MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
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SEPARATION SPECTRUMS OF GRADED DITOPOLOGICAL TEXTURE SPACES

Ramazan Ekmekçi

Abstract. The aim of this paper is to introduce some separation notions of graded ditopological texture spaces by means of spectrum idea and investigate some of their properties. Also, the relations between separation spectrums of graded ditopological texture spaces and separation axioms in ditopological structure are studied. Further, the hierarchy of the separation spectrums and a categorical aspect corresponding to the separation spectrums are given.

1. Introduction

Separation axioms play a crucial role in the study of topological structures and have been studied by several authors in numerous topological contexts. Hutton and Reilly introduced them for fuzzy topological spaces in [13] and later they were introduced for ditopological texture spaces in [6]. From [3, 5] the cetegories complemented d.t.s. **cdfDitop**, complemented simple d.t.s. **cdfSDitop** and Hutton algebras **H** are equivalent. Also, a topology on a Hutton algebra is T_k in the sense of [13] if and only if the corresponding complemented d.t.s. is T_k for $k = 0, 1, 2, 3$. The concept of interior-closure texture spaces (more general than d.t.s.) has been introduced, and the relations between the category of interior-closure textures and bicontinuous difunctions **dfIC** and both **H** and **dfDitop** are studied in [8, 9]. In addition, separation axioms in diframes (a generalization of ditopological texture spaces) have recently been studied in [14, 15].

The theory of graded ditopology was introduced by Brown and Šostak in [7], and this structure is more comprehensive than the fuzzy topology introduced independently by Šostak in [17], Kubiak in [16], and the ditopology in [2, 3]. This theory does not mention whether an element of a texture is open (closed) or not, but rather defines openness and closedness as independent grading functions. The theory of graded

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ditopologies thus offers a different and wider perspective. However, the generalizations of some properties in the theory of ditopological spaces to this theory are not self-evidently valid.

In this paper, as in [11, 18, 19], several separation spectra of graded ditopological texture spaces are introduced by the spectral theory in accordance with the separation terms in ditopological texture spaces from [6]. In addition, the properties of these separation spectra and their relations to the separation axioms in the ditopological case are investigated. Furthermore, the hierarchy of separation spectra and a categorical aspect corresponding to the separation spectra are given. Our basic motivation is to fulfill some missing parts in the theory of graded ditopologies in accordance with the theory of ditopological spaces and to study their properties comparatively.

2. Preliminaries

Ditopological texture spaces ([2, 4, 5])

Let S be a set and $\mathcal{S} \subseteq \mathcal{P}(S)$ with $S, \emptyset \in \mathcal{S}$. $\mathcal{S}$ is called a texturing of S and $(S, \mathcal{S})$ is called a texture space or simply a texture if the following conditions are met:
(i) $(\mathcal{S}, \subseteq)$ is a complete lattice which has the property that arbitrary meets coincides with intersections and finite joins coincide with unions.

(ii) $\mathcal{S}$ is completely distributive, i.e. for all index sets I, for all $i \in I$, if J_i is an index set and if $A_i^j \in \mathcal{S}$ then

$$\bigcap_{i\in I} \bigvee_{j\in J_i} A_i^j = \bigvee_{\gamma \in \Pi_i J_i} \bigcap_{i \in I} A^i_{\gamma(i)}.$$

(iii) $\mathcal{S}$ separates the points of S, i.e. if $s_1, s_2 \in S$ with $s_1 \neq s_2$ then there exists $A \in \mathcal{S}$ such that $s_1 \in A$, $s_2 \notin A$ or $s_2 \in A$, $s_1 \notin A$.

In general, a texturing of S cannot be closed under set complementation. However, if there is a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$, then σ is called a complementation on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is called a complemented texture.

The $p - sets$ given by $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and the $q - sets$, which are given by $Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S, s \notin P_u\}$ are essential for the definition of different terms in a texture space $(S, \mathcal{S})$.

Recall that $M \in \mathcal{S}$ is called a molecule if $M \neq \emptyset$ and $M \subseteq A \cup B$, $A, B \in \mathcal{S}$ implies $M \subseteq A$ or $M \subseteq B$. The sets $P_s, s \in \mathcal{S}$ are molecules, and the texture $(S, \mathcal{S})$ is called "simple" if these are the only molecules in $\mathcal{S}$. For a set $A \in \mathcal{S}$, the core of A (denoted by $A^\flat$) is defined by

$$A^\flat = \bigcap \Big\{ \bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{S},\ A = \bigvee\{A_i \mid i \in I\}\Big\}.$$

Theorem 2.1 ([4]). *In any texture space $(S, \mathcal{S})$ the following statements hold:*
1. $s \notin A \Rightarrow A \subseteq Q_s \Rightarrow s \notin A^\flat$ *for all* $s \in S$, $A \in \mathcal{S}$.

2. $A^\flat = \{s \mid A \nsubseteq Q_s\}$ *for all* $A \in \mathcal{S}$.

3. *For* $A_j \in \mathcal{S}$, $j \in J$ *we have* $(\bigvee_{j\in J} A_j)^\flat = \bigcup_{j\in J} A_j^\flat$.

4. A *is the smallest element of* $\mathcal{S}$ *that contains* $A^\flat$ *for all* $A \in \mathcal{S}$.

5. *For* $A, B \in \mathcal{S}$, *if* $A \not\subseteq B$, *then there exists* $s \in S$ *with* $A \not\subseteq Q_s$ *and* $P_s \not\subseteq B$.

6. $A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ *for all* $A \in \mathcal{S}$.

7. $A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.

Let $\mathbb{L}$ be a fuzzy lattice, i.e. a completely distributive lattice with order-reversing involution $'$ and L denote the set of molecules in $\mathbb{L}$ and $\mathcal{L} = \{\varphi(a) | a \in \mathbb{L}\}$ where $\varphi(a) = \{m \in L | m \le a\}$ for $a \in \mathbb{L}$. Then:

THEOREM 2.2 ([3]). *For the above notations,* $(L, \mathcal{L})$ *is a simple texture with complement* $\lambda(\varphi(a)) = \varphi(a')$, $a \in \mathbb{L}$ *and* $\varphi : \mathbb{L} \to \mathcal{L}$ *is a lattice isomorphism that preserves the complement.*

Conversely, any complemented simple texture can be obtained in this way from a suitable fuzzy lattice.

EXAMPLE 2.3 ([4]). 1. If $\mathcal{P}(X)$ is the powerset of a set X, then $(X, \mathcal{P}(X))$ is is the discrete texture on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$. The mapping $\pi_X : \mathcal{P}(X) \to \mathcal{P}(X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$ is a complementation on the texture $(X, \mathcal{P}(X))$.

2. Setting $\mathbb{I} = [0,1]$, $\mathcal{J} = \{[0,r), [0,r] \,| r \in \mathbb{I}\}$ results in the unit interval texture $(\mathbb{I}, \mathcal{J})$. For $r \in \mathbb{I}$, $P_r = [0,r]$ and $Q_r = [0,r)$. And the mapping $\iota : \mathcal{J} \to \mathcal{J}$, $\iota[0,r] = [0,1-r)$, $\iota[0,r) = [0,1-r]$ is a complementation on this texture.

3. The texture $(L, \mathcal{L}, \lambda)$ is defined by $L = (0,1]$, $\mathcal{L} = \{(0,r] \,|\, r \in [0,1]\}$, $\lambda((0,r]) = (0, 1-r]$. For $r \in L$, $P_r = (0,r] = Q_r$. This texture corresponds to a fuzzy lattice $(\mathbb{I} = [0,1], ')$ in the sense of Theorem 2.2.

4. Let $X \neq \emptyset$, W be the set of "fuzzy points" of $\mathbb{I}^X$, i.e. the functions

$$x_m(z) = \begin{cases} m, & z = x \\ 0, & \text{otherwise} \end{cases}.$$

for $x \in X$ and $m \in L = (0,1]$, where as before L is the set of molecules of $\mathbb{I}$. Representing x_m by the pair (x,m), you can write that $W = X \times L$. Then $(W, \mathcal{W}, \omega)$ is the texture corresponding to the fuzzy lattice $\mathbb{I}^X$ in the sense of Theorem 2.2, where $\mathcal{W} = \{\varphi(f) \,|\, f \in \mathbb{I}^X\}$, $\varphi(f) = \{(x,m) \in W \,|\, x_m \le f\} = \{(x,m) \in W \mid m \le f(x)\}$ and $\omega(\varphi(f)) = \varphi(f')$.

5. $\mathcal{S} = \{\emptyset, \{a,b\}, \{b\}, \{b,c\}, S\}$ is a simple texturing of $S = \{a,b,c\}$. $P_a = \{a,b\}$, $P_b = \{b\}$, $P_c = \{b,c\}$. It is not possible to define a complementation on $(S, \mathcal{S})$.

6. If $(S, \mathcal{S}), (V, \mathcal{V})$ are textures, then the product texturing $\mathcal{S} \otimes \mathcal{V}$ of $S \times V$ consists of arbitrary intersections of sets of the form $(A \times V) \cup (S \times B)$, $A \in \mathcal{S}, B \in \mathcal{V}$, and $(S \times V, \mathcal{S} \otimes \mathcal{V})$ is the product of $(S, \mathcal{S})$ and $(V, \mathcal{V})$. For $s \in S, v \in V$, $P_{(s,v)} = P_s \times P_v$ and $Q_{(s,v)} = (Q_s \times V) \cup (S \times Q_v)$.

Definition 2.4 ([4]). Let $(S,\mathcal{S})$ and $(V,\mathcal{V})$ be textures. Then
(i) $r \in \mathcal{P}(S) \otimes \mathcal{V}$ is called a relation on $(S,\mathcal{S})$ to $(V,\mathcal{V})$ if it satisfies

(R1) $r \not\subseteq \overline{Q}(s,v)$, $P_{s'} \not\subseteq Q_s \Rightarrow r \not\subseteq \overline{Q}(s',v)$.
(R2) $r \not\subseteq \overline{Q}(s,v) \Rightarrow \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}(s',v)$.

(ii) $R \in \mathcal{P}(S) \otimes \mathcal{V}$ is called a co-relation on $(S,\mathcal{S})$ to $(V,\mathcal{V})$ if it satisfies

(CR1) $\overline{P}(s,v) \not\subseteq R$, $P_s \not\subseteq Q_{s'} \Rightarrow \overline{P}(s',v) \not\subseteq R$.
(CR2) $\overline{P}(s,v) \not\subseteq R \Rightarrow \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}(s',v) \not\subseteq R$.

(iii) A pair (r,R), where r is a relation and R a co-relation on $(S,\mathcal{S})$ to $(V,\mathcal{V})$ is called a direlation on $(S,\mathcal{S})$ to $(V,\mathcal{V})$.

For a texture $(S,\mathcal{S})$ the identity direlation $(i_{(S,\mathcal{S})}, I_{(S,\mathcal{S})})$ is defined by $i_{(S,\mathcal{S})} = \bigvee\{\overline{P}(s,s) \mid s \in S\}$ and $I_{(S,\mathcal{S})} = \bigcap\{\overline{Q}(s,s) \mid s \in S^\flat\}$.

For $A \subseteq S$, $r^{\to}A = \bigcap\{Q_v \mid \forall s, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow A \subseteq Q_s\}$ is called the A-section of r and $R^{\to}A = \bigvee\{P_v \mid \forall s, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow P_s \subseteq A\}$ is called the A-section of R.

For $B \subseteq V$, $r^{\leftarrow}B = \bigvee\{P_s \mid \forall v, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow P_v \subseteq B\}$ is called the B-presection of r and $R^{\leftarrow}B = \bigcap\{Q_s \mid \forall v, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow B \subseteq Q_v\}$ is called the B-presection of R.

Proposition 2.5 ([4]). *If (r,R) is a direlation on $(S,\mathcal{S})$ to $(V,\mathcal{V})$ then $r^{\to}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} r^{\to}A_i$, $R^{\to}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} R^{\to}A_i$, $r^{\leftarrow}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} r^{\leftarrow}B_j$ and $R^{\leftarrow}(\bigvee_{j\in J} B_j) = \bigvee_{j\in J} R^{\leftarrow}B_j$ for any $A_i \in \mathcal{S}$, $B_j \in \mathcal{V}$, $i \in I$, $j \in J$.*

Definition 2.6 ([4]). A direlation (f,F) from $(S,\mathcal{S})$ to $(V,\mathcal{V})$, is called a difunction from $(S,\mathcal{S})$ to $(V,\mathcal{V})$ if it satisfies the following two conditions:
(DF1) For $s,s' \in S$, $P_s \not\subseteq Q_{s'} \Rightarrow \exists v \in V$ with $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s',v)} \not\subseteq F$.

(DF2) For $v,v' \in V$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s,v')} \not\subseteq F \Rightarrow P_{v'} \not\subseteq Q_v$.

(f,F) is called surjective if $\forall v,v' \in V$ $P_v \not\subseteq Q_{v'} \Rightarrow \exists s \in S$ with $f \not\subseteq \overline{Q}_{(s,v')}$ and $\overline{P}_{(s,v)} \not\subseteq F$. (f,F) is called injective if $\forall s,s' \in S$, $v \in V$ ($f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s',v)} \not\subseteq F) \Rightarrow P_s \not\subseteq Q_{s'}$.

In particular, the identity direlation (i_S, I_S) is a difunction on $(S,\mathcal{S})$.

Proposition 2.7 ([4]). *For a difunction (f,F) from $(S,\mathcal{S})$ to $(V,\mathcal{V})$, the following properties are satisfied:*
(i) $f^{\leftarrow}B = F^{\leftarrow}B$ for each $B \in \mathcal{V}$.

(ii) $f^{\leftarrow}\emptyset = F^{\leftarrow}\emptyset = \emptyset$ and $f^{\leftarrow}V = F^{\leftarrow}V = S$.

(iii) $A \subseteq F^{\leftarrow}(f^{\to}A)$ and $f^{\to}(F^{\leftarrow}B) \subseteq B$ for all $A \in \mathcal{S}$, $B \in \mathcal{V}$.

(iv) If (f,F) is surjective then $F^{\to}(f^{\leftarrow}B) = B = f^{\to}(F^{\leftarrow}B)$ for all $B \in \mathcal{V}$.

(v) If (f,F) is injective then $F^{\leftarrow}(f^{\to}A) = A = f^{\leftarrow}(F^{\to}A)$ for all $A \in \mathcal{S}$.

A ditopology on a texture $(S,\mathcal{S})$ is a pair (τ,κ), where $\tau,\kappa\subseteq\mathcal{S}$ and the set of open sets τ satisfies
(T_1) $S,\emptyset\in\tau$, (T_2) $G_1,G_2\in\tau\Rightarrow G_1\cap G_2\in\tau$, (T_3) $G_i\in\tau, i\in I\Rightarrow\bigvee_i G_i\in\tau$
and the set of closed sets κ satisfies
(CT_1) $S,\emptyset\in\kappa$, (CT_2) $K_1,K_2\in\kappa\Rightarrow K_1\cup K_2\in\kappa$, (CT_3) $K_i\in\kappa, i\in I\Rightarrow\bigcap_i K_i\in\kappa$.
In this case, $(S,\mathcal{S},\tau,\kappa)$ is called a ditopological texture spaces (or "d.t.s." for short). So a ditopology can be considered as a "topology" in which there is no need to exist a relation between the open and closed sets [2].

Let $(S,\mathcal{S},\tau,\kappa)$ be a d.t.s. For a subset $A\in\mathcal{S}$, the closure (interior) of A is defined by $[A]=\bigcap\{B\in\kappa\mid A\subseteq B\}$ ($]A[=\bigvee\{B\in\tau\mid B\subseteq A\}$) respectively [5].

Definition 2.8 ([6]). A d.t.s. $(S,\mathcal{S},\tau,\kappa)$ is said to be:
(a) R_0 if $(G\in\tau,\ G\not\subseteq Q_s)\Rightarrow[P_s]\subseteq G$.

(b) Co-R_0 if $(F\in\kappa,\ P_s\not\subseteq F)\Rightarrow F\subseteq]Q_s[$.

(c) R_1 if $(G\in\tau,\ G\not\subseteq Q_s,\ P_t\not\subseteq G)\Rightarrow\exists H\in\tau:\ H\not\subseteq Q_s,\ P_t\not\subseteq[H]$.

(d) Co-R_1 if $(F\in\kappa,\ P_s\not\subseteq F,\ F\not\subseteq Q_t)\Rightarrow\exists K\in\kappa:\ P_s\not\subseteq K,\ ]K[\not\subseteq Q_t$.

(e) *Regular* if $(G\in\tau,\ G\not\subseteq Q_s)\Rightarrow\exists H\in\tau:\ H\not\subseteq Q_s,\ [H]\subseteq G$.

(f) *Coregular* if $(F\in\kappa,\ P_s\not\subseteq F)\Rightarrow\exists K\in\kappa:\ P_s\not\subseteq K,\ F\subseteq]K[$.

$(S,\mathcal{S},\tau,\kappa)$ is called T_0 if $Q_s\not\subseteq Q_t\Rightarrow\exists C\in\tau\cup\kappa:\ P_s\not\subseteq C\not\subseteq Q_t$. A d.t.s. is called:
(i) (co-)T_1 if it is both T_0 and (co-)R_0,

(ii) (co-)T_2 if it is both T_0 and (co-)R_1 and

(iii) (co-)T_3 if it is both T_0 and (co-)*regular* respectively. For each property P, if $(S,\mathcal{S},\tau,\kappa)$ is P and co-P then we say that $(S,\mathcal{S},\tau,\kappa)$ is bi-P.

Graded ditopological texture spaces ([7])

Consider two textures $(S,\mathcal{S})$ and $(V,\mathcal{V})$. A graded ditopological texture space (or "g.d.t.s." for short) is a tuple $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ where the mappings $\mathcal{T},\mathcal{K}:\mathcal{S}\to\mathcal{V}$ satisfy following conditions:
(GT_1) $\mathcal{T}(S)=\mathcal{T}(\emptyset)=V$. (GT_2) $\mathcal{T}(A_1)\cap\mathcal{T}(A_2)\subseteq\mathcal{T}(A_1\cap A_2)$ $\forall A_1,A_2\in\mathcal{S}$.

(GT_3) $\bigcap_{j\in J}\mathcal{T}(A_j)\subseteq\mathcal{T}(\bigvee_{j\in J}A_j)$ $\forall A_j\in\mathcal{S}, j\in J$.
(GCT_1) $\mathcal{K}(S)=\mathcal{K}(\emptyset)=V$. (GCT_2) $\mathcal{K}(A_1)\cap\mathcal{K}(A_2)\subseteq\mathcal{K}(A_1\cup A_2)$ $\forall A_1,A_2\in\mathcal{S}$.

(GCT_3) $\bigcap_{j\in J}\mathcal{K}(A_j)\subseteq\mathcal{K}(\bigcap_{j\in J}A_j)$ $\forall A_j\in\mathcal{S}, j\in J$.
In this case $\mathcal{T}$ is called a $(V,\mathcal{V})$-graded topology and $\mathcal{K}$ a $(V,\mathcal{V})$-graded cotopology on $(S,\mathcal{S})$. For $v\in V$ it is defined that $\mathcal{T}^v=\{A\in\mathcal{S}\mid P_v\subseteq\mathcal{T}(A)\}$, $\mathcal{K}^v=\{A\in\mathcal{S}\mid P_v\subseteq\mathcal{K}(A)\}$. So $(\mathcal{T}^v,\mathcal{K}^v)$ is a ditopology on $(S,\mathcal{S})$ for each $v\in V$. Namely, if $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ is a g.d.t.s., then there exists a d.t.s. $(S,\mathcal{S},\mathcal{T}^v,\mathcal{K}^v)$ for each $v\in V$.

$[A]^v$ and $]A[^v$ stand for the closure and the interior of a set $A\in\mathcal{S}$ in the d.t.s. $(S,\mathcal{S},\mathcal{T}^v,\mathcal{K}^v)$ respectively, so we have $[A]^v=\bigcap\{B\in\mathcal{S}\mid A\subseteq B,\ B\in\mathcal{K}^v\}$, $]A[^v=\bigvee\{B\in\mathcal{S}\mid B\subseteq A,\ B\in\mathcal{T}^v\}$.

Let $(S, \mathcal{S}, \sigma)$ be a complemented texture. If $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is a g.d.t.s. then $(S, \mathcal{S}, \mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma, V, \mathcal{V})$ is again a g.d.t.s. Additionally, $(\mathcal{T}, \mathcal{K})$ is called complemented if $(\mathcal{T}, \mathcal{K}) = (\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$ and in this case, we say that $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, \sigma, V, \mathcal{V})$ is a complemented g.d.t.s.

If $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, \sigma, V, \mathcal{V})$ is complemented then $\sigma([A]^v) =]\sigma(A)[^v$ and $\sigma(]A[^v) = [\sigma(A)]^v$ for all $A \in \mathcal{S}$ and $v \in V$ [12].

Let $(S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{K}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be g.d.t.s., $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ difunctions. For the pair $((f, F), (h, H))$, (f, F) is called continuous w.r.t. (h, H) if $H^{\leftarrow}\mathcal{T}_2(A) \subseteq \mathcal{T}_1(F^{\leftarrow}A)$ $\forall A \in \mathcal{S}_2$, and cocontinuous w.r.t. (h, H) if $h^{\leftarrow}\mathcal{K}_2(A) \subseteq \mathcal{K}_1(f^{\leftarrow}A)$ $\forall A \in \mathcal{S}_2$. If (f, F) is continuous and cocontinuous w.r.t. (h, H) then it is said to be a bicontinuous difunction w.r.t. (h, H).

(f, F) is called open (coopen) w.r.t. (h, H) if $h^{\rightarrow}\mathcal{T}_1(A) \subseteq \mathcal{T}_2(f^{\rightarrow}A)$ $(h^{\rightarrow}\mathcal{T}_1(A) \subseteq \mathcal{T}_2(F^{\rightarrow}A))$ for all $A \in \mathcal{S}_1$ respectively. (f, F) is called closed (coclosed) w.r.t. (h, H) if $h^{\rightarrow}\mathcal{K}_1(A) \subseteq \mathcal{K}_2(f^{\rightarrow}A)$ $(h^{\rightarrow}\mathcal{K}_1(A) \subseteq \mathcal{K}_2(F^{\rightarrow}A))$ for all $A \in \mathcal{S}_1$ respectively [10].

Example 2.9. Consider the discrete texture $(V, \mathcal{V}) = (1, \mathcal{P}(1))$ (The notation 1 denotes the set $\{0\}$) and take a d.t.s. $(S, \mathcal{S}, \tau, \kappa)$. Then the mappings $\tau^g, \kappa^g : \mathcal{S} \to \mathcal{P}(1)$ defined by $\tau^g(A) = 1 \Leftrightarrow A \in \tau$ and $\kappa^g(A) = 1 \Leftrightarrow A \in \kappa$ form a g.d.t.s. $(S, \mathcal{S}, \tau^g, \kappa^g, V, \mathcal{V})$. In this case (τ^g, κ^g) is called a graded ditopology on $(S, \mathcal{S})$ corresponding to ditopology (τ, κ). Thus g.d.t.s. are more general than d.t.s.

3. Regularity and separation spectrums

Definition 3.1. Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a g.d.t.s. The families defined by

(i) $\mathcal{R}_0 = \{P_v \in \mathcal{V} \,|\, [A \in \mathcal{S},\ P_v \subseteq \mathcal{T}(A),\ A \not\subseteq Q_s] \Rightarrow [P_s]^v \subseteq A\}$

(ii) $c\mathcal{R}_0 = \{P_v \in \mathcal{V} \,|\, [A \in \mathcal{S},\ P_v \subseteq \mathcal{K}(A),\ P_s \not\subseteq A] \Rightarrow A \subseteq]Q_s[^v\}$

(iii) $\mathcal{R}_1 = \{P_v \in \mathcal{V} \,|\, [A \in \mathcal{S},\ P_v \subseteq \mathcal{T}(A),\ A \not\subseteq Q_s,\ P_t \not\subseteq A] \Rightarrow [\exists B \in \mathcal{S} :\ P_v \subseteq \mathcal{T}(B),\ B \not\subseteq Q_s,\ P_t \not\subseteq [B]^v]\}$

(iv) $c\mathcal{R}_1 = \{P_v \in \mathcal{V} \,|\, [A \in \mathcal{S},\ P_v \subseteq \mathcal{K}(A),\ P_s \not\subseteq A,\ A \not\subseteq Q_t] \Rightarrow [\exists B \in \mathcal{S} :\ P_v \subseteq \mathcal{K}(B),\ P_s \not\subseteq B,\ ]B[^v \not\subseteq Q_t]\}$

(v) $\mathcal{R} = \{P_v \in \mathcal{V} \,|\, [A \in \mathcal{S},\ P_v \subseteq \mathcal{T}(A),\ A \not\subseteq Q_s] \Rightarrow [\exists B \in \mathcal{S} :\ P_v \subseteq \mathcal{T}(B),\ B \not\subseteq Q_s,\ [B]^v \subseteq A]\}$

(vi) $c\mathcal{R} = \{P_v \in \mathcal{V} \,|\, [A \in \mathcal{S},\ P_v \subseteq \mathcal{K}(A),\ P_s \not\subseteq A] \Rightarrow [\exists B \in \mathcal{S} :\ P_v \subseteq \mathcal{K}(B),\ P_s \not\subseteq B,\ A \subseteq]B[^v\ ]\}$

are called R_0 (Co-R_0, R_1, Co-R_1, Regularity, Co-regularity) spectrums of $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ respectively. Also bi-R_0 (bi-R_1 and bi–regularity) spectrums are defined by $b\mathcal{R}_0 = \mathcal{R}_0 \cap c\mathcal{R}_0$ ($b\mathcal{R}_1 = \mathcal{R}_1 \cap c\mathcal{R}_1$ and $b\mathcal{R} = \mathcal{R} \cap c\mathcal{R}$) respectively.

In case more than one g.d.t.s. (e.g., $\mathbf{W}_k = (S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{K}_k, V_k, \mathcal{V}_k)$, $k = 1, 2, ..., n$) are mentioned, for simplicity we will use the notations $\mathcal{R}(\mathbf{W}_k)$ ($c\mathcal{R}(\mathbf{W}_k)$ etc.) for the regularity (Co-regularity etc.) spectrum of $\mathbf{W}_k$.

Proposition 3.2. *For a g.d.t.s. we have $\mathcal{R} \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_0$, $c\mathcal{R} \subseteq c\mathcal{R}_1 \subseteq c\mathcal{R}_0$ and so $b\mathcal{R} \subseteq b\mathcal{R}_1 \subseteq b\mathcal{R}_0$.*

Proof. Consider a g.d.t.s. $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$.

Let $P_v \in \mathcal{R}$ and take $A \in \mathcal{S}$ with $P_v \subseteq \mathcal{T}(A)$, $A \not\subseteq Q_s$, $P_t \not\subseteq A$. Since $P_v \in \mathcal{R}$, we have $P_v \subseteq \mathcal{T}(B)$, $B \not\subseteq Q_s$ and $[B]^v \subseteq A$ for some $B \in \mathcal{S}$. Also, using $P_t \not\subseteq A$ we get $P_v \subseteq \mathcal{T}(B)$, $B \not\subseteq Q_s$, $P_t \not\subseteq [B]^v$ for some $B \in \mathcal{S}$ and so $P_v \in \mathcal{R}_1$.

Now let $P_v \in \mathcal{R}_1$ and take $A \in \mathcal{S}$ with $P_v \subseteq \mathcal{T}(A)$, $A \not\subseteq Q_s$. We will show that "$P_t \not\subseteq A \Rightarrow P_t \not\subseteq [P_s]^v$" to show that $[P_s]^v \subseteq A$. If $P_t \not\subseteq A$, since $P_v \in \mathcal{R}_1$ we have $P_v \subseteq \mathcal{T}(B)$, $B \not\subseteq Q_s$ and $P_t \not\subseteq [B]^v$ for some $B \in \mathcal{S}$. Since $B \not\subseteq Q_s$ we have $P_s \subseteq B$ and so $[P_s]^v \subseteq [B]^v$. On the other hand, since $P_t \not\subseteq [B]^v$ and $[P_s]^v \subseteq [B]^v$ we get $P_t \not\subseteq [P_s]^v$. That is $P_v \in \mathcal{R}_0$. Hence we get $\mathcal{R} \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_0$.

Let $P_v \in c\mathcal{R}$ and take $A \in \mathcal{S}$ with $P_v \subseteq \mathcal{K}(A)$, $P_s \not\subseteq A$, $A \not\subseteq Q_t$. Since $P_v \in c\mathcal{R}$, we have $P_v \subseteq \mathcal{K}(B)$, $P_s \not\subseteq B$ and $A \subseteq]B[^v$ for some $B \in \mathcal{S}$. Since $A \not\subseteq Q_t$ and $A \subseteq]B[^v$ we have $P_v \subseteq \mathcal{K}(B)$, $P_s \not\subseteq B$, $]B[^v \not\subseteq Q_t$ and so $P_v \in c\mathcal{R}_1$.

Let $P_v \in c\mathcal{R}_1$ and take $A \in \mathcal{S}$ with $P_v \subseteq \mathcal{K}(A)$, $P_s \not\subseteq A$. We will show that "$A \not\subseteq Q_t \Rightarrow]Q_s[^v \not\subseteq Q_t$" to show that $A \subseteq]Q_s[^v$. If $A \not\subseteq Q_t$, considering $P_v \in c\mathcal{R}_1$, we have $P_v \subseteq \mathcal{K}(B)$, $P_s \not\subseteq B$ and $]B[^v \not\subseteq Q_t$ for some $B \in \mathcal{S}$. Using $P_s \not\subseteq B$ we get $B \subseteq Q_s$ and so $]B[^v \subseteq]Q_s[^v$. Thus, because of $]B[^v \not\subseteq Q_t$ and $]B[^v \subseteq]Q_s[^v$ we have $]Q_s[^v \not\subseteq Q_t$. Therefore we get $c\mathcal{R} \subseteq c\mathcal{R}_1 \subseteq c\mathcal{R}_0$. □

Corollary 3.3. *Let $\boldsymbol{W}_1 = (S, \mathcal{S}, \mathcal{T}_1, \mathcal{K}_1, V, \mathcal{V})$, $\boldsymbol{W}_2 = (S, \mathcal{S}, \mathcal{T}_1, \mathcal{K}_2, V, \mathcal{V})$ and $\boldsymbol{W}_3 = (S, \mathcal{S}, \mathcal{T}_2, \mathcal{K}_1, V, \mathcal{V})$ be three g.d.t.s. with $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$. The following hold:*
(a) $\mathcal{R}(\boldsymbol{W}_1) \subseteq \mathcal{R}(\boldsymbol{W}_2)$, $\mathcal{R}_1(\boldsymbol{W}_1) \subseteq \mathcal{R}_1(\boldsymbol{W}_2)$, $\mathcal{R}_0(\boldsymbol{W}_1) \subseteq \mathcal{R}_0(\boldsymbol{W}_2)$.

(b) $c\mathcal{R}(\boldsymbol{W}_1) \subseteq c\mathcal{R}(\boldsymbol{W}_3)$, $c\mathcal{R}_1(\boldsymbol{W}_1) \subseteq c\mathcal{R}_1(\boldsymbol{W}_3)$, $c\mathcal{R}_0(\boldsymbol{W}_1) \subseteq c\mathcal{R}_0(\boldsymbol{W}_3)$.

Proof. (a) Let $B \in \mathcal{S}$ and $v \in V$. Since $\mathcal{K}_1 \subseteq \mathcal{K}_2$ we have $\mathcal{K}_1(F) \subseteq \mathcal{K}_2(F)$ for all $F \in \mathcal{S}$. So we have

$$[B]^v_{\mathbf{W}_1} = \bigcap \{F \in \mathcal{S} \mid B \subseteq F,\, P_v \subseteq \mathcal{K}_1(F)\} \supseteq \bigcap \{F \in \mathcal{S} \mid B \subseteq F,\ P_v \subseteq \mathcal{K}_2(F)\} = [B]^v_{\mathbf{W}_2}$$

where $[B]^v_{\mathbf{W}_k}$ is $v-$closure of B in g.d.t.s. $\mathbf{W}_k$, $k = 1, 2, 3$. Thus we get $\mathcal{R}(\mathbf{W}_1) \subseteq \mathcal{R}(\mathbf{W}_2)$, $\mathcal{R}_1(\mathbf{W}_1) \subseteq \mathcal{R}_1(\mathbf{W}_2)$ and $\mathcal{R}_0(\mathbf{W}_1) \subseteq \mathcal{R}_0(\mathbf{W}_2)$ from Definition 3.1 (v), (iii) and (i) respectively.

(b) Let $B \in \mathcal{S}$ and $v \in V$. Since $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we have $\mathcal{T}_1(G) \subseteq \mathcal{T}_2(G)$ for all $G \in \mathcal{S}$. So we have

$$]B[^v_{\mathbf{W}_1} = \bigvee \{G \in \mathcal{S} \mid G \subseteq B,\, P_v \subseteq \mathcal{T}_1(G)\} \subseteq \bigvee \{G \in \mathcal{S} \mid G \subseteq B,\ P_v \subseteq \mathcal{T}_2(G)\} =]B[^v_{\mathbf{W}_3}$$

where $]B[^v_{\mathbf{W}_k}$ is $v-$interior of B in g.d.t.s. $\mathbf{W}_k$, $k = 1, 2, 3$. Therefore $c\mathcal{R}(\mathbf{W}_1) \subseteq c\mathcal{R}(\mathbf{W}_3)$, $c\mathcal{R}_1(\mathbf{W}_1) \subseteq c\mathcal{R}_1(\mathbf{W}_3)$ and $c\mathcal{R}_0(\mathbf{W}_1) \subseteq c\mathcal{R}_0(\mathbf{W}_3)$ from Definition 3.1 (vi), (iv) and (ii) respectively. □

Proposition 3.4. *For a complemented g.d.t.s. $(S, \mathcal{S}, \sigma, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ we have $\mathcal{R}_0 = c\mathcal{R}_0 = b\mathcal{R}_0$.*

Proof. Let $P_v \in \mathcal{R}_0$ and take $A \in \mathcal{S}$ with $P_v \subseteq \mathcal{K}(A)$, $P_s \not\subseteq A$. Since the g.d.t.s. is complemented we have $P_v \subseteq \mathcal{K}(A) = (\mathcal{T} \circ \sigma)(A) = \mathcal{T}(\sigma(A))$. On the other hand we have:

$$P_s \not\subseteq A \Rightarrow \sigma(A) \not\subseteq \sigma(P_s) \Rightarrow \exists s' \in S : \sigma(A) \not\subseteq Q_{s'},\ P_{s'} \not\subseteq \sigma(P_s)$$

$$\Rightarrow \exists s' \in S : \sigma(A) \nsubseteq Q_{s'},\ P_s \nsubseteq \sigma(P_{s'}) \Rightarrow \exists s' \in S : \sigma(A) \nsubseteq Q_{s'},\ \sigma(P_{s'}) \subseteq Q_s.$$

Since $\sigma(A) \in \mathcal{S}$, $P_v \subseteq \mathcal{T}(\sigma(A))$, $\sigma(A) \nsubseteq Q_{s'}$ and $P_v \in \mathcal{R}_0$ we get $[P_{s'}]^v \subseteq \sigma(A)$. This implies $A \subseteq \sigma([P_{s'}]^v) =]\sigma(P_{s'})[^v \subseteq]Q_s[^v$ by $\sigma(P_{s'}) \subseteq Q_s$. Hence we get $P_v \in c\mathcal{R}_0$, i.e. $\mathcal{R}_0 \subseteq c\mathcal{R}_0$.

Now let $P_v \in c\mathcal{R}_0$ and take $A \in \mathcal{S}$ with $P_v \subseteq \mathcal{T}(A)$, $A \nsubseteq Q_s$. Since the g.d.t.s. is complemented we have $P_v \subseteq \mathcal{T}(A) = (\mathcal{K} \circ \sigma)(A) = \mathcal{K}(\sigma(A))$. Moreover, we have:

$$A \nsubseteq Q_s \Rightarrow \sigma(Q_s) \nsubseteq \sigma(A) \Rightarrow \exists s' \in S : \sigma(Q_s) \nsubseteq Q_{s'},\ P_{s'} \nsubseteq \sigma(A)$$
$$\Rightarrow \exists s' \in S : \sigma(Q_{s'}) \nsubseteq Q_s,\ P_{s'} \nsubseteq \sigma(A) \Rightarrow \exists s' \in S : P_s \subseteq \sigma(Q_{s'}),\ P_{s'} \nsubseteq \sigma(A).$$

Since $\sigma(A) \in \mathcal{S}$, $P_v \subseteq \mathcal{K}(\sigma(A))$, $P_{s'} \nsubseteq \sigma(A)$ and $P_v \in c\mathcal{R}_0$ we get $\sigma(A) \subseteq]Q_{s'}[^v$. This implies $A \supseteq \sigma(]Q_{s'}[^v) = [\sigma(Q_{s'})]^v \supseteq]P_s[^v$ by $P_s \subseteq \sigma(Q_{s'})$. Hence we get $P_v \in \mathcal{R}_0$, i.e. $c\mathcal{R}_0 \subseteq \mathcal{R}_0$. Therefore we obtain that $\mathcal{R}_0 = c\mathcal{R}_0 = b\mathcal{R}_0$. □

Proposition 3.5. *For a complemented g.d.t.s.* $(S, \mathcal{S}, \sigma, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ *we have* $\mathcal{R}_1 = c\mathcal{R}_1 = b\mathcal{R}_1$.

Proof. Let $P_v \in c\mathcal{R}_1$ and take $A \in \mathcal{S}$ with $P_v \subseteq \mathcal{T}(A)$, $A \nsubseteq Q_s$, $P_t \nsubseteq A$. Since the g.d.t.s. is complemented we have $P_v \subseteq \mathcal{T}(A) = (\mathcal{K} \circ \sigma)(A) = \mathcal{K}(\sigma(A))$. Moreover:

$$A \nsubseteq Q_s \Rightarrow \sigma(Q_s) \nsubseteq \sigma(A) \Rightarrow \exists s' \in S : \sigma(Q_s) \nsubseteq Q_{s'}, P_{s'} \nsubseteq \sigma(A)$$
$$\Rightarrow \exists s' \in S : \sigma(Q_{s'}) \nsubseteq Q_s, P_{s'} \nsubseteq \sigma(A) \Rightarrow \exists s' \in S : P_s \subseteq \sigma(Q_{s'}), P_{s'} \nsubseteq \sigma(A)$$

and $P_t \nsubseteq A \Rightarrow \sigma(A) \nsubseteq \sigma(P_t) \Rightarrow \exists t' \in S : \sigma(A) \nsubseteq Q_{t'}, P_{t'} \nsubseteq \sigma(P_t)$

$$\Rightarrow \exists t' \in S : \sigma(A) \nsubseteq Q_{t'}, P_t \nsubseteq \sigma(P_{t'}) \Rightarrow \exists t' \in S : \sigma(A) \nsubseteq Q_{t'}, \sigma(P_{t'}) \subseteq Q_t.$$

Since $\sigma(A) \in \mathcal{S}$, $P_v \subseteq \mathcal{K}(\sigma(A))$, $P_{s'} \nsubseteq \sigma(A)$, $\sigma(A) \nsubseteq Q_{t'}$ and $P_v \in c\mathcal{R}_1$ we get

$$\exists B \in \mathcal{S} :\ P_v \subseteq \mathcal{K}(B),\ P_{s'} \nsubseteq B,\ ]B[^v \nsubseteq Q_{t'}.$$

Since the g.d.t.s. is complemented we have $P_v \subseteq \mathcal{K}(B) = (\mathcal{T} \circ \sigma)(B) = \mathcal{T}(\sigma(B))$. Also, because of $P_{s'} \nsubseteq B \Rightarrow B \subseteq Q_{s'} \Rightarrow \sigma(Q_{s'}) \subseteq \sigma(B)$ and $\sigma(Q_{s'}) \nsubseteq Q_s$ we have $\sigma(B) \nsubseteq Q_s$. Since $]B[^v \nsubseteq Q_{t'} \Rightarrow P_{t'} \subseteq]B[^v \Rightarrow [\sigma(B)]^v = \sigma(]B[^v) \subseteq \sigma(P_{t'})$ and $P_t \nsubseteq \sigma(P_{t'})$ we get $P_t \nsubseteq [\sigma(B)]^v$. Hence we get $P_v \in \mathcal{R}_1$, i.e. $c\mathcal{R}_1 \subseteq \mathcal{R}_1$.

Similarly it can be obtained that $\mathcal{R}_1 \subseteq c\mathcal{R}_1$ and so $\mathcal{R}_1 = c\mathcal{R}_1 = b\mathcal{R}_1$. □

Proposition 3.6. *For a complemented g.d.t.s.* $(S, \mathcal{S}, \sigma, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ *we have* $\mathcal{R} = c\mathcal{R} = b\mathcal{R}$.

Proof. Let $P_v \in \mathcal{R}$ and take $A \in \mathcal{S}$ with $P_v \subseteq \mathcal{K}(A)$, $P_s \nsubseteq A$. Since the g.d.t.s. is complemented we have $P_v \subseteq \mathcal{K}(A) = (\mathcal{T} \circ \sigma)(A) = \mathcal{T}(\sigma(A))$. On the other hand we have:

$$P_s \nsubseteq A \Rightarrow \sigma(A) \nsubseteq \sigma(P_s) \Rightarrow \exists s' \in S : \sigma(A) \nsubseteq Q_{s'},\ P_{s'} \nsubseteq \sigma(P_s)$$
$$\Rightarrow \exists s' \in S : \sigma(A) \nsubseteq Q_{s'},\ \sigma(P_s) \subseteq Q_{s'} \Rightarrow \exists s' \in S : \sigma(A) \nsubseteq Q_{s'},\ \sigma(Q_{s'}) \subseteq P_s.$$

Since $\sigma(A) \in \mathcal{S}$, $P_v \subseteq \mathcal{T}(\sigma(A))$, $\sigma(A) \nsubseteq Q_{s'}$ and $P_v \in \mathcal{R}$ we have

$$\exists B \in \mathcal{S} :\ P_v \subseteq \mathcal{T}(B),\ B \nsubseteq Q_{s'},\ [B]^v \subseteq \sigma(A).$$

Since the g.d.t.s. is complemented we have $P_v \subseteq \mathcal{T}(B) = (\mathcal{K} \circ \sigma)(B) = \mathcal{K}(\sigma(B))$. Besides $B \nsubseteq Q_{s'}$ implies $\sigma(Q_{s'}) \nsubseteq \sigma(B)$ so we get $P_s \nsubseteq \sigma(B)$ by the fact that

$\sigma(Q_{s'}) \subseteq P_s$. Also, since $[B]^v \subseteq \sigma(A)$ we get $A \subseteq \sigma([B]^v) =]\sigma(B)[^v$. Thus we get $P_v \in c\mathcal{R}$, i.e. $\mathcal{R} \subseteq c\mathcal{R}$.

Similarly it can be shown that $c\mathcal{R} \subseteq \mathcal{R}$ and so $\mathcal{R} = c\mathcal{R} = b\mathcal{R}$. □

Theorem 3.7. *Let $\boldsymbol{W}_j = (S_j, \mathcal{S}_j, \mathcal{T}_j, \mathcal{K}_j, V, \mathcal{V})$, $(j = 1, 2)$ be two g.d.t.s., $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction. If (f, F) is bijective, continuous and coclosed w.r.t. (i_V, I_V) then $\mathcal{R}_0(\boldsymbol{W}_1) \subseteq \mathcal{R}_0(\boldsymbol{W}_2)$ and $c\mathcal{R}_0(\boldsymbol{W}_2) \subseteq c\mathcal{R}_0(\boldsymbol{W}_1)$.*

Proof. Let $P_v \in \mathcal{R}_0(\mathbf{W}_1)$. If $A \in \mathcal{S}_2$ with $P_v \subseteq \mathcal{T}_2(A)$, $A \not\subseteq Q_{s_2}$ then we have $f^{\leftarrow}(A) \in \mathcal{S}_1$ and $P_v = {I_V}^{\leftarrow}(P_v) \subseteq {I_V}^{\leftarrow}(\mathcal{T}_2(A)) \subseteq \mathcal{T}_1(f^{\leftarrow}(A))$ by the continuity of (f, F) w.r.t. (i_V, I_V). Also, since (f, F) is surjective we have

$$A \not\subseteq Q_{s_2} \Rightarrow f^{\leftarrow}(A) \not\subseteq f^{\leftarrow}(Q_{s_2}) \Rightarrow \exists s_1 \in S_1 : \; f^{\leftarrow}(A) \not\subseteq Q_{s_1}, \; P_{s_1} \not\subseteq f^{\leftarrow}(Q_{s_2}).$$

Moreover, we have $P_{s_1} \not\subseteq f^{\leftarrow}(Q_{s_2}) \Rightarrow F^{\to}(P_{s_1}) \not\subseteq F^{\to}(f^{\leftarrow}(Q_{s_2})) = Q_{s_2}$ since (f, F) is bijective.

So, considering $f^{\leftarrow}(A) \in \mathcal{S}_1$, $P_v \subseteq \mathcal{T}_1(f^{\leftarrow}(A))$ and $f^{\leftarrow}(A) \not\subseteq Q_{s_1}$ we get $[P_{s_1}]^v \subseteq f^{\leftarrow}(A)$ by $P_v \in \mathcal{R}_0(\mathbf{W}_1)$.

Now, if we take $B \in \mathcal{S}_1$ with $P_{s_1} \subseteq B$ and $P_v \subseteq \mathcal{K}_1(B)$ then we have:

(i) $P_{s_1} \subseteq B \Rightarrow F^{\to}(P_{s_1}) \subseteq F^{\to}(B) \Rightarrow F^{\to}(B) \not\subseteq Q_{s_2} \Rightarrow P_{s_2} \subseteq F^{\to}(B)$ since $F^{\to}(P_{s_1}) \not\subseteq Q_{s_2}$.

(ii) $P_v \subseteq \mathcal{K}_1(B) \Rightarrow P_v = {i_V}^{\to}(P_v) \subseteq {i_V}^{\to}(\mathcal{K}_1(B)) \subseteq \mathcal{K}_2(F^{\to}(B)) \Rightarrow P_v \subseteq \mathcal{K}_2(F^{\to}(B))$ since (f, F) is coclosed w.r.t. (i_V, I_V).

So we get

$$\{F^{\to}(B) \in \mathcal{S}_2 \mid B \in \mathcal{S}_1, \; P_{s_1} \subseteq B, P_v \subseteq \mathcal{K}_1(B)\} \subseteq \{D \in \mathcal{S}_2 \mid P_{s_2} \subseteq D, P_v \subseteq \mathcal{K}_2(D)\}.$$

Thus, since (f, F) is surjective we have

$$\begin{aligned}
[P_{s_2}]^v &= \bigcap\{D \in \mathcal{S}_2 \mid P_{s_2} \subseteq D, P_v \subseteq \mathcal{K}_2(D)\} \\
&\subseteq \bigcap\{F^{\to}(B) \in \mathcal{S}_2 \mid B \in \mathcal{S}_1, \; P_{s_1} \subseteq B, \; P_v \subseteq \mathcal{K}_1(B)\} \\
&= F^{\to}(\bigcap\{B \in \mathcal{S}_1 \mid P_{s_1} \subseteq B, \; P_v \subseteq \mathcal{K}_1(B)\}) = F^{\to}([P_{s_1}]^v) \subseteq F^{\to}(f^{\leftarrow}(A)) = A.
\end{aligned}$$

Hence we get $P_v \in \mathcal{R}_0(\mathbf{W}_2)$.

Similarly it can be shown that $c\mathcal{R}_0(\mathbf{W}_2) \subseteq c\mathcal{R}_0(\mathbf{W}_1)$. □

Theorem 3.8. *Let $\boldsymbol{W}_j = (S_j, \mathcal{S}_j, \mathcal{T}_j, \mathcal{K}_j, V, \mathcal{V})$, $(j = 1, 2)$ be two g.d.t.s., $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction. If (f, F) is bijective, cocontinuous and open w.r.t. (i_V, I_V) then $\mathcal{R}_0(\boldsymbol{W}_2) \subseteq \mathcal{R}_0(\boldsymbol{W}_1)$ and $c\mathcal{R}_0(\boldsymbol{W}_1) \subseteq c\mathcal{R}_0(\boldsymbol{W}_2)$.*

Proof. Let $P_v \in \mathcal{R}_0(\mathbf{W}_2)$. If $A \in \mathcal{S}_1$ with $P_v \subseteq \mathcal{T}_1(A)$, $A \not\subseteq Q_{s_1}$ then we have $f^{\to}(A) \in \mathcal{S}_2$ and $P_v = {i_V}^{\to}(P_v) \subseteq {i_V}^{\to}(\mathcal{T}_1(A)) \subseteq \mathcal{T}_2(f^{\to}(A))$ by the openness of (f, F) w.r.t. (i_V, I_V). Also, since (f, F) is injective we have

$$A \not\subseteq Q_{s_1} \Rightarrow f^{\to}(A) \not\subseteq f^{\to}(Q_{s_1}) \Rightarrow \exists s_2 \in S_2 : \; f^{\to}(A) \not\subseteq Q_{s_2}, \; P_{s_2} \not\subseteq f^{\to}(Q_{s_1}).$$

Moreover, we have $P_{s_2} \not\subseteq f^{\to}(Q_{s_1}) \Rightarrow F^{\leftarrow}(P_{s_2}) \not\subseteq F^{\leftarrow}(f^{\to}(Q_{s_1})) = Q_{s_1} \Rightarrow f^{\leftarrow}(P_{s_2}) \not\subseteq Q_{s_1}$ since (f, F) is bijective.

So, considering $f^{\to}(A) \in \mathcal{S}_2$, $P_v \subseteq \mathcal{T}_2(f^{\to}(A))$ and $f^{\to}(A) \not\subseteq Q_{s_2}$ we get $[P_{s_2}]^v \subseteq f^{\to}(A)$ by $P_v \in \mathcal{R}_0(\mathbf{W}_2)$.

Now, if we take $D \in \mathcal{S}_2$ with $P_{s_2} \subseteq D$ and $P_v \subseteq \mathcal{K}_2(D)$ then we have:
(i) $P_{s_2} \subseteq D \Rightarrow f^{\leftarrow}(P_{s_2}) \subseteq f^{\leftarrow}(D) \Rightarrow f^{\leftarrow}(D) \nsubseteq Q_{s_1} \Rightarrow P_{s_1} \subseteq f^{\leftarrow}(D)$ since $f^{\leftarrow}(P_{s_2}) \nsubseteq Q_{s_1}$.

(ii) $P_v \subseteq \mathcal{K}_2(D) \Rightarrow P_v = {i_V}^{\leftarrow}(P_v) \subseteq {i_V}^{\leftarrow}(\mathcal{K}_2(D)) \subseteq \mathcal{K}_1(f^{\leftarrow}(D)) \Rightarrow P_v \subseteq \mathcal{K}_1(f^{\leftarrow}(D))$ since (f, F) is cocontinuous w.r.t. (i_V, I_V).
So we get

$$\{f^{\leftarrow}(D) \in \mathcal{S}_1 \mid D \in \mathcal{S}_2,\ P_{s_2} \subseteq D,\ P_v \subseteq \mathcal{K}_2(D)\} \subseteq \{B \in \mathcal{S}_1 \mid P_{s_1} \subseteq B,\ P_v \subseteq \mathcal{K}_1(B)\}.$$

Thus, since (f, F) is injective we have

$$\begin{aligned}[P_{s_1}]^v &= \bigcap\{B \in \mathcal{S}_1 \mid P_{s_1} \subseteq B,\ P_v \subseteq \mathcal{K}_1(B)\} \\ &\subseteq \bigcap\{f^{\leftarrow}(D) \in \mathcal{S}_1 \mid D \in \mathcal{S}_2,\ P_{s_2} \subseteq D,\ P_v \subseteq \mathcal{K}_2(D)\} \\ &= f^{\leftarrow}(\bigcap\{D \in \mathcal{S}_2 \mid P_{s_2} \subseteq D,\ P_v \subseteq \mathcal{K}_2(D)\}) = F^{\leftarrow}([P_{s_2}]^v) \subseteq F^{\leftarrow}(f^{\rightarrow}(A)) = A.\end{aligned}$$

Hence we get $P_v \in \mathcal{R}_0(\mathbf{W}_1)$.

Similarly it can be shown that $c\mathcal{R}_0(\mathbf{W}_1) \subseteq c\mathcal{R}_0(\mathbf{W}_2)$. □

The following corollary is a direct consequence of Theorem 3.7 and Theorem 3.8.

Corollary 3.9. *Let $\boldsymbol{W}_j = (S_j, \mathcal{S}_j, \mathcal{T}_j, \mathcal{K}_j, V, \mathcal{V})$, $(j = 1, 2)$ be two g.d.t.s., $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction. If (f, F) is bijective, bicontinuous, open and coclosed w.r.t. (i_V, I_V) then $\mathcal{R}_0(\boldsymbol{W}_1) = \mathcal{R}_0(\boldsymbol{W}_2)$ and $c\mathcal{R}_0(\boldsymbol{W}_1) = c\mathcal{R}_0(\boldsymbol{W}_2)$.*

Theorem 3.10. *Let $\boldsymbol{W}_j = (S_j, \mathcal{S}_j, \mathcal{T}_j, \mathcal{K}_j, V, \mathcal{V})$, $(j = 1, 2)$ be two g.d.t.s., $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a bijective difunction.*
(a) If (f, F) is continuous, coopen and coclosed w.r.t. (i_V, I_V) then $\mathcal{R}_1(\boldsymbol{W}_1) \subseteq \mathcal{R}_1(\boldsymbol{W}_2)$.

(b) If (f, F) is cocontinuous, open and closed w.r.t. (i_V, I_V) then $c\mathcal{R}_1(\boldsymbol{W}_1) \subseteq c\mathcal{R}_1(\boldsymbol{W}_2)$.

(c) If (f, F) is bicontinuous and open w.r.t. (i_V, I_V) then $\mathcal{R}_1(\boldsymbol{W}_2) \subseteq \mathcal{R}_1(\boldsymbol{W}_1)$.

(d) If (f, F) is bicontinuous and coclosed w.r.t. (i_V, I_V) then $c\mathcal{R}_1(\boldsymbol{W}_2) \subseteq c\mathcal{R}_1(\boldsymbol{W}_1)$.

Proof. We prove (b), (d) and leave the proof of (a), (c) to the reader.

(b) Let $P_v \in c\mathcal{R}_1(\mathbf{W}_1)$. If $A \in \mathcal{S}_2$ with $P_v \subseteq \mathcal{K}_2(A)$, $P_{s_2} \nsubseteq A$, $A \nsubseteq Q_{t_2}$ then we have $f^{\leftarrow}(A) \in \mathcal{S}_1$ and $P_v = {I_V}^{\leftarrow}(P_v) \subseteq {I_V}^{\leftarrow}(\mathcal{K}_2(A)) \subseteq \mathcal{K}_1(f^{\leftarrow}(A))$ by the cocontinuity of (f, F) w.r.t. (i_V, I_V). Also, since (f, F) is surjective we have

$$P_{s_2} \nsubseteq A \Rightarrow f^{\leftarrow}(P_{s_2}) \nsubseteq f^{\leftarrow}(A) \Rightarrow \exists s_1 \in S_1 :\ f^{\leftarrow}(P_{s_2}) \nsubseteq Q_{s_1},\ P_{s_1} \nsubseteq f^{\leftarrow}(A),$$
$$A \nsubseteq Q_{t_2} \Rightarrow f^{\leftarrow}(A) \nsubseteq f^{\leftarrow}(Q_{t_2}) \Rightarrow \exists t_1 \in S_1 :\ f^{\leftarrow}(A) \nsubseteq Q_{t_1},\ P_{t_1} \nsubseteq f^{\leftarrow}(Q_{t_2}).$$

So, considering $P_v \in c\mathcal{R}_1(\mathbf{W}_1)$, $f^{\leftarrow}(A) \in \mathcal{S}_1$, $P_v \subseteq \mathcal{K}_1(f^{\leftarrow}(A))$, $P_{s_1} \nsubseteq f^{\leftarrow}(A)$ and $f^{\leftarrow}(A) \nsubseteq Q_{t_1}$ we get $\exists B \in \mathcal{S}_1 :\ P_v \subseteq \mathcal{K}_1(B),\ P_{s_1} \nsubseteq B,\ ]B[^v \nsubseteq Q_{t_1}$. Since (f, F) is closed w.r.t. (i_V, I_V) we have $P_v = i_V^{\rightarrow}(P_v) \subseteq i_V^{\rightarrow}(\mathcal{K}_1(B)) \subseteq \mathcal{K}_2(f^{\rightarrow}B)$, i.e. $P_v \subseteq \mathcal{K}_2(f^{\rightarrow}B)$. Also, since (f, F) is bijective and $f^{\leftarrow}(P_{s_2}) \nsubseteq Q_{s_1}$ we have $P_{s_1} \nsubseteq B \Rightarrow B \subseteq Q_{s_1} \Rightarrow f^{\leftarrow}(P_{s_2}) \nsubseteq B \Rightarrow P_{s_2} = f^{\rightarrow}(f^{\leftarrow}(P_{s_2})) \nsubseteq f^{\rightarrow}B \Rightarrow P_{s_2} \nsubseteq f^{\rightarrow}B$.

Now, if we take $D \in \mathcal{S}_1$ with $D \subseteq B$ and $P_v \subseteq \mathcal{T}_1(D)$ then we have $f^{\rightarrow}D \subseteq f^{\rightarrow}B$ and $P_v = {i_V}^{\rightarrow}(P_v) \subseteq {i_V}^{\rightarrow}(\mathcal{T}_1(D)) \subseteq \mathcal{T}_2(f^{\rightarrow}(D))$ by the openness of (f, F) w.r.t.

(i_V, I_V). So we get

$$\begin{aligned} f^{\to}(]B[^v) &= f^{\to}(\bigvee\{D \in \mathcal{S}_1 \mid D \subseteq B,\ P_v \subseteq \mathcal{T}_1(D)\}) \\ &= \bigvee\{f^{\to}D \in \mathcal{S}_2 \mid D \in \mathcal{S}_1,\ D \subseteq B,\ P_v \subseteq \mathcal{T}_1(D)\} \\ &\subseteq \bigvee\{E \in \mathcal{S}_2 \mid E \subseteq f^{\to}B,\ P_v \subseteq \mathcal{T}_2(E)\} =]f^{\to}B[^v. \end{aligned}$$

Besides, since (f, F) is bijective, $P_{t_1} \not\subseteq f^{\leftarrow}(Q_{t_2})$ and $]B[^v \not\subseteq Q_{t_1}$ we have $P_{t_1} \subseteq]B[^v \Rightarrow]B[^v \not\subseteq f^{\leftarrow}(Q_{t_2}) \Rightarrow f^{\to}(]B[^v) \not\subseteq Q_{t_2}$. Hence we get $]f^{\to}B[^v \not\subseteq Q_{t_2}$, i.e. $P_v \in c\mathcal{R}_1(\mathbf{W}_2)$.

(d) Let $P_v \in c\mathcal{R}_1(\mathbf{W}_2)$. If $A \in \mathcal{S}_1$ with $P_v \subseteq \mathcal{K}_1(A)$, $P_{s_1} \not\subseteq A$, $A \not\subseteq Q_{t_1}$ then we have $F^{\to}(A) \in \mathcal{S}_2$ and $P_v = i_V{}^{\to}(P_v) \subseteq i_V{}^{\to}(\mathcal{K}_1(A)) \subseteq \mathcal{K}_2(F^{\to}A)$ by the coclosedness of (f, F) w.r.t. (i_V, I_V). Also, since (f, F) is injective we have

$$P_{s_1} \not\subseteq A \Rightarrow F^{\to}(P_{s_1}) \not\subseteq F^{\to}(A) \Rightarrow \exists s_2 \in S_2 :\ F^{\to}(P_{s_1}) \not\subseteq Q_{s_2},\ P_{s_2} \not\subseteq F^{\to}(A),$$
$$A \not\subseteq Q_{t_1} \Rightarrow F^{\to}(A) \not\subseteq F^{\to}(Q_{t_1}) \Rightarrow \exists t_2 \in S_2 :\ F^{\to}(A) \not\subseteq Q_{t_2},\ P_{t_2} \not\subseteq F^{\to}(Q_{t_1}).$$

So, considering $P_v \in c\mathcal{R}_1(\mathbf{W}_2)$, $F^{\to}(A) \in \mathcal{S}_2$, $P_v \subseteq \mathcal{K}_2(F^{\to}A)$, $P_{s_2} \not\subseteq F^{\to}(A)$ and $F^{\to}(A) \not\subseteq Q_{t_2}$ we get $\exists B \in \mathcal{S}_2 :\ P_v \subseteq \mathcal{K}_2(B),\ P_{s_2} \not\subseteq B,\ ]B[^v \not\subseteq Q_{t_2}$. Since (f, F) is cocontinuous w.r.t. (i_V, I_V) we have $P_v = I_V^{\leftarrow}(P_v) \subseteq I_V^{\leftarrow}(\mathcal{K}_2(B)) \subseteq \mathcal{K}_1(f^{\leftarrow}B)$, i.e. $P_v \subseteq \mathcal{K}_1(f^{\leftarrow}B)$. Also, since (f, F) is bijective and $F^{\to}(P_{s_1}) \not\subseteq Q_{s_2}$ we have $P_{s_2} \not\subseteq B \Rightarrow B \subseteq Q_{s_2} \Rightarrow F^{\to}(P_{s_1}) \not\subseteq B \Rightarrow P_{s_1} = f^{\leftarrow}(F^{\to}(P_{s_2})) \not\subseteq f^{\leftarrow}B \Rightarrow P_{s_1} \not\subseteq f^{\leftarrow}B$.

Now, if we take $D \in \mathcal{S}_2$ with $D \subseteq B$ and $P_v \subseteq \mathcal{T}_2(D)$ then we have $f^{\leftarrow}D \subseteq f^{\leftarrow}B$ and $P_v = i_V{}^{\to}(P_v) \subseteq i_V{}^{\to}(\mathcal{T}_2(D)) \subseteq \mathcal{T}_1(f^{\leftarrow}(D))$ by the continuity of (f, F) w.r.t. (i_V, I_V). So we get

$$\begin{aligned} f^{\leftarrow}(]B[^v) &= f^{\leftarrow}(\bigvee\{D \in \mathcal{S}_2 \mid D \subseteq B,\ P_v \subseteq \mathcal{T}_2(D)\}) \\ &= \bigvee\{f^{\leftarrow}D \in \mathcal{S}_1 \mid D \in \mathcal{S}_2,\ D \subseteq B,\ P_v \subseteq \mathcal{T}_2(D)\} \\ &\subseteq \bigvee\{E \in \mathcal{S}_1 \mid E \subseteq f^{\leftarrow}B,\ P_v \subseteq \mathcal{T}_1(E)\} =]f^{\leftarrow}B[^v. \end{aligned}$$

Besides, since (f, F) is bijective, $P_{t_2} \not\subseteq F^{\to}(Q_{t_1})$ and $]B[^v \not\subseteq Q_{t_2}$ we have $P_{t_2} \subseteq]B[^v \Rightarrow]B[^v \not\subseteq F^{\to}(Q_{t_1}) \Rightarrow f^{\leftarrow}(]B[^v) \not\subseteq Q_{t_1}$. Hence we get $]f^{\leftarrow}B[^v \not\subseteq Q_{t_1}$, i.e. $P_v \in c\mathcal{R}_1(\mathbf{W}_1)$. □

THEOREM 3.11. *Let $\boldsymbol{W}_j = (S_j, \mathcal{S}_j, \mathcal{T}_j, \mathcal{K}_j, V, \mathcal{V})$, $(j = 1, 2)$ be two g.d.t.s., $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a bijective difunction.*

(a) If (f, F) is continuous, coopen and coclosed w.r.t. (i_V, I_V) then $\mathcal{R}(\boldsymbol{W}_1) \subseteq \mathcal{R}(\boldsymbol{W}_2)$.

(b) If (f, F) is cocontinuous, open and closed w.r.t. (i_V, I_V) then $c\mathcal{R}(\boldsymbol{W}_1) \subseteq c\mathcal{R}(\boldsymbol{W}_2)$.

(c) If (f, F) is bicontinuous and open w.r.t. (i_V, I_V) then $\mathcal{R}(\boldsymbol{W}_2) \subseteq \mathcal{R}(\boldsymbol{W}_1)$.

(d) If (f, F) is bicontinuous and coclosed w.r.t. (i_V, I_V) then $c\mathcal{R}(\boldsymbol{W}_2) \subseteq c\mathcal{R}(\boldsymbol{W}_1)$.

Proof. We prove (a), (c) and leave the proof of (b), (d) to the reader.

(a) Let $P_v \in \mathcal{R}(\mathbf{W}_1)$. If $A \in \mathcal{S}_2$ with $P_v \subseteq \mathcal{T}_2(A)$, $A \not\subseteq Q_{s_2}$ then we have $f^{\leftarrow}(A) \in \mathcal{S}_1$ and $P_v = I_V{}^{\leftarrow}(P_v) \subseteq I_V{}^{\leftarrow}(\mathcal{T}_2(A)) \subseteq \mathcal{T}_1(f^{\leftarrow}(A))$ by the continuity of (f, F) w.r.t. (i_V, I_V). Also, since (f, F) is surjective we have

$$A \not\subseteq Q_{s_2} \Rightarrow f^{\leftarrow}(A) \not\subseteq f^{\leftarrow}(Q_{s_2}) \Rightarrow \exists s_1 \in S_1 :\ f^{\leftarrow}(A) \not\subseteq Q_{s_1},\ P_{s_1} \not\subseteq f^{\leftarrow}(Q_{s_2}).$$

So, considering $P_v \in \mathcal{R}(\mathbf{W}_1)$, $f^{\leftarrow}(A) \in \mathcal{S}_1$, $P_v \subseteq \mathcal{T}_1(f^{\leftarrow}(A))$ and $f^{\leftarrow}(A) \nsubseteq Q_{s_1}$ we get $\exists B \in \mathcal{S}_1 : P_v \subseteq \mathcal{T}_1(B),\ B \nsubseteq Q_{s_1},\ [B]^v \subseteq f^{\leftarrow}(A)$. Since (f,F) is coopen w.r.t. (i_V, I_V) we have $P_v = i_V^{\rightarrow}(P_v) \subseteq i_V^{\rightarrow}(\mathcal{T}_1(B)) \subseteq \mathcal{T}_2(F^{\rightarrow}B)$, i.e. $P_v \subseteq \mathcal{T}_2(F^{\rightarrow}B)$. Also, since (f,F) is bijective and $P_{s_1} \nsubseteq f^{\leftarrow}(Q_{s_2})$ we have $B \nsubseteq Q_{s_1} \Rightarrow P_{s_1} \subseteq B \Rightarrow B \nsubseteq f^{\leftarrow}(Q_{s_2}) \Rightarrow F^{\rightarrow}B \nsubseteq F^{\rightarrow}(f^{\leftarrow}(Q_{s_2})) = Q_{s_2} \Rightarrow F^{\rightarrow}B \nsubseteq Q_{s_2}$.

Now, if we take $D \in \mathcal{S}_1$ with $B \subseteq D$ and $P_v \subseteq \mathcal{K}_1(D)$ then we have $F^{\rightarrow}B \subseteq F^{\rightarrow}D$ and $P_v = i_V{}^{\rightarrow}(P_v) \subseteq i_V{}^{\rightarrow}(\mathcal{K}_1(D)) \subseteq \mathcal{K}_2(F^{\rightarrow}(D))$ by the coclosedness of (f,F) w.r.t. (i_V, I_V). So we get

$$\begin{aligned}[F^{\rightarrow}B]^v &= \bigcap\{E \in \mathcal{S}_2 \mid F^{\rightarrow}B \subseteq E,\ P_v \subseteq \mathcal{K}_2(E)\}\\ &\subseteq \bigcap\{F^{\rightarrow}D \in \mathcal{S}_2 \mid D \in \mathcal{S}_1\ B \subseteq D,\ P_v \subseteq \mathcal{K}_1(D)\} = F^{\rightarrow}([B]^v).\end{aligned}$$

Besides, since (f,F) is bijective and $[B]^v \subseteq f^{\leftarrow}A$ we have $[F^{\rightarrow}B]^v \subseteq F^{\rightarrow}([B]^v) \subseteq F^{\rightarrow}(f^{\leftarrow}A) = A$, i.e. $[F^{\rightarrow}B]^v \subseteq A$. Hence we get $P_v \in \mathcal{R}(\mathbf{W}_2)$.

(c) Let $P_v \in \mathcal{R}(\mathbf{W}_2)$. If $A \in \mathcal{S}_1$ with $P_v \subseteq \mathcal{T}_1(A)$, $A \nsubseteq Q_{s_1}$ then we have $f^{\rightarrow}(A) \in \mathcal{S}_2$ and $P_v = i_V{}^{\rightarrow}(P_v) \subseteq i_V{}^{\rightarrow}(\mathcal{T}_1(A)) \subseteq \mathcal{T}_2(f^{\rightarrow}A)$ by the openness of (f,F) w.r.t. (i_V, I_V). Also, since (f,F) is injective we have

$$A \nsubseteq Q_{s_1} \Rightarrow f^{\rightarrow}(A) \nsubseteq f^{\rightarrow}(Q_{s_1}) \Rightarrow \exists s_2 \in S_2 : f^{\rightarrow}(A) \nsubseteq Q_{s_2},\ P_{s_2} \nsubseteq f^{\rightarrow}(Q_{s_1}).$$

So, considering $P_v \in \mathcal{R}(\mathbf{W}_2)$, $f^{\rightarrow}(A) \in \mathcal{S}_2$, $P_v \subseteq \mathcal{T}_2(f^{\rightarrow}A)$ and $f^{\rightarrow}(A) \nsubseteq Q_{s_2}$ we get $\exists B \in \mathcal{S}_2 : P_v \subseteq \mathcal{T}_2(B),\ B \nsubseteq Q_{s_2},\ [B]^v \subseteq f^{\rightarrow}(A)$. Since (f,F) is continuous w.r.t. (i_V, I_V) we have $P_v = I_V^{\leftarrow}(P_v) \subseteq I_V^{\leftarrow}(\mathcal{T}_2(B)) \subseteq \mathcal{T}_1(f^{\leftarrow}B)$, i.e. $P_v \subseteq \mathcal{T}_1(f^{\leftarrow}B)$. Also, since (f,F) is bijective and $P_{s_2} \nsubseteq f^{\rightarrow}(Q_{s_1})$ we have $B \nsubseteq Q_{s_2} \Rightarrow P_{s_2} \subseteq B \Rightarrow B \nsubseteq f^{\rightarrow}(Q_{s_1}) \Rightarrow f^{\leftarrow}B = F^{\leftarrow}B \nsubseteq F^{\leftarrow}(f^{\rightarrow}(Q_{s_1})) = Q_{s_1} \Rightarrow f^{\leftarrow}B \nsubseteq Q_{s_1}$.

Now, if we take $D \in \mathcal{S}_2$ with $B \subseteq D$ and $P_v \subseteq \mathcal{K}_2(D)$ then we have $f^{\leftarrow}B \subseteq f^{\leftarrow}D$ and $P_v = I_V{}^{\leftarrow}(P_v) \subseteq I_V{}^{\leftarrow}(\mathcal{K}_2(D)) \subseteq \mathcal{K}_1(f^{\leftarrow}(D))$ by the cocontinuity of (f,F) w.r.t. (i_V, I_V). So we get

$$\begin{aligned}[f^{\leftarrow}B]^v &= \bigcap\{E \in \mathcal{S}_1 \mid f^{\leftarrow}B \subseteq E,\ P_v \subseteq \mathcal{K}_1(E)\}\\ &\subseteq \bigcap\{f^{\leftarrow}D \in \mathcal{S}_1 \mid D \in \mathcal{S}_2,\ B \subseteq D,\ P_v \subseteq \mathcal{K}_2(D)\}\\ &= f^{\leftarrow}(\bigcap\{D \in \mathcal{S}_2 \mid B \subseteq D,\ P_v \subseteq \mathcal{K}_2(D)\}) = f^{\leftarrow}([B]^v)\end{aligned}$$

Besides, since (f,F) is bijective and $[B]^v \subseteq f^{\rightarrow}(A)$ we have $[f^{\leftarrow}B]^v \subseteq f^{\leftarrow}([B]^v) \subseteq f^{\leftarrow}(f^{\rightarrow}(A)) = F^{\leftarrow}(f^{\rightarrow}(A)) = A$, i.e. $[f^{\leftarrow}B]^v \subseteq A$. Hence we get $P_v \in \mathcal{R}(\mathbf{W}_1)$. □

Definition 3.12. Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a g.d.t.s. The families defined by
(i) $\mathrm{T}_0 = \{P_v \in \mathcal{V} \mid [s,t \in S,\ Q_s \nsubseteq Q_t] \Rightarrow [\exists D \in (\mathcal{T}^v \cup \mathcal{K}^v) : P_s \nsubseteq D \nsubseteq Q_t]\}$,
(ii) $\mathrm{T}_1 = \mathrm{T}_0 \cap \mathcal{R}_0$, (iii) $c\mathrm{T}_1 = \mathrm{T}_0 \cap c\mathcal{R}_0$, (iv) $b\mathrm{T}_1 = \mathrm{T}_0 \cap b\mathcal{R}_0$,
are called T_0 (T_1, co-T_1, bi-T_1) spectrums of $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ respectively.

Proposition 3.13. *For a g.d.t.s. $W = (S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ the following are satisfied:*
(a) $\mathrm{T}_1(W) = \{P_v \in \mathcal{V} \mid [s,t \in S,\ Q_s \nsubseteq Q_t] \Rightarrow [\exists D \in \mathcal{K}^v : P_s \nsubseteq D \nsubseteq Q_t]\}$,
(b) $c\mathrm{T}_1(W) = \{P_v \in \mathcal{V} \mid [s,t \in S,\ Q_s \nsubseteq Q_t] \Rightarrow [\exists D \in \mathcal{T}^v : P_s \nsubseteq D \nsubseteq Q_t]\}$.

Proof. (a) Let $P_v \in \mathrm{T}_1(W)$. So, we have $P_v \in \mathrm{T}_0(W)$ and $P_v \in \mathcal{R}_0(W)$. Let $s,t \in S$ with $Q_s \nsubseteq Q_t$. Since $P_v \in \mathrm{T}_0(W)$, $P_s \nsubseteq D \nsubseteq Q_t$ for some $D \in (\mathcal{T}^v \cup \mathcal{K}^v)$. If

$D \in \mathcal{K}^v$ then $P_v \in \{P_v \in \mathcal{V} \,|\, [s,t \in S,\ Q_s \not\subseteq Q_t] \Rightarrow [\exists D \in \mathcal{K}^v : P_s \not\subseteq D \not\subseteq Q_t]\}$. If $D \in \mathcal{T}^v$, since $P_v \subseteq \mathcal{T}(D)$, $D \not\subseteq Q_t$, $P_v \in \mathcal{R}_0(W)$ we have $[P_t]^v \subseteq D$. Considering $P_s \not\subseteq D \Rightarrow D \subseteq Q_s$, we get $P_t \subseteq [P_t]^v \subseteq Q_s$. That is, $P_t \subseteq B \subseteq Q_s$ for some $B \in \mathcal{K}^v$. So, $P_s \not\subseteq B \not\subseteq Q_t$ for some $B \in \mathcal{K}^v$ (from [6] Theorem 4.4.). Hence we get $\mathrm{T}_1(W) \subseteq \{P_v \in \mathcal{V} \,|\, [s,t \in S,\ Q_s \not\subseteq Q_t] \Rightarrow [\exists D \in \mathcal{K}^v : P_s \not\subseteq D \not\subseteq Q_t]\}$.

Now, let $P_v \in \{P_v \in \mathcal{V} \,|\, [s,t \in S,\ Q_s \not\subseteq Q_t] \Rightarrow [\exists D \in \mathcal{K}^v : P_s \not\subseteq D \not\subseteq Q_t]\}$. So, $P_v \in \mathrm{T}_0(W)$. Let $A \in \mathcal{S}$, $P_v \subseteq \mathcal{T}(A)$ and $A \not\subseteq Q_s$. If $P_t \not\subseteq A$ then $A \subseteq Q_t$ and $Q_t \not\subseteq Q_s$. So, $\exists B_t \in \mathcal{K}^v : P_t \not\subseteq B_t \not\subseteq Q_s$ from the hypothesis. This implies $\exists B_t \in \mathcal{K}^v : P_s \subseteq B_t \subseteq Q_t$ for each $P_t \not\subseteq A$. Thus we get $P_s \subseteq \bigcap_{P_t \not\subseteq A} B_t \subseteq \bigcap_{P_t \not\subseteq A} Q_t = A$. Since $\bigcap_{P_t \not\subseteq A} B_t \in \mathcal{K}^v$ we obtain that $[P_s]^v \subseteq A$, i.e. $P_v \in \mathcal{R}_0(W)$. So $\{P_v \in \mathcal{V} \,|\, [s,t \in S,\ Q_s \not\subseteq Q_t] \Rightarrow [\exists D \in \mathcal{K}^v : P_s \not\subseteq D \not\subseteq Q_t]\} \subseteq \mathrm{T}_1(W)$.

(b) This can be shown similarly. □

The next corollary follows from Proposition 3.4.

Corollary 3.14. *For a complemented g.d.t.s.* $(S, \mathcal{S}, \sigma, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ *we have* $\mathrm{T}_1 = c\mathrm{T}_1 = b\mathrm{T}_1$.

Definition 3.15. Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a g.d.t.s. The families defined by
(i) $\mathrm{T}_2 = \mathrm{T}_0 \cap \mathcal{R}_1$, (ii) $c\mathrm{T}_2 = \mathrm{T}_0 \cap c\mathcal{R}_1$, (iii) $b\mathrm{T}_2 = \mathrm{T}_0 \cap b\mathcal{R}_1$,
are called T_2 (co-T_2, bi-T_2) spectrums of $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ respectively.

Proposition 3.16. *For a g.d.t.s.* $W = (S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ *we have*

$$b\mathrm{T}_2(W) = \{P_v \in \mathcal{V} \,|\, [s,t \in S,\ Q_s \not\subseteq Q_t] \Rightarrow [\exists A \in \mathcal{T}^v\ \exists B \in \mathcal{K}^v : A \subseteq B, P_s \not\subseteq B, A \not\subseteq Q_t]\}.$$

Proof. Let $P_v \in b\mathrm{T}_2(W)$. So, we have $P_v \in \mathrm{T}_0(W)$, $P_v \in \mathcal{R}_1(W)$ and $P_v \in c\mathcal{R}_1(W)$. Let $s,t \in S$ with $Q_s \not\subseteq Q_t$. Since $P_v \in \mathrm{T}_0(W)$, $P_s \not\subseteq D \not\subseteq Q_t$ for some $D \in (\mathcal{T}^v \cup \mathcal{K}^v)$. If $D \in \mathcal{T}^v$, since $P_v \in \mathcal{R}_1(W)$ there exists $A \in \mathcal{S}$ such that $P_v \subseteq \mathcal{T}(A)$, $A \not\subseteq Q_t$ and $P_s \not\subseteq [A]^v$. So, if we take $B = [A]^v$ then $A \in \mathcal{T}^v$, $B \in \mathcal{K}^v$, $A \subseteq B$, $P_s \not\subseteq B$ and $A \not\subseteq Q_t$. Hence $P_v \in \mathcal{Y} = \{P_v \in \mathcal{V} \,|\, [s,t \in S,\ Q_s \not\subseteq Q_t] \Rightarrow [\exists A \in \mathcal{T}^v\ \exists B \in \mathcal{K}^v : A \subseteq B,\ P_s \not\subseteq B,\ A \not\subseteq Q_t]\}$. If $D \in \mathcal{K}^v$, since $P_v \in c\mathcal{R}_1(W)$ there exists $B \in \mathcal{S}$ such that $P_v \subseteq \mathcal{K}(B)$, $P_s \not\subseteq B$ and $]B[^v \not\subseteq Q_t$. So, if we take $A =]B[^v$ then $A \in \mathcal{T}^v$, $B \in \mathcal{K}^v$, $A \subseteq B$, $P_s \not\subseteq B$ and $A \not\subseteq Q_t$. Hence we get $P_v \in \mathcal{Y}$ and so $b\mathrm{T}_2(W) \subseteq \mathcal{Y}$.

Now, let $P_v \in \mathcal{Y}$. **(1)** If $s,t \in S$ with $Q_s \not\subseteq Q_t$ then $A \subseteq B$, $P_s \not\subseteq B$ and $A \not\subseteq Q_t$ for some $A \in \mathcal{T}^v$, $B \in \mathcal{K}^v$. If we take $D = B$ then we have $D \in (\mathcal{T}^v \cup \mathcal{K}^v)$, and $P_s \not\subseteq D \not\subseteq Q_t$, i.e. $P_v \in \mathrm{T}_0(W)$. **(2)** If $C \in \mathcal{S}$, $P_v \subseteq \mathcal{T}(C)$, $C \not\subseteq Q_s$ and $P_t \not\subseteq C$ then we have $Q_t \not\subseteq Q_s$. So, considering $P_v \in \mathcal{Y}$, we have $A \subseteq B$, $P_t \not\subseteq B$ and $A \not\subseteq Q_s$ for some $A \in \mathcal{T}^v$, $B \in \mathcal{K}^v$. This follows $P_v \subseteq \mathcal{T}(A)$, $A \not\subseteq Q_s$ and $P_t \not\subseteq [A]^v$, i.e. $P_v \in \mathcal{R}_1(W)$. **(3)** If $C \in \mathcal{S}$, $P_v \subseteq \mathcal{K}(C)$, $P_s \not\subseteq C$ and $C \not\subseteq Q_t$ then we have $Q_s \not\subseteq Q_t$. So, considering $P_v \in \mathcal{Y}$, we have $A \subseteq B$, $P_s \not\subseteq B$ and $A \not\subseteq Q_t$ for some $A \in \mathcal{T}^v$, $B \in \mathcal{K}^v$. This follows $P_v \subseteq \mathcal{K}(B)$, $P_s \not\subseteq B$ and $]B[^v \not\subseteq Q_t$, i.e. $P_v \in c\mathcal{R}_1(W)$. Thus we get $\mathcal{Y} \subseteq b\mathrm{T}_2(W)$. □

The next follows from Proposition 3.5.

Corollary 3.17. *For a complemented g.d.t.s.* $(S, \mathcal{S}, \sigma, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ *we have* $\mathrm{T}_2 = c\mathrm{T}_2 = b\mathrm{T}_2$.

Definition 3.18. Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a g.d.t.s. The families defined by
(i) $\mathrm{T}_3 = \mathrm{T}_0 \cap \mathcal{R}$, (ii) $c\mathrm{T}_3 = \mathrm{T}_0 \cap c\mathcal{R}$, (iii) $b\mathrm{T}_3 = \mathrm{T}_0 \cap b\mathcal{R}$,
are called T_3 (co-T_3, bi-T_3) spectrums of $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ respectively.

The next corollaries follow from Proposition 3.6 and Proposition 3.2 respectively.

Corollary 3.19. *For a complemented g.d.t.s.* $(S, \mathcal{S}, \sigma, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ *we have* $\mathrm{T}_3 = c\mathrm{T}_3 = b\mathrm{T}_3$.

Corollary 3.20. *For a g.d.t.s. we have* $\mathrm{T}_3 \subseteq \mathrm{T}_2 \subseteq \mathrm{T}_1 \subseteq \mathrm{T}_0$, $c\mathrm{T}_3 \subseteq c\mathrm{T}_2 \subseteq c\mathrm{T}_1 \subseteq \mathrm{T}_0$ *and* $b\mathrm{T}_3 \subseteq b\mathrm{T}_2 \subseteq b\mathrm{T}_1 \subseteq \mathrm{T}_0$.

Example 3.21. (i) Let $(S, \mathcal{S}, \tau, \kappa)$ be a d.t.s. and $(V, \mathcal{V}) = (1, \mathcal{P}(1))$ the discrete texture on a singleton. If $(S, \mathcal{S}, \tau, \kappa)$ is R_0, (R_1, regular, T_i, $i = 0, 1, 2, 3$) then for the g.d.t.s. $W = (S, \mathcal{S}, \tau^g, \kappa^g, V, \mathcal{V})$, $P_v \in \mathcal{R}_0$ ($P_v \in \mathcal{R}_1$, $P_v \in \mathcal{R}$, $P_v \in \mathrm{T}_i$, $i = 0, 1, 2, 3$) respectively for all $v \in V$, i.e. $v = 0$.

(ii) For a g.d.t.s. $W = (S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$, the following hold:

- (a) $P_v \in \mathcal{R}_i(W)$ ($P_v \in c\mathcal{R}_i(W)$) $\Leftrightarrow (S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ is R_i (co-R_i) for $i = 0, 1$.
- (b) $P_v \in \mathcal{R}(W)$ ($P_v \in c\mathcal{R}(W)$) $\Leftrightarrow (S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ is regular (co-regular).
- (c) $P_v \in \mathrm{T}_0 \Leftrightarrow (S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ is T_0.
- (d) $P_v \in \mathrm{T}_i$ ($P_v \in c\mathrm{T}_i(W)$) $\Leftrightarrow (S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ is T_i (co-T_i) for $i = 1, 2, 3$.

(iii) Let $(S, \mathcal{S} = \mathcal{P}(S))$ and $(V, \mathcal{V} = \mathcal{P}(V))$ be discrete textures with $V = \{v, y, z\}$ where S has more than one element. If we define $\mathcal{T}_1, \mathcal{T}_2, \mathcal{K}_1, \mathcal{K}_2 : \mathcal{S} \to \mathcal{V}$ by

$$\mathcal{T}_1(A) = \begin{cases} V, & A = \emptyset \text{ or } A = S \\ \{v\}, & \text{otherwise} \end{cases} \qquad \mathcal{K}_1(A) = \begin{cases} V, & A = \emptyset \text{ or } A = S \\ \{y\}, & \text{otherwise} \end{cases}$$

$$\mathcal{T}_2(A) = \begin{cases} V, & A = \emptyset \text{ or } A = S \\ \{v, z\}, & \text{otherwise} \end{cases} \qquad \mathcal{K}_2(A) = \begin{cases} V, & A = \emptyset \text{ or } A = S \\ \{y, z\}, & \text{otherwise} \end{cases}$$

for all $A \in \mathcal{S}$ then we have four g.d.t.s. $W_{ij} = (S, \mathcal{S}, \mathcal{T}_i, \mathcal{K}_j, V, \mathcal{V})$. Note that $\mathcal{T}_1^v = \mathcal{S} = \mathcal{P}(S)$, $\mathcal{T}_1^y = \mathcal{T}_1^z = \{S, \emptyset\}$, $\mathcal{K}_1^y = \mathcal{S} = \mathcal{P}(S)$, $\mathcal{K}_1^v = \mathcal{K}_1^z = \{S, \emptyset\}$, $\mathcal{T}_2^v = \mathcal{T}_2^z = \mathcal{S} = \mathcal{P}(S)$, $\mathcal{T}_2^y = \{S, \emptyset\}$, $\mathcal{K}_2^v = \{S, \emptyset\}$, $\mathcal{K}_2^y = \mathcal{K}_2^z = \mathcal{S} = \mathcal{P}(S)$. So we get:

(a) $\mathcal{R}_0(W_{11}) = \mathcal{R}_1(W_{11}) = \mathcal{R}(W_{11}) = \{P_y, P_z\}$, $c\mathcal{R}_0(W_{11}) = c\mathcal{R}_1(W_{11}) = c\mathcal{R}(W_{11}) = \{P_v, P_z\}$, $\mathrm{T}_0(W_{11}) = \{P_v, P_y\}$, $\mathrm{T}_1(W_{11}) = \mathrm{T}_2(W_{11}) = \mathrm{T}_3(W_{11}) = \{P_y\}$ and $c\mathrm{T}_1(W_{11}) = c\mathrm{T}_2(W_{11}) = c\mathrm{T}_3(W_{11}) = \{P_v\}$.

(b) $\mathcal{R}_0(W_{12}) = \mathcal{R}_1(W_{12}) = \mathcal{R}(W_{12}) = \{P_y, P_z\}$, $c\mathcal{R}_0(W_{12}) = c\mathcal{R}_1(W_{12}) = c\mathcal{R}(W_{12}) = \{P_v\}$.

(c) $\mathcal{R}_0(W_{21}) = \mathcal{R}_1(W_{21}) = \mathcal{R}(W_{21}) = \{P_y\}$, $c\mathcal{R}_0(W_{21}) = c\mathcal{R}_1(W_{21}) = c\mathcal{R}(W_{21}) = \{P_v, P_z\}$.

(d) $\mathcal{R}_0(W_{22}) = \mathcal{R}_1(W_{22}) = \mathcal{R}(W_{22}) = \{P_y, P_z\}$, $c\mathcal{R}_0(W_{22}) = c\mathcal{R}_1(W_{22}) = c\mathcal{R}(W_{22}) = \{P_v, P_z\}$, $\mathrm{T}_0(W_{22}) = V$, $b\mathrm{T}_1(W_{22}) = b\mathrm{T}_2(W_{22}) = b\mathrm{T}_3(W_{22}) = \{P_z\}$.

(iv) *(Example for Theorem 3.7, Theorem 3.10(a) and Theorem 3.11(a))* Consider $((i_S, I_S), (i_V, I_V)) : W_{11} \to W_{12}$. (i_S, I_S) is continuous, open, coopen, closed, coclosed (but not cocontinuous) w.r.t. (i_V, I_V) and $\mathcal{R}_0(W_{11}) \subseteq \mathcal{R}_0(W_{12})$, $c\mathcal{R}_0(W_{12}) \subseteq c\mathcal{R}_0(W_{11})$, $\mathcal{R}_1(W_{11}) \subseteq \mathcal{R}_1(W_{12})$, $\mathcal{R}(W_{11}) \subseteq \mathcal{R}(W_{12})$.

(v) *(Example for Theorem 3.8)* Consider $((i_S, I_S), (i_V, I_V)) : W_{12} \to W_{21}$. (i_S, I_S) is cocontinuous, open, coopen (but not continuous, closed, coclosed) w.r.t. (i_V, I_V) and $\mathcal{R}_0(W_{21}) \subseteq \mathcal{R}_0(W_{12})$, $c\mathcal{R}_0(W_{12}) \subseteq c\mathcal{R}_0(W_{21})$.

(vi) *(Example for Theorem 3.10(b) and Theorem 3.11(b))* Consider $((i_S, I_S), (i_V, I_V)) : W_{12} \to W_{22}$. (i_S, I_S) is cocontinuous, open, coopen, closed, coclosed (but not continuous) w.r.t. (i_V, I_V) and $c\mathcal{R}_1(W_{12}) \subseteq c\mathcal{R}_1(W_{22})$, $c\mathcal{R}_1(W_{12}) \subseteq c\mathcal{R}_1(W_{22})$.

(vii) *(Example for Theorem 3.10(c) and Theorem 3.11(c))* Consider $((i_S, I_S), (i_V, I_V)) : W_{22} \to W_{21}$. (i_S, I_S) is bicontinuous, open, coopen (but not closed, coclosed) w.r.t. (i_V, I_V) and $\mathcal{R}_1(W_{21}) \subseteq \mathcal{R}_1(W_{22})$, $\mathcal{R}(W_{21}) \subseteq \mathcal{R}(W_{22})$.

(viii) *(Example for Theorem 3.10(d) and Theorem 3.11(d))* Consider $((i_S, I_S), (i_V, I_V)) : W_{21} \to W_{11}$. (i_S, I_S) is bicontinuous, closed, coclosed (but not open, coopen) w.r.t. (i_V, I_V) and $c\mathcal{R}_1(W_{11}) \subseteq c\mathcal{R}_1(W_{21})$, $c\mathcal{R}(W_{11}) \subseteq c\mathcal{R}(W_{21})$.

4. Categorical aspects

In this section, we examine the relations between separation spectra of g.d.t.s. and separation properties of d.t.s. from a category-theoretic point of view. Our reference for category theory is [1].

The class of d.t.s. and bicontinuous difunctions between them form a category denoted by **dfDitop** [5]. T_k d.t.s. form a full concrete subcategory $\mathbf{dfDitop}_k$ of **dfDitop** for $k = 0, 1, 2, 3$ [6]. The class of g.d.t.s. and the relatively bicontinuous difunction pairs between them form a category denoted by **dfGDitop**, moreover **dfDitop** can be embedded in **dfGDitop** [7].

Since we define separation properties of g.d.t.s. by using spectrum idea (compatible with the grading setting) it is not possible to mention about "T_k *g.d.t.s.*" but "T_k *spectrum of g.d.t.s.*" for $k = 0, 1, 2, 3$. So, as a subcategory of **dfGDitop**, let $\mathbf{dfGDitop}_k$ denote the category of g.d.t.s. with nonempty $b\mathrm{T}_k$ spectrum for $k = 0, 1, 2, 3$, i.e. $Ob\mathbf{dfGDitop}_k = \{W \in Ob\mathbf{dfGDitop} \mid b\mathrm{T}_k(W) \neq \emptyset\}$.

Proposition 4.1. *The functor* $\mathfrak{F}_k : \mathbf{dfDitop}_k \to \mathbf{dfGDitop}_k$ *defined by*

$$\mathfrak{F}_k((f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2))$$
$$((f, F), (i, I)) : (S_1, \mathcal{S}_1, \tau_1^g, \kappa_1^g, 1, \mathcal{P}(1)) \to (S_2, \mathcal{S}_2, \tau_2^g, \kappa_2^g, 1, \mathcal{P}(1)).$$

is a full embedding for $k = 0, 1, 2, 3$.

Proof. From Example 3.21(ii), $\mathfrak{F}_k$ is injective on objects. For any morphisms (f, F), $(g, G) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$; we have $\mathfrak{F}_k((f, F)) = \mathfrak{F}_k((g, G)) \Rightarrow ((f, F), (i, I)) = ((g, G), (i, I)) \Rightarrow (f, F) = (g, G)$. So, $\mathfrak{F}_k$ is an embedding. Moreover, for d.t.s. $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$, if $((f, F), (i, I)) : (S_1, \mathcal{S}_1, \tau_1^g, \kappa_1^g, 1, \mathcal{P}(1)) \to (S_2, \mathcal{S}_2, \tau_2^g, \kappa_2^g, 1, \mathcal{P}(1))$ is a morphism in $\mathbf{dfGDitop}_k$ then $\mathfrak{F}_k((f, F)) = ((f, F), (i, I))$ and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is a morphism in $\mathbf{dfDitop}_k$. Hence, $\mathfrak{F}_k$ is a full embedding for $(k = 0, 1, 2, 3)$. $\square$

Now, we fix a texture $(Z,\mathcal{Z})$ and think about $(Z,\mathcal{Z})$-g.d.t.s. as a subcategory $\mathbf{dfGDitop}_k(Z,\mathcal{Z})$ of $\mathbf{dfGDitop}_k$. For a $(Z,\mathcal{Z})$-g.d.t.s. $W=(S,\mathcal{S},\mathcal{T},\mathcal{K},Z,\mathcal{Z})\in Ob\mathbf{dfGDitop}_k(Z,\mathcal{Z})$ we define a d.t.s. $(S,\mathcal{S},\tau^{\cup},\kappa^{\cup})$ by

$$\tau^{\cup}=\ll\bigcup_{P_z\in b\mathrm{T}_k(W)}\mathcal{T}^z\gg \quad\text{and}\quad \kappa^{\cup}=\ll\bigcup_{P_z\in b\mathrm{T}_k(W)}\mathcal{K}^z\gg .$$

That is, $\tau^{\cup}$ is a topology on $(S,\mathcal{S})$ with subbase $\bigcup_{P_z\in b\mathrm{T}_k(W)}\mathcal{T}^z$ and $\kappa^{\cup}$ is a cotopology on $(S,\mathcal{S})$ with subbase $\bigcup_{P_z\in b\mathrm{T}_k(W)}\mathcal{K}^z$. So, $(S,\mathcal{S},\tau^{\cup},\kappa^{\cup})\in\mathbf{dfDitop}_k$ for $(k=0,1,2,3)$ by Example 3.21(ii) and [6].

However, it is an open problem that whether $b\mathrm{T}_k(W_2)\subseteq b\mathrm{T}_k(W_1)$ under the relatively bicontinuity of $((f,F),(i_Z,I_Z))$ is valid or not. So, if we naturally consider the mapping $\mathfrak{H}_k:\mathbf{dfGDitop}_k(Z,\mathcal{Z})\to\mathbf{dfDitop}_k$ defined by

$$\begin{aligned}\mathfrak{H}_k(((f,F),(i_Z,I_Z)):W_1=(S_1,\mathcal{S}_1,\mathcal{T}_1,\mathcal{K}_1,Z,\mathcal{Z})\to W_2=(S_2,\mathcal{S}_2,\mathcal{T}_1,\mathcal{K}_1,Z,\mathcal{Z}))\\ =(f,F):(S_1,\mathcal{S}_1,\tau_1^{\cup},\kappa_1^{\cup})\to(S_2,\mathcal{S}_2,\tau_2^{\cup},\kappa_2^{\cup})\end{aligned}$$

then it is an open problem that whether $\mathfrak{H}_k$ is a functor or not.

5. Conclusion

In this paper, different separation spectra of graded ditopological texture spaces, the properties of these separation spectra and their relations to the separation axioms in the ditopological case are investigated. As expected, the hierarchy of separation spectra and their fundamental role in complemented structures are compatible with the ditopological case (see Propositions 3.2, 3.4, 3.5, 3.6 and Corollary 3.20). Obviously, the separation spectra of g.d.t.s. are more general than the separation axioms in d.t.s. (see Example 3.21(i)) and, of course, some generalizations of the properties of the separation axioms in d.t.s. are not valid for the graded ditopological case. For example, the separation properties are preserved under further conditions (see Theorems 3.7, 3.8, 3.10, 3.11 and [6]). In Section 4, we state an open problem that determines whether $\mathfrak{H}_k$ is a functor or not. Although we expect that $\mathfrak{H}_k$ is not a functor, it is not so easy to find a counterexample for it.

The construction of separations in the theory of graded ditopological texture spaces can be useful to study compactness in this theory and to discover new properties in this theory. Considering the interrelations between the structures g.d.t.s., d.t.s., Hutton algebras, fuzzy topological spaces, interior- closure textures and diframes, this work has the potential to improve these areas of study. Other separation notions such as normality, complete regularity, and relations between separation spectra and compactness spectra in the theory of graded ditopological texture spaces can be investigated in further studies.

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Department of Mathematics, Kırklareli University, 39100, Kırklareli, Turkey
E-mail: rekmekci@klu.edu.tr
ORCID iD: https://orcid.org/0000-0001-6496-7358