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UNIQUENESS OF SOME DELAY-DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION WITH FINITE WEIGHTS

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Abstract. In this paper, we study the uniqueness problems of $f^n(z)L(g)$ and $g^n(z)L(f)$ when they share a non-zero small function $\alpha(z)$ with finite weights, where $L(h)$ represents any one of $h^{(k)}(z),\ h(z+c),\ h(z+c)-h(z)$ and $h^{(k)}(z+c)$, $k \geq 1$ and c is a non-zero constant. Here $f(z)$ and $g(z)$ are transcendental meromorphic (or entire) functions and $\alpha(z)$ is a small function with respect to both $f(z)$ and $g(z)$. Our results improve and supplement the recent results due to Gao and Liu [Bull. Korean Math. Soc. 59 (2022), 155-166].

1. Introduction

In this paper, by a meromorphic function we will always mean a meromorphic function in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [9, 12, 22]. Let E be an arbitrary set of positive real numbers of finite linear measure, which do not necessarily have to be the same for each occurrence. For a non-constant meromorphic function f we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $(r \to \infty, r \notin E)$. A meromorphic function $a(z)(\not\equiv \infty)$ is called a small function with respect to f if $T(r, a) = S(r, f)$. It is said that two meromorphic functions $f(z)$ and $g(z)$ share a small function $a(z)$ CM (taking into account the multiples) if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiples. If we do not take the multiplicities, then we say that $f(z)$ and $g(z)$ share the small function $a(z)$ IM (ignoring multiplicities). The order $\rho(f)$ and hyperorder $\rho_2(f)$ of a meromorphic function f is defined as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

At the beginning of this century, I. Lahiri [10, 11] introduced the concept of weighted sharing, which reads as follows:

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Definition 1.1. Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with the weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$, and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g *share* (a, k) to mean that f, g share the value a with the weight k. Clearly, if f, g share (a, k), then f, g share (a, p) for every integer p, $0 \leq p \leq k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2 ([20]). Let $f(z)$, $g(z)$ and $\alpha(z)$ be meromorphic functions on a domain D. We say that the functions f and g share the function α with weight k if for every $m = 1, 2, \ldots, k$ the m-fold zeros of $f - \alpha$ coincide with the m-fold zeros of $g - \alpha$, and the zeros of $f - \alpha$ of a multiplicity greater than k coincide with the zeros of $g - \alpha$ of a multiplicity greater than k.

Definition 1.3 ([10, 11]). We denote by $N(r, a; f| \geq k)$ the counting function of those $a-$points of f whose multiplicities are not smaller than k, where each $a-$point is counted according to its multiplicity. $\overline{N}(r, a; f| \geq k)$ is the counting function of those $a-$points of f whose multiplicities are not smaller than k, where each $a-$point is counted only once, without taking its multiplicity into account.

Definition 1.4 ([10,11]). We denote by $N_2(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2)$.

In 1959 Hayman [8] proved the following result with respect to the zero distribution of a special type of complex differential polynomials.

Theorem 1.5. *If $f(z)$ is a transcendental entire function and $n \geq 2$ is a positive integer, then $f^n(z)f'(z) - a$ has infinitely many zeros, where a is a non-zero constant.*

In 1967, Clunie [5] proved that Theorem 1.5 also holds if $n = 1$. The analogous result for meromorphic functions is known as the Hayman Conjecture [8] and is as follows:

If $f(z)$ is a transcendental meromorphic function and n is a positive integer, then $f^n(z)f'(z) - a$ has infinitely many zeros, where a is a non-zero constant.

It should be noted that the above conjecture has been fully proved by many researchers. Hayman himself proved the conjecture for $n \geq 3$. Mues [19] proved the conjecture for $n = 2$. For $n = 1$ it was proved in [3,4,23]. In 2007, Laine and Yang [14] proved a result for the zero distribution of zeros of a complex difference polynomial.

Theorem 1.6 ([14]). *If $f(z)$ is a transcendental entire function of finite order and $n \geq 2$, then $f^n(z)f(z + c) - a$ has infinitely many zeros, where a, c are non-zero constants.*

Some improvements of this result have been made in [18]. Analogous theorems for transcendental meromorphic functions with $\rho_2(f) < 1$ were proved in [17, 18, 21]. In 2009, Liu and Yang [16] considered the zero distribution of $f^n(z)(f(z+c)-f(z))-p(z)$, where $p(z)$ is a nonzero polynomial. In 2020, Laine and Latreuch [13] proved the following result, which relates to the delay differential form of the Hayman Conjecture.

THEOREM 1.7. *Let $f(z)$ be a transcendental meromorphic (resp. entire) function with $\rho_2(f) < 1$ and $a(z)$ a non-zero small function with respect to $f(z)$. If $n \geq k+4$ (resp. $n \geq 3$), then $f^n(z)f^{(k)}(z+c) - a(z)$ has infinitely many zeros, where c is a non-zero constant.*

In the following, we denote $\mathcal{M}$ as the class of transcendental meromorphic functions and $\mathcal{M}'$ as the class of transcendental meromorphic functions of hyperorder less than 1. Similarly, we denote $\mathcal{E}$ as the class of transcendental entire functions and $\mathcal{E}'$ as the class of transcendental entire functions of hyperorder less than 1.

Recently, Gao and Liu [6] proved a result related to the paired Hayman conjecture for complex delay-differential polynomials.

THEOREM 1.8 ([6]). *If one of the following conditions is satisfied:*
(i) $L(h) = h^{(k)}(z), n \geq k+4$ *and* $h \in \mathcal{M}$ *or* $n \geq 3$ *and* $h \in \mathcal{E}$;

(ii) $L(h) = h(z+c), n \geq 4$ *and* $h \in \mathcal{M}'$ *or* $n \geq 3$ *and* $h \in \mathcal{E}'$;

(iii) $L(h) = h(z+c) - h(z), n \geq 5$ *and* $h \in \mathcal{M}'$ *or* $n \geq 3$ *and* $h \in \mathcal{E}'$;

(iv) $L(h) = h^{(k)}(z+c), n \geq k+4$ *and* $h \in \mathcal{M}'$ *or* $n \geq 3$ *and* $h \in \mathcal{E}'$;
then at least one of $f^n(z)L(g) - a(z)$ and $g^n(z)L(f) - a(z)$ has infinitely many zeros, where $a(z)$ is a non-zero small function with respect to both $f(z)$ and $g(z)$, $k \geq 1$ and c is a non-zero constant.

REMARK 1.9. The above theorem is not true for $n = 1$. The condition $\rho_2(f) < 1$ cannot be lifted either. Counterexamples to this can be found in [6].

In the same paper, Gao and Liu [6] proved the following two theorems about the uniqueness of delay differential polynomials.

THEOREM 1.10. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and n, k two positive integers. If $f^n(z)L(g)$ and $g^n(z)L(f)$ share a non-zero small function $\alpha(z)$ CM, and one of the*
(i) $L(h) = h^{(k)}(z), n \geq 3k+16$ *and* $f, g \in \mathcal{M}$;

(ii) $L(h) = h(z+c), n \geq 16$ *and* $f, g \in \mathcal{M}'$;

(iii) $L(h) = h(z+c) - h(z), n \geq 19$ *and* $f, g \in \mathcal{M}'$;

(iv) $L(h) = h^{(k)}(z+c), n \geq 3k+16$ *and* $f, g \in \mathcal{M}'$;
is satisfied, then either $f^n(z)L(g) = g^n(z)L(f)$ or $f^n(z)L(g)g^n(z)L(f) = \alpha^2(z)$.

THEOREM 1.11. *Let $f(z)$ and $g(z)$ be two transcendental entire functions and n, k be two positive integers. If $f^n(z)L(g)$ and $g^n(z)L(f)$ share a non-zero small function $\alpha(z)$ CM and one of* *(i)* $L(h) = h^{(k)}(z), n \geq 8$ *and* $f, g \in \mathcal{E}$;

(ii) $L(h) = h(z+c), n \geq 8$ *and* $f, g \in \mathcal{E}'$;

(iii) $L(h) = h(z+c) - h(z), n \geq 8$ *and* $f, g \in \mathcal{E}'$;

(iv) $L(h) = h^{(k)}(z+c), n \geq 8$ *and* $f, g \in \mathcal{E}'$;
is satisfied then either $f^n(z)L(g) = g^n(z)L(f)$ *or* $f^n(z)L(g)g^n(z)L(f) = \alpha^2(z)$.

Regarding Theorems 1.10 and 1.11 it is natural to ask following questions: Whether the nature of sharing can be relaxed in Theorems 1.10 and 1.11? What will be the effect of such relaxation?

We prove that the conclusions of Theorems 1.10 and 1.11 remain unchanged if we replace the CM sharing by the weighted sharing with weight 2. We also prove the related results for the sharing of small functions with weights 1 and 0. We now state our main results.

THEOREM 1.12. *Let* $f(z)$ *and* $g(z)$ *be two non-constant transcendental meromorphic functions,* n, k *be two positive integers, and* $\alpha(z)$ *be a non-zero small function with respect to both* $f(z)$ *and* $g(z)$. *If* $f^n(z)L(g)$ *and* $g^n(z)L(f)$ *share* $(\alpha, l)(l \geq 2$, *an integer)* *and one of*
(i) $L(h) = h^{(k)}(z), n \geq 3k + 16$ *and* $f, g \in \mathcal{M}$;

(ii) $L(h) = h(z+c), n \geq 14$ *and* $f, g \in \mathcal{M}'$;

(iii) $L(h) = h(z+c) - h(z), n \geq 19$ *and* $f, g \in \mathcal{M}'$;

(iv) $L(h) = h^{(k)}(z+c), n \geq 3k + 16$ *and* $f, g \in \mathcal{M}'$;
holds, then either $f^n(z)L(g) = g^n(z)L(f)$ *or* $f^n(z)L(g)g^n(z)L(f) = \alpha^2(z)$.

THEOREM 1.13. *Let* $f(z)$ *and* $g(z)$ *be two non-constant transcendental meromorphic functions,* n, k *be two positive integers, and* $\alpha(z)$ *be a non-zero small function with respect to both* $f(z)$ *and* $g(z)$. *If* $f^n(z)L(g)$ *and* $g^n(z)L(f)$ *share* $(\alpha, 1)$ *and one of*
(i) $L(h) = h^{(k)}(z), n \geq \frac{7k}{2} + 18$ *and* $f, g \in \mathcal{M}$;

(ii) $L(h) = h(z+c), n \geq 16$ *and* $f, g \in \mathcal{M}'$;

(iii) $L(h) = h(z+c) - h(z), n \geq 22$ *and* $f, g \in \mathcal{M}'$;

(iv) $L(h) = h^{(k)}(z+c), n \geq \frac{7k}{2} + 18$ *and* $f, g \in \mathcal{M}'$;
holds, then either $f^n(z)L(g) = g^n(z)L(f)$ *or* $f^n(z)L(g)g^n(z)L(f) = \alpha^2(z)$.

THEOREM 1.14. *Let* $f(z)$ *and* $g(z)$ *be two non-constant transcendental meromorphic functions,* n, k *be two positive integers, and* $\alpha(z)$ *be a non-zero small function with respect to both* $f(z)$ *and* $g(z)$. *If* $f^n(z)L(g)$ *and* $g^n(z)L(f)$ *share* $(\alpha, 0)$ *and one of*
(i) $L(h) = h^{(k)}(z), n \geq 6k + 28$ *and* $f, g \in \mathcal{M}$;

(ii) $L(h) = h(z+c), n \geq 26$ *and* $f, g \in \mathcal{M}'$;

(iii) $L(h) = h(z+c) - h(z), n \geq 37$ *and* $f, g \in \mathcal{M}'$;

(iv) $L(h) = h^{(k)}(z+c), n \geq 6k + 28$ *and* $f, g \in \mathcal{M}'$;
holds, then either $f^n(z)L(g) = g^n(z)L(f)$ *or* $f^n(z)L(g)g^n(z)L(f) = \alpha^2(z)$.

For transcendental entire functions f and g we obtain the following corollaries.

Corollary 1.15. *Under the same hypothesis as in Theorem 1.12, the same conclusions hold in each of the following cases:*
(i) $L(h) = h^{(k)}(z), n \geq 8$ *and* $f, g \in \mathcal{E}$;

(ii) $L(h) = h(z+c), n \geq 8$ *and* $f, g \in \mathcal{E}'$;

(iii) $L(h) = h(z+c) - h(z), n \geq 8$ *and* $f, g \in \mathcal{E}'$;

(iv) $L(h) = h^{(k)}(z+c), n \geq 8$ *and* $f, g \in \mathcal{E}'$.

Corollary 1.16. *Under the same hypothesis as in Theorem 1.13, the same conclusions hold in each of the following cases:*
(i) $L(h) = h^{(k)}(z), n \geq 9$ *and* $f, g \in \mathcal{E}$;

(ii) $L(h) = h(z+c), n \geq 9$ *and* $f, g \in \mathcal{E}'$;

(iii) $L(h) = h(z+c) - h(z), n \geq 9$ *and* $f, g \in \mathcal{E}'$;

(iv) $L(h) = h^{(k)}(z+c), n \geq 9$ *and* $f, g \in \mathcal{E}'$.

Corollary 1.17. *Under the same hypothesis as in Theorem 1.14, the same conclusions hold in each of the following cases:*
(i) $L(h) = h^{(k)}(z), n \geq 14$ *and* $f, g \in \mathcal{E}$;

(ii) $L(h) = h(z+c), n \geq 14$ *and* $f, g \in \mathcal{E}'$;

(iii) $L(h) = h(z+c) - h(z), n \geq 14$ *and* $f, g \in \mathcal{E}'$;

(iv) $L(h) = h^{(k)}(z+c), n \geq 14$ *and* $f, g \in \mathcal{E}'$.

2. Lemmas

We now give some lemmas which will be needed in the sequel. We define

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are non-constant meromorphic functions defined in the complex plane.

Lemma 2.1 ([22]). *Let f be a non-constant meromorphic function and k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}(z)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + k\overline{N}(r, f(z)) + S(r, f(z)).$$

Lemma 2.2 ([15]). *Suppose that $T : [0, \infty) \to [0, \infty)$ is a non-decreasing continuous function with $\rho_2(T) < 1$ and c is a non-zero real number. If $\delta \in (0, 1 - \rho_2(T))$, then*

$$T(r+c) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right).$$

The next lemma can be proved easily by using Lemma 2.2.

Lemma 2.3. *Let f be a transcendental meromorphic function with $\rho_2(f) < 1$ and c be a non-zero constant. Then the following inequalities hold.*
(i) $N(r,0;f(z+c)) \le N(r,0;f) + S(r,f)$;

(ii) $N(r,\infty;f(z+c)) \le N(r,\infty;f) + S(r,f)$;

(iii) $\overline{N}(r,0;f(z+c)) \le \overline{N}(r,0;f) + S(r,f)$;

(iv) $\overline{N}(r,\infty;f(z+c)) \le \overline{N}(r,\infty;f) + S(r,f)$.

The following lemma gives the characteristic function of $L(h)$ when $L(h)$ takes the difference $h(z+c)-h(z)$ or the delay-differential $h^{(k)}(z+c)$. These results can easily be obtained by [7, Lemma 8.3] and the first fundamental theorem of Nevanlinna.

Lemma 2.4. *(i)* $T\left(r, \frac{1}{h(z+c)-h(z)}\right) \le 2T(r,h(z)) + S(r,h(z))$, $h \in \mathcal{M}'$ *and* $T\left(r, \frac{1}{h(z+c)-h(z)}\right) \le T(r,h(z)) + S(r,h(z))$, $h \in \mathcal{E}'$.

(ii) $T\left(r, \frac{1}{h^{(k)}(z+c)}\right) \le (k+1)T(r,h(z)) + S(r,h(z))$, $h \in \mathcal{M}'$ *and* $T\left(r, \frac{1}{h^{(k)}(z+c)}\right) \le T(r,h(z)) + S(r,h(z))$, $h \in \mathcal{E}'$.

Lemma 2.5 ([6]). *(i) If $f,\ g \in \mathcal{M}$, then*
$$nT(r,f) - (k+1)T(r,g) \le T(r, f^n g^{(k)}) + S(r,g) \le nT(r,f) + (k+1)T(r,g).$$

(ii) If $f,\ g \in \mathcal{E}$, then
$$nT(r,f) - T(r,g) \le T(r, f^n g^{(k)}) + S(r,g) \le nT(r,f) + T(r,g).$$

Lemma 2.6 ([6]). *If $f,\ g \in \mathcal{M}'$ or $\mathcal{E}'$, then*
$$nT(r,f) - T(r,g) \le T(r, f^n(z)g(z+c)) + S(r,g) \le nT(r,f) + T(r,g).$$

Lemma 2.7 ([6]). *(i) If $f,\ g \in \mathcal{M}'$ and $g(z+c) - g(z) \not\equiv 0$, then*
$$nT(r,f) - 2T(r,g) \le T(r, f^n(z)(g(z+c) - g(z))) + S(r,g) \le nT(r,f) + 2T(r,g).$$

(ii) If $f,\ g \in \mathcal{E}'$ and $g(z+c) - g(z) \not\equiv 0$, then
$$nT(r,f) - T(r,g) \le T(r, f^n(z)(g(z+c) - g(z))) + S(r,g) \le nT(r,f) + T(r,g).$$

Lemma 2.8 ([6]). *(i) If $f,\ g \in \mathcal{M}'$, then*
$$nT(r,f) - (k+1)T(r,g) \le T(r, f^n(z)g^{(k)}(z+c)) + S(r,g) \le nT(r,f) + (k+1)T(r,g).$$

(ii) If $f,\ g \in \mathcal{E}'$, then
$$nT(r,f) - T(r,g) \le T(r, f^n(z)g^{(k)}(z+c)) + S(r,g) \le nT(r,f) + T(r,g).$$

The proof of the following lemma is exactly the same as the proof of [1, Lemma 2].

Lemma 2.9. *Let $f,\ g$ be two non-constant meromorphic functions and let α be a non-zero small function with respect to both f and g. If f and g share $(\alpha, 2)$, then one of the following holds:*

(i) $T(r,f)+T(r,g)\leq 2\{N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)\}+S(r,f)+S(r,g)$;

(ii) $f=g$; *(iii)* $fg=\alpha^2$.

Lemma 2.10. *Let f, g be two non-constant meromorphic functions and let α be a non-zero small function with respect to both f and g. If f and g share $(\alpha,1)$ and $H\not\equiv 0$, then*

$$T(r,f)+T(r,g)\leq 2\{N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)\}$$
$$+\frac{1}{2}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+S(r,f)+S(r,g).$$

Proof. Let $F=f/\alpha$ and $G=g/\alpha$. If f and g do not share a zero or a pole with α, then F and G share $(1,1)$. Now $N_2(r,\infty;F)=N_2(r,\infty;f)+N_2(r,0;\alpha)$. Also $T(r,f)=T(r,F.\alpha)\leq T(r,F)+S(r,f)$ and $T(r,g)=T(r,G.\alpha)\leq T(r,G)+S(r,g)$. With [2, Lemma 2.15] we get the result. □

Lemma 2.11. *Let f, g be two non-constant meromorphic functions and let α be a non-zero small function with respect to both f and g. If f and g share $(\alpha,0)$ and $H\not\equiv 0$, then*

$$T(r,f)+T(r,g)\leq 2\{N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)\}$$
$$+3\{\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+S(r,f)+S(r,f).$$

Proof. Let $F=f/\alpha$ and $G=g/\alpha$. Then F and G share $(1,0)$. Also $T(r,f)=T(r,F.\alpha)\leq T(r,F)+S(r,f)$ and $T(r,g)=T(r,G.\alpha)\leq T(r,G)+S(r,g)$. We get the result with [2, Lemma 2.14]. □

3. Proof of the Theorems

Proof (Theorem 1.12). Let $F(z)=f^n(z)L(g),G(z)=g^n(z)L(f)$. Then F and G share $(\alpha,2)$. Suppose that (i) of Lemma 2.9 holds. Then

$$T(r,F)+T(r,G)$$
$$\leq 2\{N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\}+S(r,F)+S(r,G). \quad (1)$$

Part I. Let $L(h)=h^{(k)}(z)$. Then $F(z)=f^n g^{(k)}(z),G(z)=g^n f^{(k)}(z)$. Therefore

$$N_2(r,\infty;F)\leq 2\overline{N}(r,\infty;F)\leq 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}; \quad (2)$$
$$N_2(r,\infty;G)\leq 2\overline{N}(r,\infty;G)\leq 2\{\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)\}. \quad (3)$$

Using Lemma 2.1 we have

$$N_2(r,0;F)\leq 2\overline{N}(r,0;f)+N(r,0;g^{(k)})$$
$$\leq 2\overline{N}(r,0;f)+N(r,0;g)+k\overline{N}(r,\infty;g)+S(r,g), \quad (4)$$

and

$$N_2(r,0;G)\leq 2\overline{N}(r,0;g)+N(r,0;f^{(k)})$$
$$\leq 2\overline{N}(r,0;g)+N(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f). \quad (5)$$

Using (i) of Lemma 2.5 and (2)-(5) in (1) we get

$$\begin{aligned}
&(n-k-1)\{T(r,f)+T(r,g)\}\\
&\leq (2k+8)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+6\{N(r,0;f)+N(r,0;g)\}+S(r,f)+S(r,g)\\
&\leq (2k+14)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),
\end{aligned}$$

i.e. $\{n-(3k+15)\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$,

contradicts the assumption that $n \geq 3k+16$ for $L(h)=h^{(k)}(z)$.

Part II. Let $L(h)=h(z+c)$. Then $F(z)=f^n g(z+c), G(z)=g^n f(z+c)$. Using Lemma 2.3 we get

$$N_2(r,\infty;F)\leq 2\overline{N}(r,\infty;f)+N(r,\infty;g(z+c))\leq 2\overline{N}(r,\infty;f)+N(r,\infty;g)+S(r,g); \quad (6)$$

$$N_2(r,\infty;G)\leq 2\overline{N}(r,\infty;g)+N(r,\infty;f(z+c))\leq 2\overline{N}(r,\infty;g)+N(r,\infty;f)+S(r,f); \quad (7)$$

$$N_2(r,0;F)\leq 2\overline{N}(r,0;f)+N(r,0;g(z+c))\leq 2\overline{N}(r,0;f)+N(r,0;g)+S(r,g); \quad (8)$$

$$N_2(r,0;G)\leq 2\overline{N}(r,0;g)+N(r,0;f(z+c))\leq 2\overline{N}(r,0;g)+N(r,0;f)+S(r,f). \quad (9)$$

Using Lemma 2.6 and (6)-(9) in (1) we get

$$\begin{aligned}
&(n-1)\{T(r,f)+T(r,g)\}\\
&\leq 6\{N(r,\infty;f)+N(r,\infty;g)+N(r,0;f)+N(r,0;g)\}+S(r,f)+S(r,g)\\
&\leq 12\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),
\end{aligned}$$

i.e. $(n-13)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$,

that is a contradiction to the assumption that $n \geq 14$ for $L(h)=h(z+c)$.

Part III. Let $L(h)=h(z+c)-h(z)$. Then $F(z)=f^n(g(z+c)-g(z)), G(z)=g^n(f(z+c)-f(z))$. Using Lemma 2.3 we get

$$\begin{aligned}
N_2(r,\infty;F) &\leq 2\overline{N}(r,\infty;f)+N(r,\infty;g(z+c))+N(r,\infty;g)\\
&\leq 2\{\overline{N}(r,\infty;f)+N(r,\infty;g)\}+S(r,g); \qquad (10)\\
N_2(r,\infty;G) &\leq 2\overline{N}(r,\infty;g)+N(r,\infty;f(z+c))+N(r,\infty;f)\\
&\leq 2\{\overline{N}(r,\infty;g)+N(r,\infty;f)\}+S(r,f). \qquad (11)
\end{aligned}$$

Using (i) of Lemma 2.4 we get

$$\begin{aligned}
N_2(r,0;F) &\leq 2\overline{N}(r,0;f)+N(r,0;g(z+c)-g(z))+S(r,g)\\
&\leq 2\{\overline{N}(r,0;f)+T(r,g)\}+S(r,g); \qquad (12)\\
N_2(r,0;G) &\leq 2\overline{N}(r,0;g)+N(r,0;f(z+c)-f(z))+S(r,f)\\
&\leq 2\{\overline{N}(r,0;g)+T(r,f)\}+S(r,f). \qquad (13)
\end{aligned}$$

Using (i) of Lemma 2.7 and (10)-(13) in (1) we get

$$\begin{aligned}
(n-2)\{T(r,f)+T(r,g)\} &\leq 8\{N(r,\infty;f)+N(r,\infty;g)\}+4\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\
&\quad +4\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\
&\leq 16\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).
\end{aligned}$$

This gives $(n-18)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$. Since $n \geq 19$ for $L(h)=$

$h(z+c)-h(z)$, we again get a contradiction.

Part IV. Let $L(h)=h^{(k)}(z+c)$. Then $F(z)=f^n g^{(k)}(z+c), G(z)=g^n f^{(k)}(z+c)$. Using Lemma 2.3 we get

$$\begin{aligned} N_2(r,\infty;F) &\le 2\overline{N}(r,\infty;F)\le 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g(z+c))\}\\ &\le 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+S(r,g); \quad (14)\\ N_2(r,\infty;G) &\le 2\overline{N}(r,\infty;G)\le 2\{\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f(z+c))\}\\ &\le 2\{\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)\}+S(r,f). \quad (15)\end{aligned}$$

Using (ii) of Lemma 2.4 we get

$$N_2(r,0;F)\le 2\overline{N}(r,0;f)+N(r,0;g^{(k)}(z+c))\le 2\overline{N}(r,0;f)+(k+1)T(r,g); \quad (16)$$

$$N_2(r,0;G)\le 2\overline{N}(r,0;g)+N(r,0;f^{(k)}(z+c))\le 2\overline{N}(r,0;g)+(k+1)T(r,f). \quad (17)$$

Using (i) of Lemma 2.8 and (14)-(17) in (1) we get

$$\begin{aligned}(n-k-1)\{T(r,f)+T(r,g)\} &\le 8\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+4\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\ &\quad+2(k+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\le (2k+14)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

This gives $\{n-(3k+15)\}\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g)$. Since $n\ge 3k+16$ for $L(h)=h^{(k)}(z+c)$, we get a contradiction.

Therefore either $F=G$ or $FG=\alpha^2$. This means that either $f^n(z)L(g)=g^n(z)L(f)$ or $f^n(z)L(g)g^n(z)L(f)=\alpha^2(z)$. This completes the proof of Theorem 1.12. □

Proof (Theorem 1.13). Let $F,\ G$ be defined as in Theorem 1.12. Then F and G share $(\alpha,1)$. Let $H\not\equiv 0$. Then by Lemma 2.10 we get

$$\begin{aligned} T(r,F)+T(r,G) &\le 2\{N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\}\\ &+\frac{1}{2}\{\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\}+S(r,F)+S(r,G). \quad (18)\end{aligned}$$

Part I. Let $L(h)=h^{(k)}(z)$. Then

$$\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f^n g^{(k)})=\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g); \quad (19)$$

$$\overline{N}(r,\infty;G)=\overline{N}(r,\infty;g^n f^{(k)})=\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f). \quad (20)$$

Using Lemma 2.1 we get

$$\begin{aligned}\overline{N}(r,0;F)&=\overline{N}(r,0;f^n g^{(k)})\le \overline{N}(r,0;f)+\overline{N}(r,0;g^{(k)})\\ &\le \overline{N}(r,0;f)+\overline{N}(r,0;g)+k\overline{N}(r,\infty;g)+S(r,f)+S(r,g). \quad (21)\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{N}(r,0;G)&=\overline{N}(r,0;g^n f^{(k)})\\ &\le \overline{N}(r,0;f)+\overline{N}(r,0;g)+k\overline{N}(r,\infty;f)+S(r,f)+S(r,g). \quad (22)\end{aligned}$$

Therefore using (i) of Lemma 2.5, (2)-(5) and (19)-(22) in (18) we get

$$\begin{aligned}&(n-k-1)\{T(r,f)+T(r,g)\}\\ \le\ &\left(\frac{5k}{2}+9\right)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+7\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g)\end{aligned}$$

$$\leq \left(\frac{5k}{2}+16\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

i.e. $$\left\{n-\left(\frac{7k}{2}+17\right)\right\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

Since $n \geq \frac{7k}{2}+18$ for $L(h) = h^{(k)}(z)$, we get a contradiction.

Part II. Let $L(h) = h(z + c)$. Then using Lemma 2.3 we have

$$\overline{N}(r,\infty;F) \leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g(z+c)) \leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,g); \quad (23)$$

$$\overline{N}(r,\infty;G) \leq \overline{N}(r,\infty;g)+\overline{N}(r,\infty;f(z+c)) \leq \overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)+S(r,f); \quad (24)$$

$$\overline{N}(r,0;F) \leq \overline{N}(r,0;f)+\overline{N}(r,0;g(z+c)) \leq \overline{N}(r,0;f)+\overline{N}(r,0;g)+S(r,g); \quad (25)$$

$$\overline{N}(r,0;G) \leq \overline{N}(r,0;g)+\overline{N}(r,0;f(z+c)) \leq \overline{N}(r,0;g)+\overline{N}(r,0;f)+S(r,f). \quad (26)$$

Therefore using Lemma 2.6, (6)-(9) and (23)-(26) in (18) we obtain

$$\begin{aligned}&(n-1)\{T(r,f)+T(r,g)\}\\ &\leq 7\{N(r,\infty;f)+N(r,\infty;g)+N(r,0;f)+N(r,0;g)\}+S(r,f)+S(r,g)\\ &\leq 14\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

i.e. $$(n-15)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

contradicts with the fact $n \geq 16$ for $L(h) = h(z + c)$.

Part III. Let $L(h) = h(z + c) - h(z)$. Then using Lemma 2.3 we have

$$\begin{aligned}\overline{N}(r,\infty;F) &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g(z+c))+\overline{N}(r,\infty;g)+S(r,g)\\ &\leq \overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+S(r,g); \quad (27)\\ \overline{N}(r,\infty;G) &\leq \overline{N}(r,\infty;g)+\overline{N}(r,\infty;f(z+c))+\overline{N}(r,\infty;f)+S(r,f)\\ &\leq \overline{N}(r,\infty;g)+2\overline{N}(r,\infty;f)+S(r,f). \quad (28)\end{aligned}$$

From Lemma 2.4 follows:

$$\overline{N}(r,0;F) \leq \overline{N}(r,0;f)+\overline{N}(r,0;g(z+c)-g(z)) \leq \overline{N}(r,0;f)+2T(r,g)+S(r,g); \quad (29)$$

$$\overline{N}(r,0;G) \leq \overline{N}(r,0;g)+\overline{N}(r,0;f(z+c)-f(z)) \leq \overline{N}(r,0;g)+2T(r,f)+S(r,f). \quad (30)$$

Therefore by Lemma 2.7 (i), (10)-(13), (27)-(30) and (18) we obtain

$$(n-2)\{T(r,f)+T(r,g)\} \leq 19\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

i.e. $$(n-21)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

Since $n \geq 22$ for $L(h) = h(z + c) - h(z)$, we get a contradiction.

Part IV. Let $L(h) = h^{(k)}(z + c)$. Then using Lemma 2.3 we get

$$\overline{N}(r,\infty;F)\leq\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g(z+c))\leq\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,g); \quad (31)$$

$$\overline{N}(r,\infty;G)\leq\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f(z+c))\leq\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)+S(r,f). \quad (32)$$

Using Lemma 2.4, we get

$$\overline{N}(r,0;F)\leq\overline{N}(r,0;f)+\overline{N}(r,0;g^{(k)}(z+c))\leq\overline{N}(r,0;f)+(k+1)T(r,g)+S(r,g); \quad (33)$$

$$\overline{N}(r,0;G)\leq\overline{N}(r,0;g)+\overline{N}(r,0;f^{(k)}(z+c))\leq\overline{N}(r,0;g)+(k+1)T(r,f)+S(r,f). \quad (34)$$

Therefore using Lemma 2.8 (i), (14)-(17) and (31)-(34) in (18) we obtain

$$(n-k-1)\{T(r,f)+T(r,g)\} \leq \left(\frac{5k}{2}+16\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

i.e. $\left\{n-\left(\frac{7k}{2}+17\right)\right\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$

that contradicts our assumption that $n \geq \frac{7k}{2}+18$ for $L(h)=h^{(k)}(z+c)$.

Thus we have $H=0$. Then $\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1}$. Integrating twice, we get

$$F=\frac{(B-1)G-(A+B-1)}{BG-(A+B)} \text{ and } G=\frac{(A+B)F-(A+B-1)}{BF-(B-1)},$$

where $A \neq 0$ and B are constants. Now we consider the following two cases.

Case 1. Let $B=0$. Then $F=\frac{1}{A}(G-(1-A))$ and $G=A(F-\frac{A-1}{A})$.

Subcase 1.1. If $A \neq 1$, then $N(r,1-A;G)=N(r,0;F)$ and $N\big(r,\frac{A-1}{A};F\big)=N(r,0;G)$. Using second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}\left(r,\frac{A-1}{A};F\right)+\overline{N}(r,\infty;F)+S(r,F)\\ &= \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,F),\end{aligned}$$

and
$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}(r,1-A;G)+\overline{N}(r,\infty;G)+S(r,G)\\ &= \overline{N}(r,0;G)+\overline{N}(r,0;F)+\overline{N}(r,\infty;G)+S(r,G).\end{aligned}$$

Therefore

$$\begin{aligned} &T(r,F)+T(r,G)\\ &\leq 2\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,F)+S(r,G). \end{aligned} \quad (35)$$

Part I. Let $L(h)=h^{(k)}(z)$. Then using Lemma 2.5 (i) and (19)-(22) in (35) we get

$$\begin{aligned}(n-k-1)\{T(r,f)+T(r,g)\} &\leq (2k+2)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}\\ &\quad +4\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g),\end{aligned}$$

i.e. $\{n-(3k+7)\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$

contradicting the assumption that $n \geq \frac{7k}{2}+18$ for $L(h)=h^{(k)}(z)$.

Part II. Let $L(h)=h(z+c)$. Then using Lemma 2.6 and (23)-(26) in (35) we get $(n-7)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$ which is a contradiction as $n \geq 16$ when $L(h)=h(z+c)$.

Part III. Let $L(h)=h(z+c)-h(z)$. Then using Lemma 2.7 (i) and (27)-(30) in (35) we get $(n-11)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$. Since $n \geq 22$ when $L(h)=h(z+c)-h(z)$, we get a contradiction.

Part IV. Let $L(h)=h^{(k)}(z+c)$. Then using Lemma 2.8 (i) and (31)-(34) in (35) we get $\{n-(3k+7)\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$. Since $n \geq \frac{7k}{2}+18$ when $L(h)=h^{(k)}(z+c)$, we get a contradiction.

Subcase 1.2. If $A=1$, then $F=G$, that is $f^n(z)L(g)=g^n(z)L(f)$.

Case 2. Let $B \neq 0$. Now we consider the following three subcases.

Subcase 2.1. Assume that $B \neq 1$. Then $N\big(r, \frac{B-1}{B}; F\big) = N(r, \infty; G)$ and $N\big(r, \frac{A+B}{B}; G\big) = N(r, \infty; F)$. Using Nevanlinna's second fundamental theorem we obtain

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, 0; F) + \overline{N}\big(r, \frac{B-1}{B}; F\big) + \overline{N}(r, \infty; F) + S(r, F) \\ &= \overline{N}(r, 0; F) + \overline{N}(r, \infty; G) + \overline{N}(r, \infty; F) + S(r, F), \end{aligned}$$

and

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}\big(r, \frac{A+B}{B}; G\big) + \overline{N}(r, \infty; G) + S(r, G) \\ &= \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, G). \end{aligned}$$

Therefore

$$\begin{aligned} &T(r, F) + T(r, G) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + 2\{\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)\} + S(r, F + S(r, G). \quad (36) \end{aligned}$$

Part I. Let $L(h) = h^{(k)}(z)$. Then using (i) of Lemma 2.5 and (19)-(22) in (36) we get $\{n-(2k+7)\}\{T(r, f)+T(r, g)\} \leq S(r, f)+S(r, g)$. This contradicts the fact that $n \geq \frac{7k}{2} + 18$ when $L(h) = h^{(k)}(z)$.

Part II. Let $L(h) = h(z + c)$. Then using Lemma 2.6 and (23)-(26) in (36) we get $(n - 7)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g)$. This contradicts the fact that $n \geq 16$ when $L(h) = h(z + c)$.

Part III. Let $L(h) = h(z + c) - h(z)$. Then using Lemma 2.7 (i) and (27)-(30) in (36) we get $(n - 11)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g)$. Since $n \geq 22$ when $L(h) = h(z + c) - h(z)$, we get a contradiction.

Part IV. Let $L(h) = h^{(k)}(z + c)$. Then using Lemma 2.8 (i) and (31)-(34) in (36) we get $\{n - (2k + 7)\}\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g)$. Since $n \geq \frac{7k}{2} + 18$ when $L(h) = h^{(k)}(z + c)$, we get a contradiction.

Subcase 2.2. Assume that $B = 1$, $A \neq -1$. Then $F = -\frac{A}{G-(A+1)}$ and $G = \frac{(A+1)F-A}{F}$. Hence $N(r, 0; F) = N(r, A + 1; G)$ and $N(r, 0; G) = N\big(r, \frac{A}{A+1}; F\big)$. Using Nevanlinna's second fundamental theorem we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, 0; F) + \overline{N}\left(r, \frac{A}{A+1}; F\right) + \overline{N}(r, \infty; F) + S(r, F) \\ &= \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + S(r, F); \\ T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}(r, A + 1; G) + \overline{N}(r, \infty; G) + S(r, G) \\ &= \overline{N}(r, 0; G) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; G) + S(r, G). \end{aligned}$$

Therefore

$$\begin{aligned} &T(r, F) + T(r, G) \\ &\leq 2\{\overline{N}(r, 0; F) + \overline{N}(r, 0; G)\} + \{\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)\} + S(r, F + S(r, G). \end{aligned}$$

If we now proceed in a similar way to **Subcase 2.1.**, we arrive at a contradiction.

Subcase 2.3. Let B=1, A=−1. Then FG=1 and $f^n(z)L(g)g^n(z)L(f)$=$\alpha^2(z)$.

This proves the theorem. □

Proof (Theorem 1.14). Let F, G be defined as in Theorem 1.12. Then F and G share $(\alpha, 0)$. Assume that $H \not\equiv 0$. Therefore by Lemma 2.11 we have

$$T(r,F)+T(r,G) \le 2\{N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\}$$
$$+3\{\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\}+S(r,F)+S(r,G). \quad (37)$$

Part I. Let $L(h) = h^{(k)}(z)$. Then we obtain with (i) of Lemma 2.5, (2)-(5) and (19)-(22) in (37):

$$(n-k-1)\{T(r,f)+T(r,g)\}$$
$$\le (5k+14)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+12\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g)$$
$$\le (5k+26)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

i.e. $\{n-(6k+27)\}\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$.

This contradicts our assumption that $n \ge 6k+28$ when $L(h) = h^{(k)}(z)$.

Part II. Let $L(h) = h(z+c)$. Then we obtain with Lemma 2.6, (6)-(9) and (23)-(26) in (37): $(n-25)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$. This contradicts our assumption that $n \ge 26$ when $L(h) = h(z+c)$.

Part III. Let $L(h) = h(z+c) - h(z)$. Then using (i) of Lemma 2.7, (10)-(13) and (27)-(30) in (37) we get:

$$(n-2)\{T(r,f)+T(r,g)\} \le 17\{N(r,\infty;f)+N(r,\infty;g)\}+7\{N(r,0;f)+N(r,0;g)\}$$
$$+10\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)$$
$$\le 34\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

i.e. $(n-36)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$.

Since $n \ge 37$ when $L(h) = h(z+c) - h(z)$, we arrive at a contradiction.

Part IV. Let $L(h) = h^{(k)}(z+c)$. Now using Lemma 2.8 (i), (14)-(17) and (31)-(34) in (37) we get $\{n-(6k+27)\}\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$. Since $n \ge 6k+28$ when $L(h) = h^{(k)}(z+c)$, we arrive at a contradiction.
Therefore, $H = 0$.

Rest of the proof is similar to that of the case $H = 0$ in Theorem 1.13. This proves the theorem. □

Proof (Corollary 1.15). Since f and g are entire functions, $L(h)$ is also an entire function. Consequently, F and G are also entire functions. Therefore

$$N(r,\infty;f) = 0,\ N(r,\infty;g) = 0,\ N(r,\infty;F) = 0 \text{ and } N(r,\infty:G) = 0. \quad (38)$$

Then the proof follows from the proof of Theorem 1.12. □

Proof (Corollary 1.16). Let us assume that $H \not\equiv 0$. Since f and g are entire functions, we obtain with (38) from Lemma 2.10

$$T(r,F)+T(r,G) \le 2\{N_2(r,0;F)+N_2(r,0;G)\}$$
$$+\frac{1}{2}\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+S(r,F)+S(r,G).$$

Now the proof follows from the proof of Theorem 1.13. □

Proof (Corollary 1.17). Let us assume that $H \not\equiv 0$. Since f and g are entire functions, we obtain with (38) from Lemma 2.11

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2\{N_2(r,0;F)+N_2(r,0;G)\} \\ &\quad + 3\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+S(r,F)+S(r,G). \end{aligned}$$

Now the proof follows from the proof of the Theorem 1.14. □

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