V_n|^{k+2} dx - \tau \int_\Omega \operatorname{div}(\Phi(\nabla V_n - \Theta(V_n))) |V_n|^k V_n dx + \tau \int_\Omega \beta(V_n) |V_n|^k V_n dx$$
$$= \int_\Omega (\tau f_n + V_{n-1}) |V_n|^k V_n dx.$$

By Holder's inequality, (H1), (H2) and (H3), using also $\Phi(\nabla V_i - \Theta(V_i)).\nabla V_i$ is monotone, we obtain

$$\|V_n\|_{k+2}^{k+2} \le \tau c_1 \|V_n\|_{k+1}^{k+1} + \|V_{n-1}\|_{k+2} \|V_n\|_{k+2}^{k+1}.$$

Hence
$$\|V_n\|_{k+2} \le \tau c_1 \|V_n\|_{k+1}^{k+1} + \|V_{n-1}\|_{k+2}.$$

By using simple induction, we have that $\|V_n\|_{k+2} \le Nc_2T + \|V_0\|_{k+2}$. Lastly, as $k \to \infty$, we get the result (6).

For (7). Let $1 \le i \le N$, replacing φ by V_i as test function in (4), we get that

$$\int_\Omega (V_i - V_{i-1}) V_i dx + \tau \int_\Omega \Phi(\nabla V_i - \Theta(V_i)).\nabla V_i dx + \tau \int_\Omega \beta(V_i) V_i dx = \int_\Omega \tau f_i V_i dx. \tag{9}$$

By the elementary identity, $a(a-b)=\frac{1}{2}a^2-\frac{1}{2}b^2+\frac{1}{2}(a-b)^2$, we obtain from (9) that

$$\frac{1}{2}\left\|V_i\right\|_2^2-\frac{1}{2}\left\|V_{i-1}\right\|_2^2+\frac{1}{2}\left\|V_i-V_{i-1}\right\|_2^2+\tau\int_\Omega \Phi(\nabla V_i-\Theta(V_i)).\nabla V_i dx\leq \tau c_3\left\|V_i\right\|_2. \quad (10)$$

Taking the sum (10) from $i=1$ to n to obtain

$$\frac{1}{2}\left\|V_n\right\|_2^2-\frac{1}{2}\left\|V_0\right\|_2^2+\frac{1}{2}\sum_{i=1}^n\left\|V_i-V_{i-1}\right\|_2^2+\tau\sum_{i=1}^n\int_\Omega \Phi(\nabla V_i-\Theta(V_i)).\nabla V_i dx\leq c_4.$$

Thus, $$\frac{1}{2}\sum_{i=1}^n\left\|V_i-V_{i-1}\right\|_2^2+\tau\sum_{i=1}^n\int_\Omega \Phi(\nabla V_i-\Theta(V_i)).\nabla V_i dx\leq c_4+\frac{1}{2}\left\|V_0\right\|_2^2.$$

So, $$\frac{1}{2}\sum_{i=1}^n\left\|V_i-V_{i-1}\right\|_2^2+\tau\sum_{i=1}^n\int_\Omega \Phi(\nabla V_i-\Theta(V_i)).\nabla V_i dx\leq c_5. \quad (11)$$

Hence, $$\frac{1}{2}\sum_{i=1}^n\left\|V_i-V_{i-1}\right\|_2^2\leq c_5.$$

This implies the stability result (7).

For (8). In view of (11) and (7), we get the stability result (8). □

THEOREM 3.3. *Let the hypotheses (H1), (H2) and (H3) hold. Then, there exists a positive constant $C(v_0,f,F)$ depending on the data but not on N such that for all $n=1,\dots,N$, we have*

$$\tau\sum_{i=1}^n\|\beta(V_i)\|_1\leq C(v_0,f,F), \quad (12)$$

$$\lim_{k\to 0}\sum_{i=1}^n\frac{\tau}{k}\int_{\{|V_i|\leq k\}}\Phi(\nabla V_i-\Theta(V_i)).\nabla V_i\leq C(v_0,f,F), \quad (13)$$

$$\sum_{i=1}^n\|V_i-V_{i-1}\|_1\leq C(v_0,f,F). \quad (14)$$

Proof. **For** (12) **and** (13). Given a constant $k>0$, we define the cut function $T_k:\mathbb{R}\to\mathbb{R}$ as

$$T_k(s):=\begin{cases} s & \text{if} \quad |s|\leq k,\\ k\quad \text{sign}(s) & \text{if}\quad |s|>k,\end{cases}$$

where $$\text{sign}(s):=\begin{cases} 1 & \text{if}\quad s>0,\\ 0 & \text{if}\quad s=0,\\ -1 & \text{if}\quad s<0.\end{cases}$$

Replacing φ by $T_k(V_i)$ as test function in (4), and dividing this equality by k, taking limits when k goes to 0, we get that

$$\left\|V_i\right\|_1+\tau\left\|\beta\left(V_i\right)\right\|_1+\lim_{k\to 0}\frac{\tau}{k}\int_{\{|V_i|\leq k\}}\Phi(\nabla V_i-\Theta(V_i)).\nabla V_i\leq\tau\left\|f_i\right\|_1+\left\|V_{i-1}\right\|_1. \quad (15)$$

Summing (15) from $i = 1$ to n, we conclude the stability results (12) and (13).

For (14). Replacing φ by $T_\tau(V_i - V_{i-1})$ in (3.2) and dividing this equality by τ we have

$$\int_\Omega (V_i - V_{i-1}) \frac{T_\tau(V_i - V_{i-1})}{\tau} dx + \int_{B_\tau^i} \Phi(\nabla V_i - \Theta(V_i)).(\nabla V_i - \nabla V_{i-1}) dx$$
$$\leq \tau \|\beta(V_i)\|_1 + \tau \|f_i\|_1 ,$$

where $B_\tau^i = \{|V_i - V_{i-1}| \leq \tau\}$. By applying Lemma 2.5, we obtain

$$\frac{1}{q(x)}|\nabla V_i - \theta(V_i)|^{q(x)} - \frac{1}{q(x)}|\nabla V_{i-1} - \theta(V_i)|^{q(x)}$$
$$\leq |\nabla V_i - \theta(V_i)|^{q(x)-2}(\nabla V_i - \theta(V_i)).(\nabla V_i - \nabla V_{i-1}).$$

So

$$\int_\Omega (V_i - V_{i-1}) \frac{T_\tau(V_i - V_{i-1})}{\tau} dx + \int_{B_\tau^i} (\frac{1}{q(x)}|\nabla V_i - \theta(V_i)|^{q(x)} - \frac{1}{q(x)}|\nabla V_{i-1} - \theta(V_i)|^{q(x)}) dx$$
$$\leq \tau \|\beta(V_i)\|_1 + \tau \|f_i\|_1 .$$

Summing the inequality above from $i = 1$ to n, using to the stability result (12), we obtain

$$\sum_{i=1}^{n} \int_\Omega (V_i - V_{i-1}) \frac{T_\tau(V_i - V_{i-1})}{\tau} dx \leq \frac{1}{q(x)} \int_\Omega |\nabla V_0|^{q(x)} dx + c_6$$
$$\leq \frac{1}{q_-} \int_\Omega |\nabla V_0|^{q(x)} dx + c_6.$$

Then, we let τ tends to 0 in the inequality above, we deduce the stability result (14). □

4. Convergence and existence results

In this section and from the above results, we build a weak solution of problem (P) and we show that this solution is unique.

4.1 Proof of existence

Let us introduce a piecewise linear extension, called Rothe function by

$$\begin{cases} v_N(0) := v_0 \\ v_N(t) := V_{n-1} + (V_n - V_{n-1}) \frac{(t - t_{n-1})}{\tau}, \forall t \in \left] t_{n-1}, t_n \right], n = 1, \dots, N \quad \text{in} \quad \Omega, \end{cases}$$

and a piecewise constant function

$$\begin{cases} \bar{v}_N(0) := v_0 \\ \bar{v}_N(t) := V_n \quad \forall t \in]\, t_{n-1}, t_n], \quad n = 1, \dots, N \quad \text{in} \quad \Omega, \end{cases}$$

where $t_n := n\tau$. As already shown, for any $N \in \mathbb{N}$, the solution $(V_n)_{1 \leq n \leq N}$ of problems (P_n) is unique. Thus, v_N and $\bar{v}_N$ are uniquely defined and by construction, for any $t \in]t_{n-1}, t_n]$, $n = 1, \ldots, N$, we have that

(i) $\frac{\partial v_N(t)}{\partial t} = \frac{(V_n - V_{n-1})}{\tau}$.

(ii) $\bar{v}_N(t) - v_N(t) = (V_n - V_{n-1}) \frac{t_n - t}{\tau}$.

From Theorem 3.2, for any $N \in \mathbb{N}$, the solution $(V_n)_{1 \le n \le N}$ of problems (3) is unique. Thus, v_N and $\bar{v}_N$ are uniquely defined.

By using the stability results of Theorem 3.3, we deduce the following a priori estimates concerning the Rothe function v_N and the function $\bar{v}_N$.

Lemma 4.1. *Under the hypotheses (H1), (H2) and (H3), there exists a positive constant $C(T, v_0, f, F)$ not depending on N such that for all $N \in \mathbb{N}$, we get*

$$\|\bar{v}_N - v_N\|^2_{L^2(Q_T)} \le \frac{1}{N} C(T, v_0, f, F), \tag{16}$$

$$\|\bar{v}_N\|_{L^\infty(0,T,L^2(\Omega))} \le C(T, v_0, f, F), \tag{17}$$

$$\|v_N\|_{L^\infty(0,T,L^2(\Omega))} \le C(T, v_0, f, F), \tag{18}$$

$$\|\bar{v}_N\|_{L^{q(x)}\left(0,T,W^{1,q(x)}(\Omega)\right)} \le C(T, v_0, f, F), \tag{19}$$

$$\|\beta(\bar{v}_N)\|_{L^1(Q_T)} \le C(T, v_0, f, F), \tag{20}$$

$$\left\|\frac{\partial v_N}{\partial t}\right\|^2_{L^2(Q_T)} \le C(T, v_0, f, F). \tag{21}$$

Proof. **For** (16). We have

$$\begin{aligned}\|\bar{v}_N - v_N\|^2_{L^2(Q_T)} &= \int_0^T \int_\Omega |\bar{v}_N - v_N|^2 \, dxdt \\ &\le \sum_{i=1}^{i=N} \int_{t_{n-1}}^{t_n} \int_\Omega |V_n - V_{n-1}|^2 \left(\frac{t_n - t}{\tau}\right)^2 dxdt \le \frac{1}{N} C(T, v_0, f, F).\end{aligned}$$

We follow the same techniques used above to show the estimates (17)–(20).

For (21). We have for $n = 1, \ldots, N$ and $t \in (t_{n-1}, t_n]$ $\frac{\partial v_N(t)}{\partial t} = \frac{(V_n - V_{n-1})}{\tau}$.

Thanks to the result (14), we conclude the estimate (21). At last the proof of Lemma 4.1 is obtained. □

Now, using the two results (17) and (18) of Lemma 4.1, the sequences $(v_N)_{N\in\mathbb{N}}$ and $(\bar{v}_N)_{N\in\mathbb{N}}$ are uniformly bounded in $L^\infty\left(0, T, L^2(\Omega)\right)$. Therefore, there exists two elements v and w in $L^\infty\left(0, T, L^2(\Omega)\right)$ such that

$$\begin{aligned}\bar{v}_N &\to^* v \quad \text{in} \quad L^\infty\left(0, T, L^2(\Omega)\right), \\ v_N &\to^* w \quad \text{in} \quad L^\infty\left(0, T, L^2(\Omega)\right).\end{aligned}$$

Finally, from the result (16) of Lemma 4.1, it follows that $v \equiv w$.

Furthermore, by Lemma 4.1 and the hypotheses (H2), we get that

$$\begin{aligned}\frac{\partial v_N}{\partial t} &\to \frac{\partial v}{\partial t} \quad \text{in} \quad L^2(Q_T), \\ \bar{v}_N &\to v \quad \text{in} \quad L^{q(x)}\left(0, T, W^{1,q(x)}(\Omega)\right).\end{aligned}$$

From the hypotheses (H1), we know that $\beta(\bar{v}_N) \to \beta(v)$ a.e. in Q_T, and $|\beta(\bar{v}_N)| \leq M|\bar{v}_N| \in L^1(Q_T)$. Then, thanks to the Lebesgue dominated convergence theorem, we deduce that $\beta(\bar{v}_N) \to \beta(v)$ in $L^1(Q_T)$.

On the other hand, since $\{\nabla \bar{v}_N - \Theta(\bar{v}_N)\}$ is equiintegrable by the assumption (H3) and the boundedness of $(\bar{v}_N)$ it result that $\Phi(\nabla \bar{v}_N - \Theta(\bar{v}_N)) \to \Phi(\nabla v - \Theta(v))$ weakly in $L^1(Q_T)$.

By the reflexivity of $L^{q'(x)}(\Omega)$ and the boundedness of $\{\Phi(\nabla \bar{v}_N - \Theta(\bar{v}_N))\}$, we deduce that $\Phi(\nabla \bar{v}_N - \Theta(\bar{v}_N)) \to \Phi(\nabla v - \Theta(v))$ weakly in $\left(L^{q'(x)}(Q_T)\right)^d$. According to Lemma 4.1 and Aubin-Simons compactness result, we get that $v_N \to v$ in $C\left(0, T, L^2(\Omega)\right)$.

Now, we show that the limit function v is a weak solution of problem (P). Firstly, we have $v_N(0) = V_0 = v_0$ for all $N \in \mathbb{N}$, then $v(0,.) = v_0$. Secondly, let $\varphi \in C^1(Q_T)$, we rewrite (2) in the form

$$\int_0^T \int_\Omega \frac{\partial v_N}{\partial t} \varphi dxdt + \int_0^T \int_\Omega \Phi(\nabla \bar{v}_N - \Theta(\bar{v}_N)).\nabla \varphi dxdt + \int_0^T \int_\Omega \beta(\bar{v}_N) \varphi dxdt = \int_0^T \int_\Omega f_N \varphi dxdt, \tag{22}$$

where $f_N(t,x) = f_n(x), \forall t \in]t_{n-1}, t_n], n = 1, \ldots, N$. Taking limits as $N \to \infty$ in (22) and using the above results, we deduce that v is a weak solution of nonlinear parabolic problem (P).

4.2 Proof of uniqueness

We assume there exist two weak solutions v and w of nonlinear parabolic problem (P), replacing φ by $v - w$ as test function for solution v in (2) and substituting φ by $w - v$ as test function for solution w in (2), we get that

$$\int_0^T \int_\Omega \frac{\partial v}{\partial t}(v - w) dxdt + \int_0^T \int_\Omega \Phi(\nabla v - \Theta(v)).\nabla(v - w) dxdt + \int_0^T \int_\Omega \beta(v)(v - w) dxdt = \int_0^T \int_\Omega f(v - w) dxdt,$$

and

$$\int_0^T \int_\Omega \frac{\partial w}{\partial t}(w - v) dxdt + \int_0^T \int_\Omega \Phi(\nabla w - \Theta(w)).\nabla(w - v) dxdt + \int_0^T \int_\Omega \beta(w)(w - v) dxdt = \int_0^T \int_\Omega f(w - v) dxdt.$$

Summing up the two above equalities, we obtain

$$\int_0^T \int_\Omega \frac{\partial(v - w)}{\partial t}(v - w) dxdt + \int_0^T \int_\Omega (\Phi(\nabla v - \Theta(v)) - \Phi(\nabla w - \Theta(w))).\nabla(v - w) dxdt + \int_0^T \int_\Omega (\beta(v) - \beta(w))(v - w) dxdt = 0.$$

By the hypotheses (H1), (H3) and since $\Phi(\nabla v - \Theta(v))$ is monotone, we have that

$|v-w|^2 \leq c\int_0^T |v-w|^2\, dt$. We finally deduce from Gronwall's inequality, $|v-w|^2 \leq |v_0-w_0|^2 \exp(cT)$, $\forall t \in (0,T)$. Thus, we conclude that $v \equiv w$.

Acknowledgement. The authors are grateful to the Editor-in-Chief and the anonymous referee for their valuable comments, which helped improve the quality of this article.

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(received 16.11.2022; in revised form 16.07.2024; available online 21.12.2024)

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