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ON $\mathcal{I}^{\mathcal{K}}$-CONVERGENCE IN TOPOLOGICAL SPACES VIA SEMI-OPEN SETS

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Abstract. A sequence $\{x_n\}$ is s-$\mathcal{I}^{\mathcal{K}}$-convergent to ξ, if there exists a 'big enough' subsequence $\{x_{n_k}\}$ which $\mathcal{K}$-converges to ξ via semi-open sets. In this paper, we introduce the concept of s-$\mathcal{I}^{\mathcal{K}}$-convergence which generalizes $\mathcal{S}$-$\mathcal{I}$-convergence and discuss some properties, as well as its relation with compact sets. For two given ideals $\mathcal{I}$ and $\mathcal{K}$, we justify the existence of an ideal such that $\mathcal{I}^{\mathcal{K}}$-convergence and convergence with the third ideal coincides for semi-open sets. Moreover, the notion of s-$\mathcal{I}^{\mathcal{K}}$-cluster point of a sequence is defined and studied here. We characterize the collection of s-$\mathcal{I}^{\mathcal{K}}$-cluster points of a sequence as semi-closed subsets of the space.

1. Introduction

After Kuratowski introduced ideals in 1933, the term became known as a collection of sets considered to be "small" or "negligible". In an ordinary space, three basic topological notions, namely convergence, closure and neighborhood, play a crucial role in determining other topological properties. In the recent past, ideal theory has been used together with convergence theory to develop some promising generalizations of existing concepts in Point-Set Topology.

In particular, two notions for the convergence of a sequence were introduced in 2000 by Kostyrko et al. [7], called $\mathcal{I}$ and $\mathcal{I}^*$-convergence for the real numbers and later, in 2005, by Lahiri and Das [8] for a topological space. Undoubtedly, it was actively practiced in the following period, and some work from it can be found in [4]. Later, in 2011, Macaz and Sleziak [10] introduced the notion of $\mathcal{I}^{\mathcal{K}}$-convergence of a function in a topological space. Although it appears in the context of $\mathcal{I}^*$-convergence, $\mathcal{I}^{\mathcal{K}}$-convergence further extends the notion of ideal convergence. In particular, if the ideals $\mathcal{I}$ and $\mathcal{K}$ coincide, then the terms $\mathcal{I}^{\mathcal{K}}$ and $\mathcal{I}$-convergence also apply. In the last decade, $\mathcal{I}^{\mathcal{K}}$-convergence has been studied in detail in several articles, namely, [3, 10].

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On the other hand, Levine introduced semi-open sets [9] in a topological space in 1963, and afterwards it was used to generalize several concepts in Point-Set Topology. Recently, Guevara et al. [5] used the concept of semi-open set to define and study the notion of $\mathcal{S}$-$\mathcal{I}$-convergence in topological spaces. In this paper, we define $\mathcal{I}^{\mathcal{K}}$-convergence using semi-open sets in a topological space and denote it by $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence.

An ideal on a set S is a collection of subsets of the given set that is closed under subset inclusion and finite union. Fin is a basic ideal that includes all finite subsets of S. For a given ideal $\mathcal{I} \subset P(\mathbb{N})$, two additional subsets of $P(\mathbb{N})$ namely, $\mathcal{I}^*$ or $F(\mathcal{I})$ and $\mathcal{I}^+$ of $P(\mathbb{N})$, are defined, namely a: $\mathcal{I}^* := \{A \subset \mathbb{N} : A^{\complement} \in \mathcal{I}\}$ and $\mathcal{I}^+$:= collection of all subsets that do not belong to $\mathcal{I}$. We say that two ideals $\mathcal{I}$ and $\mathcal{K}$ on S fulfill the ideality condition if $\mathcal{I} \cup \mathcal{K}$ is a proper ideal [14], alternatively, $S \neq I \cup K$, for all $I \in \mathcal{I}$, $K \in \mathcal{K}$.

In this paper, we deal with the proper ideals (not containing $\mathbb{N}$) on the set of natural numbers $\mathbb{N}$, which are admissible ideals (containing all finite subsets of $\mathbb{N}$), to study different aspects of $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence in a topological space. Thus, in the following part of this paper, all considered ideals are proper and admissible. The main results in this paper are divided into two sections.

Section 2 introduces the $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence for two ideals $\mathcal{I}$, $\mathcal{K}$ satisfying the ideality condition and some basic properties are investigated. Section 4 deals with the definition and basic properties of $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster points of a sequence. The two sections 3 and 5 contribute to answering an existence problem (Theorem 3.14, Theorem 5.5), which can be formulated as follows: Whether there exists an ideal $\mathcal{J}$ for two given ideals $\mathcal{I}$, $\mathcal{K}$ such that $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence and $\mathcal{S}$-$\mathcal{J}$-convergence coincide under certain assumptions. Section 5 also contains the characterization of the collection $sC_x(\mathcal{I}^{\mathcal{K}})$ of $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster points of a sequence as semi-closed subsets (Theorem 5.8) and further, we evaluate a condition for coincidence for the collection of $sC_x(\mathcal{I}^{\mathcal{K}})$ and $sL(\mathcal{I}^{\mathcal{K}})$ (Theorem 5.3).

2. Preliminaries

For a given function $f : S \to X$, which is in fact a generalization of a sequence, Macaz and Sleziak [10] defined the $\mathcal{I}^{\mathcal{K}}$-convergence for two ideals $\mathcal{I}$ and $\mathcal{K}$ on S.

Definition 2.1 ([10]). A function $f : S \to X$ is said to be $\mathcal{K}$-convergent to $x \in X$, if for any nonempty open set U containing x, we have $\{s \in S : f(s) \notin U\} \in \mathcal{K}$.

Definition 2.2 ([10]). A function $f : S \to X$ is said to be $\mathcal{I}^{\mathcal{K}}$-convergent to $x \in X$, if there exists a set $M \in F(\mathcal{I})$ such that the function $g : S \to X$ given by $g(s) = f(s)$, if $s \in M$ and $g(s) = x$, if $s \notin M$, is $\mathcal{K}$-convergent to x. If f is $\mathcal{I}^{\mathcal{K}}$-convergent to x, then we write $\mathcal{I}^{\mathcal{K}}$-$\lim f = x$.

In particular, following is the definition of $\mathcal{I}^{\mathcal{K}}$-convergence of a sequence in a space.

Definition 2.3. A sequence $\{x_n\}$ is said to be $\mathcal{I}^{\mathcal{K}}$-convergent to an element $\xi \in X$ if there exists a set $M = \{n_1, n_2, \dots, n_k, \dots\} \in \mathcal{I}^*$ such that the subsequence $\{x_{n_k}\}$ is $\mathcal{K}$-convergent to ξ.

Proposition 2.4 ([14, Proposition 2.1]). *Let X be a topological space and $f : S \to X$ be a function. Let $\mathcal{I}, \mathcal{K}$ be two ideals on S such that $\mathcal{I} \cup \mathcal{K}$ is an ideal. Then*
(i) $\mathcal{I}^{\mathcal{K}^*}$-$\lim f = x$ if and only if $(\mathcal{I} \cup \mathcal{K})^*$-$\lim f = x$.

(ii) $\mathcal{I}^{\mathcal{K}}$-$\lim f = x$ implies $\mathcal{I} \cup \mathcal{K}$-$\lim f = x$.

Some of the definitions and concepts of generalized open sets in [2,5,9,11,12] that are used in the content of the accompanying sections are listed below.

Definition 2.5. Let X be a topological space. Then
(i) $O \subset X$ is said to be semi-open [9] if there exists an open set U such that $U \subset O \subset \overline{U}$. The collection of all semi-open subsets of X is denoted by $SO(X)$.

(ii) The complement set of a semi-open set is termed as a semi-closed set.

(iii) The semi-closure [9] of a subset F of X, denoted by $sCl(F)$, is defined as the intersection of all semi-closed set containing F. Otherwise, a point $x \in sCl(A)$ if and only if for every semi-open set U containing x, $U \cap A \neq \emptyset$.

(iv) An element $x \in F \subset X$ is said to be semi-limit point [2] of F, if for every semi-open set O containing x, $O \cap F \neq \phi$.

(v) A topological space X is said to be semi-Hausdorff [11] if for every distinct pair of elements x, $y \in X$, there exists a disjoint pair of semi-open sets U and V containing x and y respectively.

(vi) A function $f : X \to Y$ is said to be irresolute [5] if $f^{-1}(O) \in SO(X)$ for each $O \in SO(Y)$. A function $f : X \to Y$ is irresolute [5] if and only if for each $x \in X$ and each $V \in SO(Y)$ containing $f(x)$, there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subset V$.

(vii) A function $f : X \to Y$ is said to be semi-continuous [9] if $f^{-1}(O) \in SO(X)$ for each open set $O \in Y$. A function $f : X \to Y$ is semi-continuous [9] if and only if for each $x \in X$ and each open V in Y containing $f(x)$, there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subset V$.

Theorem 2.6 ([5, Theorem 3.5]). *Let X be a space and $\mathcal{I}$ be an ideal. If every sequence $\{x_n\}$ in X has an $\mathcal{S}$-$\mathcal{I}$-cluster point, then every infinite subset of X has a semi-ω-accumulation point. The converse is true if $\mathcal{I}$ does not contain any infinite sets.*

Meanwhile, we refer to [6, 8] for the basic general topological and ideal theoretic terminologies, definitions and results mentioned in the content. In the following, unless otherwise stated, we denote X as topological space and $\mathcal{I}$ and $\mathcal{K}$ as ideals on $\mathbb{N}$.

3. Some properties of $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence

As a generalization of the work on $\mathcal{S}$-$\mathcal{I}$-convergence, we consider $\mathcal{I}^{\mathcal{K}}$-convergence, which is one of the most generalized among all ideal convergences, to study its properties using semi-open sets. We denote it as $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence in a space X. A sequence $x = \{x_n\}$ is called $\mathcal{S}$-$\mathcal{I}$-convergent [5] to an element $\xi \in X$ if for every non-empty semi-open set U containing ξ, the set $\{n \in \mathbb{N} : x_n \notin U\}$ belongs to $\mathcal{I}$. Here we define the notion of $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence of a sequence in a topological space.

Definition 3.1. A sequence $\{x_n\}$ is said to be $\mathcal{S}$-$\mathcal{I}^{\mathcal{M}}$-convergent to an element $\xi \in X$ if there exists a set $M = \{n_1, n_2, \ldots, n_k, \ldots\} \in \mathcal{I}^*$ such that the subsequence $\{x_{n_k}\}$ is $\mathcal{S}$-$\mathcal{M}$-convergent to ξ, where $\mathcal{M}$ is an ideal convergence mode.

If $\mathcal{K}$ is an ideal and $\mathcal{M} = \mathcal{K}^*$, then we say that $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}^*}$-convergent to an element $\xi \in X$. Also, if $\mathcal{M} = \mathcal{K}$, then $\{x_n\}$ is said to be $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to an element $\xi \in X$. If the ideal $\mathcal{K}$ does not contain an infinite set, then Definition 3.1 exhibits the $\mathcal{S}$-$\mathcal{I}^*$-convergence. Also, if $\mathcal{K}$ is a P-ideal [3] (condition AP [8]), then $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}^*}$ and $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence both coincide. Basically, $\mathcal{K}$-convergence implies $\mathcal{I}^{\mathcal{K}}$-convergence (Definition 2.3), analogously we have the following lemma.

Lemma 3.2. *$\mathcal{S}$-$\mathcal{K}$ convergence implies $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence.*

Proof. Let $\{x_n\}$ be an $\mathcal{S}$-$\mathcal{K}$-convergent sequence to an element ξ in a space X. Then, for any semi-open set U containing ξ, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{K}$. Then, for any $M \in \mathcal{I}^*$, the set $\{n \in M : x_n \notin U\} \subset \{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{K}$. Thus, $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent in X. □

Proposition 3.3. *Let $\{x_n\}$ be a sequence in X. For $\mathcal{I}, \mathcal{K}$ be two ideals on $\mathbb{N}$ such that $\mathcal{I} \cup \mathcal{K}$ is an ideal. Then*
(i) $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}^*}$-converges to x if and only if $\{x_n\}$ is $\mathcal{S}$-$(\mathcal{I} \cup \mathcal{K})^*$-converges to x.
(ii) Also, $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-converges to x implies $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I} \cup \mathcal{K}$-converges to x.

Proof. Consider two ideals $\mathcal{I}$ and $\mathcal{K}$ such that $\mathcal{I} \cup \mathcal{K}$ is an ideal.

(i) Let $\{x_n\}$ be a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}^*}$-convergent to x. So, there exists $M = \{n_1, n_2, n_3, \ldots\}$ such that $\{x_{n_k}\}$ is $\mathcal{S}$-$\mathcal{K}^*$-convergent to x. Then, for any semi open set U containing x, there exists $N \in \mathcal{K}^*$ such that $\{n_k \in N : x_{n_k} \notin U\} \in Fin$, i.e., $\{n \in M \cap N : x_n \notin U\} \in Fin$. But $M \cap N \in (\mathcal{I} \cap \mathcal{K})^*$. Hence, $\{x_n\}$ is $\mathcal{S}$-$(\mathcal{I} \cup \mathcal{K})^*$-convergent to x.

Conversely, let $\{x_n\}$ be $\mathcal{S}$-$(\mathcal{I} \cup \mathcal{K})^*$-convergent to x. So, for a semi-open set U containing x, there exists $M \cap N \in (\mathcal{I} \cup \mathcal{K})^*$, where $M \in \mathcal{I}^*, N \in \mathcal{K}^*$, such that $\{n \in M \cap N : x_n \notin U\} \in Fin$. Consider a subsequence $\{x_{n_k}\}_{n_k \in M}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that $\{n_k \in N : x_{n_k} \notin U\} \in Fin$. Thus, $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}^*}$-convergent to x.

(ii) Let $\{x_n\}$ be $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to x. So, there exists $M \in \mathcal{I}^*$ such that $\{n \in M : x_n \notin U_x\} \in \mathcal{K}$, where U is an semi open set containing x. Then, $\{n \in \mathbb{N} : x_n \notin U\} \subset \{n \in M : x_n \notin U\} \cup \{n : n \notin M\}$. Therefore, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I} \cup \mathcal{K}$. Thus, $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I} \cup \mathcal{K}$-convergent to x. □

Following results are immediate consequences of Proposition 3.3.

Corollary 3.4. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals on $\mathbb{N}$ provided $\mathcal{I} \cup \mathcal{K}$ is an ideal. Then (i) $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}^*}$-convergence implies both $\mathcal{S}$-$\mathcal{I}$-convergence as well as $\mathcal{S}$-$\mathcal{K}$-convergence.*

(ii) $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence implies $\mathcal{S}$-$\mathcal{I}$-convergence provided $\mathcal{K} \subseteq \mathcal{I}$.

(iii) $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence implies $\mathcal{S}$-$\mathcal{K}$-convergence provided $\mathcal{I} \subseteq \mathcal{K}$.

If $\mathcal{K} = Fin$, then $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence implies $\mathcal{S}$-$\mathcal{I}$-convergence. A simple observation is that if the ideals $\mathcal{I}$ and $\mathcal{K}$ both coincide, then $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence appear as the $\mathcal{S}$-$\mathcal{I}$-convergence. Eventually, if $\mathcal{I}$, $\mathcal{K} = Fin$, then $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence coincide with $\mathcal{S}$-convergence and hence, further implies usual convergence. Following is an example to show that even if $\mathcal{I} = \mathcal{K} = Fin$, usual convergence does not coincide with the $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence.

Example 3.5. Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$ such that $\mathcal{I} \cup \mathcal{K}$ is an ideal. Let $\mathbb{R}$ be the set of real numbers with usual topology and let $\{x_n\}$ be defined as $x_n = (\frac{1}{n})$. Then $x_n \to is0$. Consider the semi-open set $U = (-1, 0]$ containing 0. But $\{n \in \mathbb{N} : x_n \notin U\} = \mathbb{N} \notin \mathcal{I} \cup \mathcal{K}$ for any ideals $\mathcal{I}$ and $\mathcal{K}$. So $\{x_n\}$ is not $\mathcal{S}$-$\mathcal{I} \cup \mathcal{K}$-convergent to 0. Therefore, $\{x_n\}$ is not $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to 0 by Proposition 3.3.

The proof of the next lemma follows immediately from the definition of $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence.

Lemma 3.6. *$\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence implies $\mathcal{I}^{\mathcal{K}}$-convergence, for any ideals $\mathcal{I}$ and $\mathcal{K}$.*

The following example shows that the converse of Lemma 3.6 is not necessarily true.

Example 3.7. Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$ such that $\mathcal{I} \cup \mathcal{K}$ is an ideal. Let $[-1, 1]$ be the interval in $\mathbb{R}$ with usual subspace topology and $\{x_n\}$ a sequence defined as $x_n = (\frac{1}{n})\sin(\frac{1}{n})$. Thus, for any open set U containing 0, we have $\{n \in \mathbb{N} : x_n \notin U\}$ is finite, that implies $x_n \to_{\mathcal{K}} 0$. Therefore, $\{x_n\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to 0. Let us now consider the semi-open set $V = (-1, 0]$ in $[-1, 1]$. Then $\{n \in \mathbb{N} : x_n \notin V\} = \mathbb{N} \notin \mathcal{I} \cup \mathcal{K}$. Hence, $x_n \nrightarrow_{\mathcal{I} \cup \mathcal{K}} 0$. It therefore follows from Proposition 2.4 (ii) that $\{x_n\}$ is not $\mathcal{I}^{\mathcal{K}}$-convergent to 0.

Theorem 3.8. *In a semi-Hausdorff space X, each $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent sequence has a unique $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-limit in X, provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

Proof. Consider a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent sequence $\{x_n\}$ in a semi-Hausdorff space X. Suppose that $\{x_n\}$ has two distinct $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-limits, say a and b. Being X a semi-Hausdorff space, there exists $U, V \in SO(X)$ with $U \cap V = \emptyset$ such that $a \in U$, $b \in V$. As $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to a and b, there exist $M_1, M_2 \in \mathcal{I}^*$ such that $\{n \in M_1 : x_n \notin U\} \in \mathcal{K}$ and $\{n \in M_2 : x_n \notin V\} \in \mathcal{K}$. So, for $M = M_1 \cap M_2 \in \mathcal{I}^*$ $(\neq \emptyset)$, the sets $\{n \in M : x_n \notin U\}$ and $\{n \in M : x_n \notin V\}$ belong to $\mathcal{K}$. Now, we have

$$\{n \in M : x_n \notin U \cap V\} = \{n \in M : x_n \notin U\} \cup \{n \in M : x_n \notin V\} \in \mathcal{K}.$$

Again,

$$\{n \in \mathbb{N} : x_n \notin U \cap V\} = \{n \in M : x_n \notin U \cap V\} \cup \{n \in M^{\complement} : x_n \notin U \cap V\}$$

$$\subseteq \{n \in M : x_n \notin U \cap V\} \cup M^{\complement} \in \mathcal{I} \cup \mathcal{K}.$$

But, $\mathcal{I} \cup \mathcal{K}$ is an ideal, which implies $\{n \in \mathbb{N} : x_n \notin U \cap V\} \neq \mathbb{N}$. Therefore, we conclude that $\{n \in \mathbb{N} : x_n \in U \cap V\} \neq \emptyset$, which is a contradiction. □

COROLLARY 3.9. *In a Hausdorff space X, each $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent sequence has a unique $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-limit in X, provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

REMARK 3.10. Since sequences in a general space are inadequate, the unique $\mathcal{I}^{\mathcal{K}}$-limit of every X-valued sequence does not necessarily imply that X is Hausdorff. But the space X will be at least T_1. If not, then there exists one distinct pair x, $y \in X$ such that for U (open set) containing x, it also contains y, or for U (open set) containing y, it also contains x. Without loss of generality, we assume that for U (open set) containing x implies that $y \in U$. Consider the sequence $\{x_n\}$ such that $x_n = y$, $\forall n \in \mathbb{N}$. Undoubtedly, $\{x_n\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to y. However, since x shares all its open sets with y, every sequence $\mathcal{I}^{\mathcal{K}}$ that converges to y also converges to x. Therefore, $\{x_n\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to x. This contradicts our assumption. Thus X is a T_1-space. If X is a first countable space, then X is also Hausdorff.

Immediately, we observe that the above remark is true for $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence in a semi-Hausdroff space. So, using Theorem 3.8, we have the following result.

THEOREM 3.11. *If each sequence in a space X has a unique $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-limit, then X has the semi-T_1 [11] property. Moreover, X is a semi-Hausdorff space provided X is first countable.*

In a space, uniqueness of $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-limit of sequences is a stronger property than that of usual limit. So, following question is relevant in this context: **Does the uniqueness of $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-limit of every sequence in X imply that the space X is Hausdorff?**

Proposition 3.3 hints at an interesting existence scenario which can be stated as whether there exists an ideal $\mathcal{J}$ such that $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence coincide with $\mathcal{S}$-$\mathcal{J}$-convergence. Similar results for $\mathcal{I}^{\mathcal{K}}$-convergence is affirmatively answered for Hausdorff spaces in article [14]. Following results contribute to the problem of interlinking between $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergence and usual ideal convergence.

REMARK 3.12. If $\mathcal{I}$ and $\mathcal{K}$ be two ideals on $\mathbb{N}$ satisfying the ideality condition. Consider the set $\{K \cup J : K \in \mathcal{K}\}$, for any $J \in \mathcal{I}$. If $\mathcal{J}$ is the collections of sets under the operations finite union and subset inclusion among the members of $\{K \cup J : K \in \mathcal{K}\}$. Then, $\mathcal{J}$ is an ideal on $\mathbb{N}$. Also, $\mathcal{J} \subseteq \mathcal{I} \cup \mathcal{K}$.

LEMMA 3.13. *Let $\mathcal{I}$, $\mathcal{K}$ be two ideals on $\mathbb{N}$ satisfying ideality condition. Let $\{x_n\}$ be a X-valued sequence. If $\mathcal{J}$ = ideal generated by the set $\{K \cup J : K \in \mathcal{K})$, for any $J \in \mathcal{I}$. Then $\{x_n\}$ is $\mathcal{S}$-$\mathcal{J}$-convergent to x implies $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to x.*

Proof. Let $x = \{x_n\}$ be $\mathcal{S}$-$\mathcal{J}$-convergent in X, where $\mathcal{J}$= ideal generated by the set $\{K \cup J : K \in \mathcal{K})$, for any $J \in \mathcal{I}$. Assuming $M = J^{\complement}$ and $V \in SO(X)$, we can observe that $\{n_k \in M : x_{n_k} \notin V\} \subseteq \{n \in \mathbb{N} : x_n \notin V\} \setminus \{n_k \in \mathbb{N} : n_k \notin M\}$. Again,

$\{n_k \in M : x_{n_k} \notin V\} \setminus J \subseteq (K \cup J) \setminus J \in \mathcal{K}$. Subsequently, x_{n_k} is $\mathcal{S}$-$\mathcal{K}$-convergent to x. Hence, $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to x. □

Theorem 3.14. *Let X be a Hausdorff Space. Let $\{x_n\}$ be $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to x. Then, there exists an ideal $\mathcal{J}$ such that $x \in X$ is an $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-limit of the sequence $\{x_n\}$ if and only if x is also a $\mathcal{J}$-limit of $\{x_n\}$ provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

Proof. Suppose that $\{x_n\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to x. So, there exists a set $M \in \mathcal{I}^*$ such that $\{n_k \in M : x_{n_k} \notin U_x\} \in \mathcal{K}$ where $U_x \in SO(X)$. Now, let $J = M^c$. Since, $(\mathcal{I} \cup \mathcal{K})$ is an ideal, the set $\{K \cup J : K \in \mathcal{K}\}$ generates an ideal, say $\mathcal{J}$. Then

$$\{n \in \mathbb{N} : x_n \notin U_x\} \subseteq \{n_k \in M : x_{n_k} \notin U_x\} \cup M^{\complement} \in \mathcal{J}.$$

Hence, $\{x_n\}$ is $\mathcal{J}$-convergent to x.

Conversely, suppose that $\{x_n\}$ is $\mathcal{S}$-$\mathcal{J}$-convergent to x, where $\mathcal{J}$ = ideal generated by $(\mathcal{K} \cup J)$, for any $J \in \mathcal{I}$. Then, by Lemma 3.13, $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to x. □

Theorem 3.15. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals and $F \subset X$. If there exists a sequence $\{x_n\}$ in F (with distinct elements) which is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to $\xi \in X$, then ξ is a semi-limit point of F, in essence, $\xi \in sCl(F)$, the semi-closure of F.*

Proof. Let U be any semi-open subset of X containing the point ξ. Since $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to $\xi \in X$, so, there exists a set $M \in \mathcal{I}^*$ such that $\{n \in M : x_n \notin U\} \in \mathcal{K}$. In other words, $\{n \in M : x_n \in U\} \notin \mathcal{K}$ since $\mathcal{K}$ is an ideal. Then choose $n_0 \in \{n \in M : x_n \in U\}$ such that $x_{n_0} \neq \xi$, then $x_{n_0} \in F \cap (U - \{\xi\})$ and hence, $F \cap (U - \{\xi\}) \neq \emptyset$. This shows that ξ is a semi-limit point of F. □

Corollary 3.16. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals and consider $F \subset X$. If there exists a sequence $\{x_n\}$ in F (with distinct elements) which is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to $\xi \in X$, then $\xi \in Cl(F)$.*

Theorem 3.17. *If $F \subset X$ is a semi-closed set, then for any sequence in F which is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to a, we have $a \in F$.*

Proof. Suppose $F \subset X$ is a semi-closed set and $\{x_n\}$ is any sequence in F that is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to the element a, but $a \notin F$. Since F is semi-closed, we have $sCl(F) = F$ and therefore $a \notin sCl(F)$. Then there exists a semi-open set U containing a such that $F \cap U \neq \emptyset$. Since $\{x_n\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to a, there exists $M \in \mathcal{I}^*$ such that $\{n \in M : x_n \notin U\} \in \mathcal{K}$. Furthermore, $\{n \in M : x_n \in U\} \notin \mathcal{K}$, which implies that $\{n \in M : x_n \in U\} \neq \emptyset$. According to our hypothesis, $x_n \in F$, that implies $F \cap U \neq \emptyset$. This is a contradiction. □

Theorem 3.18. *Let $f : X \to Y$ be a semi-continuous function. If $\{x_n\}$ is a sequence in X which is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to $\xi \in X$, then $\{f(x_n)\}$ is an $\mathcal{I}^{\mathcal{K}}$-convergent sequence to $f(\xi)$.*

Proof. Consider a sequence $\{x_n\}$ in X which is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to $\xi \in X$. We claim that $\{f(x_n)\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to $f(\xi)$. Suppose not, that is $\{f(x_n)\}$ is not $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to $f(\xi)$. Then there exists a open set $V \subseteq Y$, containing $f(\xi)$ and for each

$M \in \mathcal{I}^*$, we have $\{n \in M : f(x_n) \notin V\} \notin \mathcal{K}$. Now by Definition 2.5 (vii), there exists $U \in SO(X)$ such that $\xi \in U$ and $f(U) \subset V$. Now, $\{n \in M : f(x_n) \notin V\} \subset \{n \in M : x_n \notin U\}$. Then $\{n \in M : x_n \notin U\} \notin \mathcal{K}$, that implies $\{x_n\}$ is not $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to ξ. That is a contradiction to our assumption. Hence, $\{f(x_n)\}$ is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to $f(\xi)$. □

THEOREM 3.19. *Let $f : X \to Y$ be an irresolute function. If $\{x_n\}$ is a sequence in X which is $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-convergent to $\xi \in X$, then $\{f(x_n)\}$ $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-converges to $f(\xi)$.*

Proof. The proof is similar to that of Theorem 3.18 with the use of the characterization of an irresolute function that is shown in Definition 2.5 (vi). □

4. $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster points and several properties

In this section we introduce the terminology of $\mathcal{I}^{\mathcal{K}}$ cluster points of a sequence for semi-open sets in a topological space. An element p in a space X is said to be an $\mathcal{I}^*$-cluster point of a sequence $\{x_n\}$ if there exists $M = \{m_1, m_2, \ldots, m_k, \ldots\} \in \mathcal{I}^*$ such that, that the subsequence $\{x_{m_k}\}$ has a cluster point p, more precisely, for any open set U containing p, the set $\{n \in \mathbb{N} : x_{m_k} \in U\}$ is an infinite subset of $\mathbb{N}$.

DEFINITION 4.1. Let X be a space and $\{x_n\}$ be a sequence in X. A point $p \in X$ is called a $\mathcal{S}$-$\mathcal{I}^{\mathcal{M}}$-cluster point of $\{x_n\}$ in X if there exists $M \in \mathcal{I}^*$ such that for any $U \in SO(X)$ containing p, we have $\{n \in M : x_n \in U\} \notin \mathcal{M}$, where $\mathcal{M}$ is an ideal convergence mode.

If $\mathcal{M} = \mathcal{K}$ and $\mathcal{M} = \mathcal{K}^*$, then Definition 4.1 refers to the definitions of $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point and $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}^*}$-cluster point of a sequence correspondingly. It is doubtless that each $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-limit of a sequence is a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point, but the converse is not always true. But if the ideal $\mathcal{K}$ is a maximal ideal (for any $A \subset \mathbb{N}$ it implies that $A \in \mathcal{K}$ or $A^{\complement} \in \mathcal{K}$), then the converse is also true i.e. $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-limit and $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point of a sequence coincide.

REMARK 4.2. Let X be a space and $\{x_n\}$ be a sequence in X. Then
1. If $\mathcal{I} = \mathcal{K}$, then $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster points of $\{x_n\}$ coincide with that of $\mathcal{S}$-$\mathcal{I}$-cluster points.
2. If $\mathcal{K} = Fin$, then $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster points of $\{x_n\}$ are also $\mathcal{S}$-$\mathcal{I}$-cluster points.

We denote the collection of all $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster points of a sequence $x = \{x_n\}$ in a space by $sC_x(\mathcal{I}^{\mathcal{K}})$. Consequently, from Definition 4.1, it straightaway follows that $sC_x(\mathcal{I}^{\mathcal{K}}) \subseteq sC_x(\mathcal{K})$.

LEMMA 4.3. *$sC_x(\mathcal{I} \cup \mathcal{K}) \subseteq sC_x(\mathcal{I}^{\mathcal{K}})$, for two ideals $\mathcal{I}$ and $\mathcal{K}$ such that $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

Proof. Let y be not a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point of $x = \{x_n\}$. Then for all $M \in \mathcal{I}^*$ there exists $U \in SO(X)$ containing y such that $\{n \in M : x_n \in U\} \in \mathcal{K}$. Hence, $\{n \in \mathbb{N} : x_n \in U\} \subseteq \{n \in M : x_n \in U\} \cup M^{\complement} \in \mathcal{I} \cup \mathcal{K}$. Hence, y is not a $\mathcal{S}$-$(\mathcal{I} \cup \mathcal{K})$-cluster point of x. □

Theorem 4.4. *Let $\mathcal{I}$, $\mathcal{K}$, $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{K}_1$ and $\mathcal{K}_2$ be ideals on $\mathbb{N}$. Suppose that $x = \{x_n\}$, $y = \{y_n\}$ are two sequences in a space X. Then*
(i) If $\mathcal{K}_1 \subset \mathcal{K}_2$, then $sC_x(\mathcal{I}^{\mathcal{K}_2}) \subseteq sC_x(\mathcal{I}^{\mathcal{K}_1})$,

(ii) If $\mathcal{I}_1 \subset \mathcal{I}_2$, then $sC_x({\mathcal{I}_1}^{\mathcal{K}}) \subseteq sC_x({\mathcal{I}_2}^{\mathcal{K}})$

(iii) and if $\{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{K}$, then $sC_x(\mathcal{I}^{\mathcal{K}}) = sC_y(\mathcal{I}^{\mathcal{K}})$.

Proof. The proof of (i) and (ii) follows directly from the Definition 4.1.

(iii) Consider, $N = \{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{K}$. Suppose that $a \in sC_x(\mathcal{I}^{\mathcal{K}})$, then there exists $M \in \mathcal{I}^*$ such that for each $U_a \in SO(X)$, we have $\{n \in M : x_n \in U_a\} \notin \mathcal{K}$. We negate the possibility of $M \cap N = \emptyset$, as in that case the result follows immediately. Again, $M \cap N \in \mathcal{K}$ and

$$\{n \in M : y_n \in U_a\} = \{n \in M \cap N : y_n \in U_a\} \cup \{n \in M \setminus N : x_n \in U\}.$$

However, $\{n \in M \cap N : y_n \in U\} \in \mathcal{K}$. Thus, $\{n \in M : y_n \in U_a\} \notin \mathcal{K}$. Thus, $a \in sC_y(\mathcal{I}^{\mathcal{K}})$. Since the sequences x, y are taken arbitrarily, hence $sC_x(\mathcal{I}^{\mathcal{K}}) = sC_y(\mathcal{I}^{\mathcal{K}})$. □

Recall that an element $p \in X$ is a semi-ω-accumulation point [5] of $A \subset X$ if for every semi-open set U containing p, $U \cap A$ is an infinite set.

Theorem 4.5. *Let X be a space and $\mathcal{I}$, $\mathcal{K}$ be two ideals. If every sequence $\{x_n\}$ in X has a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point, then every infinite subset of X has a semi-ω-accumulation point.*

Proof. Let F (infinite) $\subset X$, then there exists a sequence $\{x_n\}$ of distinct points in F. Suppose that every sequence in X has a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point. Let a be a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point of $\{x_n\}$. Then, for $U \in SO(X)$ containing a, we have $\{n \in M : x_n \in U\} \notin \mathcal{K}$. Since as per assumption, $Fin \subset \mathcal{K}$ then the set $\{n \in M : x_n \in U\}$ is infinite. Since $x_n \in F$, $U \cap F$ is a infinite set. Hence a is semi-ω-accumulation point of F. □

Corollary 4.6. *Let X be a space. If every sequence $\{x_n\}$ has a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point, then every infinite subset of X has a ω-accumulation point.*

Recall that a space X is semi-compact [13] if every semi-open cover of X possesses a finite subcover. Similarly, X is said to be semi-Lindelöf if every semi-open cover of X possesses a countable subcover.

Theorem 4.7. *If X is a semi-Lindelöf space and each sequence in X has an $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point, then X is a semi-compact space.*

Proof. Suppose that X is a semi-Lindelöf space and that each X-valued sequence has a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point. Let $\mathcal{S} = \{S_\lambda : \lambda \in \Lambda\}$ be a semi-open cover of X. So, there exists a countable subcover $\mathcal{S}' = \{S_1, S_2, \ldots, S_m, \ldots\}$. If possible, consider the sequence $U = \{U_m\}$ such that $U_1 = S_1$ and for each $m > 1$, let $U_m = S_m$, where S_m is the first member of the sequence of U's such that $S_m \nsubseteq U_1 \cup U_2 \cup \ldots \cup U_{m-1}$. Assuming the axiom of choice, consider a sequence $x = \{x_m\}$ such that $x_1 \in U_1$ and for each $m > 1$, let $x_m \in U_m - (U_1 \cup U_2 \cup \ldots \cup U_{m-1})$. Now by our hypothesis, $\{x_m\}$ has a $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point, say l. Then, there exists j such that $l \in U_j$. Consequently, there exists a set $M \in \mathcal{I}^*$ such that $\{n \in M : x_n \in U_j\} \notin \mathcal{K}$. So, the set $\{n \in M : x_n \in U_j\}$ must be a infinite subset of $\mathbb{N}$. Thus, there exists $k > j$ such that $k \in \{n \in M : x_n \in U_j\}$; that is $x_k \in U_j$, which is a contradiction. Subsequently, there must exists $m_0 \in \mathbb{N}$ such that the process of induction for $\{U_m\}$ is impossible to continue after $m = m_0$. Therefore, $\{U_1, U_2, \ldots, U_{m_0}\}$ is a finite subcover of X for given cover $\mathcal{S}$. □

Corollary 4.8. *If X is a semi-Lindeloff space and each X-valued sequence has an $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-cluster point, then X is a compact space.*

5. Some characterizations

Considering that semi-closed subsets of a semi-compact space is again semi-compact [13], we compare the collection of semi-limits of a sequence and that of semi-cluster points of a sequence in a topological space.

Theorem 5.1. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals satisfying ideality condition and K be a semi-compact subset of a space X. For any sequence $\{x_n\}$ in X, if $\{n \in \mathbb{N} : x_n \in K\} \notin (\mathcal{I} \cup \mathcal{K})$, then $K \cap sC_x(\mathcal{I}^{\mathcal{K}}) \neq \emptyset$.*

Proof. If possible, consider $K \cap sC_x(\mathcal{I}^{\mathcal{K}}) \neq \emptyset$. Then for each $c \in K$, there exists a set $M_c \in \mathcal{I}^*$ such that for any $U_c \in SO(X)$ containing c, we have $R_c = \{n \in M_c : x_n \in U_c\} \in \mathcal{K}$. Now, $K \subset \bigcup_{c \in K} U_c$, hence the semi open cover $\{U_c : c \in K\}$ contains a finite subcover $U_{c_1}, U_{c_2}, \ldots, U_{c_k}$. Consider $M = M_1 \cap M_2 \cap \ldots \cap M_k \in \mathcal{I}^*$ such that for each $c \in K$, we have $R_c = \{n \in M : x_n \in U_c\} \in \mathcal{K}$. So, $\{n \in M : x_n \in K\} \subset R_{c_1} \cup R_{c_2} \cup \ldots \cup R_{c_k}$. Now, right hand side of the above set inequality belongs to $\mathcal{K}$, that implies $\{n \in M : x_n \in K\} \in \mathcal{K}$. But, $\{n \in \mathbb{N} : x_n \in K\} \subset \{n \in M : x_n \in K\} \cup M^{\complement} \in (\mathcal{I} \cup \mathcal{K})$. This is a contradiction. □

Theorem 5.2. *Let X be a semi-compact space. Suppose that $\{x_n\}$ is a sequence in X such that $sC_x(\mathcal{I}^{\mathcal{K}}) = \{l\}$. Then $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-$\lim x_n = l$. Further, if the ideal $\mathcal{K}$ is a P-ideal, then $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}^*}$-$\lim x_n = l$.*

Proof. Consider a semi-open set U_l containing l. Then $X' = X \setminus U_l$ is a semi-closed subset of X. Now, for each $z \in X' \implies z \notin sC_x(\mathcal{I}^{\mathcal{K}})$. So, for all $M \in \mathcal{I}^*$, there exists $U_z \in SO(X)$ containing z such that $\{n \in M : x_n \in U_z\} \in \mathcal{K}$. Now, the semi-open cover $\{U_z\}_{z \in X'}$ of X' has a finite subcover, say $\{U_{z_1}, U_{z_2}, \ldots, U_{z_k}\}$. Then $\bigcup_{i \le k} \{n \in M : x_n \in U_{z_i}\} \in \mathcal{K}$. That implies $\{n \in M : x_n \in U_l\} \in \mathcal{K}^*$. Therefore, $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-$\lim x_n = l$. □

The following result on the collection $sC_x(\mathcal{I}^{\mathcal{K}})$ and $sL(\mathcal{I}^{\mathcal{K}})$ (the collection of semi-limits of a sequence) summarise the Theorem 5.1 and Theorem 5.2.

Theorem 5.3. *Suppose that $\{x_n\}$ is a sequence in X such that $\{n \in \mathbb{N} : x_n \in K\} \notin (\mathcal{I} \cup \mathcal{K})$, for any semi-compact $K \subset X$. Then $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-$\lim x_n = l$ if and only if $sC_x(\mathcal{I}^{\mathcal{K}}) = \{l\}$.*

In this segment, the existence problem mentioned in earlier sections is discussed further and the following results are obtained. Here we discuss the existence of the ideal $\mathcal{J}$ for a given $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-clustered sequence such that their corresponding semi-cluster point sets coincides.

Lemma 5.4. *Let $\mathcal{I}$, $\mathcal{K}$ be two ideals on $\mathbb{N}$ satisfying ideality condition. Let $x = x_n$ be a sequence in X. If $\mathcal{J}$ = ideal generated by $(\mathcal{K} \cup J)$, for any $J \in \mathcal{I}$. Then $a \in sC_x(\mathcal{J})$ implies $a \in sC_x(\mathcal{I}^{\mathcal{K}})$.*

Proof. Contrapositively, let $a \notin sC_x(\mathcal{I}^{\mathcal{K}})$. Then there exists at least an open set $V \in SO(X)$ containing a for which, for all $M \in \mathcal{I}^*$ we have $\{n \in M : x_n \in V\} \in \mathcal{K}$. Particularly, for $M_j = J^{\mathsf{C}} \in \mathcal{I}^*$, we have $\{n \in \mathbb{N} : x_n \in V\} \subseteq \{n \in M_j : x_n \in V\} \cup {M_j}^{\mathsf{C}}$. That implies $\{n \in \mathbb{N} : x_n \in V\} \in (\mathcal{K} \cup J)$. Hence, $a \notin sC_x(\mathcal{J})$. □

Theorem 5.5. *Let X be a Hausdorff space and $x = \{x_n\}$ be a sequence in X. Then there exists an ideal $\mathcal{J}$ such that $a \in sC_x(\mathcal{J})$ if and only if $a \in sC_x(\mathcal{I}^{\mathcal{K}})$ provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

Proof. Consider, $a \notin sC_x(\mathcal{J})$, where $\mathcal{J}$ is the ideal generated by $\mathcal{K} \cup J$, for any $J \in \mathcal{I}$, provided $(\mathcal{I} \cup \mathcal{K})$ is an ideal. We claim that $a \notin sC_x(\mathcal{I}^{\mathcal{K}})$. So, there exists $U_a \in SO(X)$ containing a such that $\{n \in \mathbb{N} : x_n \in U_a\} \in \mathcal{J}$. Let $J = M^{\mathsf{C}}$. Thus,

$$\{n \in M : x_n \in U_a\} \subseteq \{n \in \mathbb{N} : x_n \in U_a\} \setminus M^{\mathsf{C}}.$$

Therefore, $\{n \in M : x_n \in U_a\} \subseteq (\mathcal{K} \cup J) \setminus J \in \mathcal{K}$, for any $M \in \mathcal{I}^*$. Thus $a \notin sC_x(\mathcal{I}^{\mathcal{K}})$. Converse part of the proof is immediate by Lemma 3.13. □

For a non Hausdorff space, does there exist an ideal $\mathcal{J}$ for a given $\mathcal{S}$-$\mathcal{I}^{\mathcal{K}}$-clustered sequence such that their corresponding set of semi-cluster points coincides?

Here we characterize the collection of semi-cluster points of a sequence as a known subsets in a topological space. Recall that a subset D of a topological space X is said to be dense in X if for any open set U, $U \cap D \neq \emptyset$. In an arbitrary space, the notion of dense set for semi-open sets is equivalent to that of open sets.

Theorem 5.6 ([12, Theorem 2.4]). *Let X be a space and $D \subset X$. Then D is dense in X if and only if $U \cap D \neq \emptyset$ for every $U \in SO(X)$.*

Definition 5.7. Let X be an arbitrary space. We say that X is a semi-closed hereditarily separable space if every semi-closed subsets of X is separable.

Theorem 5.8. *Let $\mathcal{I}$, $\mathcal{K}$ be two ideals on $\mathbb{N}$ and X be a space. Then*
(i) For $x = \{x_n\}_{n \in \mathbb{N}}$, a sequence in X; $sC_x(\mathcal{I}^{\mathcal{K}})$ is a semi-closed set.

(ii) If X is semi-closed hereditarily separable and there exists a disjoint sequence of sets $\{D_n\}$ such that $D_n \subset \mathbb{N}$, $D_n \notin \mathcal{I}, \mathcal{K}$ for all n, then for every non empty semi-closed subset F of X, there exists a sequence x in X such that $F = sC_x(\mathcal{I}^{\mathcal{K}})$ provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.

Proof. Consider the sequence $x = \{x_n\}$ in X and let $\mathcal{I}$, $\mathcal{K}$ be the two ideals on $\mathbb{N}$.

(i) Let $y \in sCl(C_x(\mathcal{I}^{\mathcal{K}}))$; the semi-closure of $C_x(\mathcal{I}^{\mathcal{K}})$. Let U be a semi-open set containing y. It is clear that $U \cap C_x(\mathcal{I}^{\mathcal{K}}) \neq \emptyset$. Let $q \in U \cap C_x(\mathcal{I}^{\mathcal{K}})$ i.e., $q \in U$ and $q \in C_x(\mathcal{I}^{\mathcal{K}})$. Now there exists a set $M \in \mathcal{I}^*$ such that $\{n \in M : y_n \in U\} \notin \mathcal{K}$. Thus, $y \in C_x(\mathcal{I}^{\mathcal{K}})$.

(ii) F is separable as a semi-closed subset of X. Then by Definition 2.3 of [15] and Theorem 5.6, there exists a countable set $S = \{s_1, s_2, \ldots\} \subset F$ such that $sCl(S) = F$. Consider $x_n = s_i$ for $n \in D_i$. Then, we have a subsequence $\{k_n\}$ of the sequence $\{n\}$. Now, consider the sequence $x = \{x_{n_k}\}$ and let $y \in sC_x(\mathcal{K})$ (taking $y \neq s_i$ otherwise if $y = s_i$ for some i, then y is eventually (except finite elements) in F). We claim that $sC_x(\mathcal{I}^{\mathcal{K}}) \subset F$. Let U be any semi-open set containing y. Then $\{n : x_{n_k} \in U\} \notin \mathcal{K}$ $(\neq \emptyset)$. So, $s_i \in U$ for some i. Therefore, $F \cap U$ is non empty. So y is a semi-limit point of F and semi-closedness of F implies $y \in F$. Hence $sC_x(\mathcal{K}) \subset F$. Further $sC_x(\mathcal{I}^{\mathcal{K}}) \subseteq sC_x(\mathcal{K}) \subset F$. For the converse argument, consider $a \in F$ and U be a semi-open set containing a, then there exists $s_i \in S$ such that $s_i \in U$. Then $\{n : x_{n_k} \in U\} \supset D_i$ $(\notin \mathcal{K}, \mathcal{I})$. Thus $\{n : x_{n_k} \in U\} \notin (\mathcal{I} \cup \mathcal{K})$ i.e., $a \in sC_x(\mathcal{I} \cup \mathcal{K})$. Again, by Lemma 4.3, $sC_f(\mathcal{I} \cup \mathcal{K}) \subseteq sC_f(\mathcal{I}^{\mathcal{K}})$. Thus, we get the reverse implication. □

Remark 5.9. Theorem 5.8 extends [8, Theorem 10] to semi-open sets, it follows by letting $\mathcal{I}$, $\mathcal{K}$ coincide and using open sets instead of semi-open sets in the above theorem.

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