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$(\mathbb{C}^*)^k$-ACTION ON $\mathbb{C}P^N$

Svjetlana Terzić

Abstract. For any natural numbers k and $l < k$, it is defined the action of the algebraic torus $(\mathbb{C}^*)^k$ on $\mathbb{C}P^{N-1}$, where $N = \binom{k}{l}$, which is given as the l-symmetric power representation of $(\mathbb{C}^*)^k$ in $(\mathbb{C}^*)^N$ and the standard action of $(\mathbb{C}^*)^N$ on $\mathbb{C}P^{N-1}$. It is well known question about description of toric manifolds arising from this action. In this note we solve this problem for $k = 2n$ and $l = n$, that is we describe the orbits of this action and their closures, that is the corresponding toric manifolds. In this context we also discuss the $(\mathbb{C}^*)^{2n}$-action on the complex Grassmann manifolds $G_{2n,n}$ by the Plücker embedding $G_{2n,n} \to \mathbb{C}P^{N-1}$, where $N = \binom{2n}{n}$. The explicit expressions of torus orbit closures we obtain can be further used in description of singularities of toric varieties.

1. Introduction

Toric varieties are a subclass of algebraic varieties whose geometric properties can be described in terms of combinatorial objects. A toric variety is "tori" since it can be defined as an algebraic torus orbit closure. Toric varieties form a bridge between algebraic geometry and the theory of polytopes, they enable to translate many notions of algebraic geometry into the combinatorial language of polytopes, and they are a rich source of examples. Toric varieties were first introduced by Demazure [10] and then extensively studied by many authors [8, 9, 11]. They were generalized in a few directions, which still form areas of extensive study today, such as in the symplectic setting [2, 7], or in the topology and combinatorial setting [3]. A comprehensive encyclopedic reference work on modern developments in various areas of torus action is the recently published book [4].

In this note we explicitly describe the toric varieties obtained as algebraic orbit completions of the action of the algebraic torus $(\mathbb{C}^*)^{2n}$ on the complex projective space $\mathbb{C}P^{N-1}$, given by the n-th symmetric power representation of $(\mathbb{C}^*)^{2n}$ in $(\mathbb{C}^*)^N$, where $N = \binom{2n}{n}$ and the standard $(\mathbb{C}^*)^N$ action on $\mathbb{C}P^{N-1}$.

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We also consider the complex Grassmann manifolds $G_{2n,n}$ since they admit a canonical action of the torus $(\mathbb{C}^*)^{2n}$. Using Plücker coordinates, these manifolds can be embedded equivariantly in $\mathbb{C}P^{N-1}$, $N = \binom{2n}{n}$, where $\mathbb{C}P^{N-1}$ is endowed with the $(\mathbb{C}^*)^{2n}$-action, which is given as the n-th symmetric power representation $(\mathbb{C}^*)^{2n}$ in $(\mathbb{C}^*)^N$ and the standard $(\mathbb{C}^*)^N$-action. Consequently, we describe the closures of the principal orbits in Grassmannians $G_{2n,n}$.

We note that the question of describing the closures of the algebraic torus orbits, mainly principal, for the canonical action of the compact torus on complex homogeneous spaces has been the focus of interest and study of many specialists in the common areas [15–18].

2. Orbits of $(\mathbb{C}^*)^{2n}$-action on $\mathbb{C}P^{\binom{2n}{n}-1}$

Consider an algebraic torus $(\mathbb{C}^*)^{2n}$ and its n-th symmetric power representation $\rho^{n,2n} : (\mathbb{C}^*)^{2n} \to (\mathbb{C}^*)^N$, where $N = \binom{2n}{n}$. This representation can be defined by induction, i.e $\rho^0 = 1$, $\rho^{n,n}(t_1, \dots, t_n) = t_1 \cdots t_n$ and

$$\begin{aligned}\rho^{n,2n}(t_1, \dots t_{2n}) &= (t_1 \rho^{n-1,2n-1}(t_2, \dots, t_{2n}), \rho^{n,2n-1}(t_2, \dots, t_{2n})) \\ &= (t_1 t_2 \rho^{n-2,2n-2}(t_3, \dots, t_{2n}), t_1 \rho^{n-1,2n-1}(t_2, \dots, t_n), \\ &\quad t_2 \rho^{n-1,2n-2}(t_3, \dots, t_{2n}), \rho^{n,2n-2}(t_3, \dots, t_{2n})).\end{aligned}$$

Furthermore, $\rho^{n,2n} = (\rho_1, \rho_2, \dots, \rho_N)$, where $\rho_i : (\mathbb{C}^*)^{2n} \to \mathbb{C}^*$ are its characters. According to the inductive definition of $\rho^{n,2n}$, the characters ρ_i are given by

$$\rho_1 = t_1 \cdots t_n,\ \rho_2 = t_1 \cdots t_{n-1} t_{n+1}, \dots,\ \rho_{\frac{N}{2}} = t_1 t_{n+2} \cdots t_{2n},$$

$$\rho_{\frac{N}{2}+1} = t_2 \cdots t_{n+1}, \dots,\ \rho_N = t_{n+1} \cdots t_{2n}.$$

We denote by $\theta_{2n,n}$ the action of the torus $(\mathbb{C}^*)^{2n}$ on the complex projective space $\mathbb{C}P^{N-1}$, which is given by the representation $\rho^{n,2n}$ and the standard action of $(\mathbb{C}^*)^N$.

Let $\Lambda_1, \dots, \Lambda_N$ be the weight vectors of the representation $\rho^{n,2n}$. They are $2n$ vectors with n coordinates equal to 1 and n coordinates equal to 0. We have [13] the classical notion of moment map $\mu : \mathbb{C}P^N \to \mathbb{R}^{2n}$ defined by

$$\mu(a_1 : \dots : a_N) = \frac{1}{\sum_{i=1}^N |a_i|^2} (|a_1|^2 \Lambda_1 + \dots + |a_N|^2 \Lambda_N). \tag{1}$$

2.1 Principal orbits

We now want to describe a principal orbit of $\theta_{2n,n}$, i.e. an orbit of a point in $\mathbb{C}P^{N-1}$ whose homogeneous coordinates are all non-zero. Let $\mathbf{a} = (a_1 : \dots : a_N) \in \mathbb{C}P^{N-1}$ such that $a_1 \cdots a_N \neq 0$. Its $\theta_{2n,n}$-orbit is given by

$$\emptyset(\mathbf{a}) = (\rho_1 a_1 : \rho_2 a_2 : \dots : \rho_N a_N).$$

The image $\mu(\emptyset(\mathbf{a}))$ of this orbit through the moment map (1) is the convex hull over the vectors $\Lambda_1, \dots, \Lambda_N$, which is known to be the interior of the hypersimplex $\Delta_{2n,n}$ [19].

PROPOSITION 2.1. *The principal orbit* $\emptyset(\mathbf{a})$ *for the* $\theta_{2n,n}$*-action on* $\mathbb{C}P^{N-1}$ *is given by the intersection of the family of the following hypersurfaces in* $\mathbb{C}P^{N-1}$*:*

$$c_{kl}z_iz_j = c_{ij}z_kz_l \quad \textit{iff} \quad \rho_i\rho_j = \rho_k\rho_l, \tag{2}$$

with $z_i \neq 0$, $1 \leq i \leq N$, *then* $c_{ij} = a_ia_j$, $c_{kl} = a_ka_l$ *and* $1 \leq i,j,k,l \leq 2n$, $\{i,j\} \neq \{k,l\}$.

Proof. Let $\mathbf{z} = (z_1, \ldots, z_N) \in \emptyset(\mathbf{a})$, then let $z_i = \rho_ia_i$, which gives $z_iz_j = \rho_i\rho_ja_ia_j$. So if $\rho_i\rho_j = \rho_k\rho_l$, then $a_ka_lz_iz_j = a_ia_jz_kz_l$, which means that $\mathbf{z}$ belongs to the intersection of the family of hypersurfaces (2). Conversely, it is also true that a point τ belongs to the image of the representation ρ if and only if the following is true: $\tau_i\tau_j = \tau_k\tau_l$ if and only if $\rho_i\rho_j = \rho_k\rho_l$. Since the coordinates of $\mathbf{z}$ satisfy this condition, it follows that the point $\mathbf{z}$ belongs to the orbit $\emptyset(\mathbf{a})$. ☐

REMARK 2.2. If we consider the characters ρ_i and ρ_j such that $\rho_i\rho_j = t_1\cdots t_{2n}$, we see that the family of hypersurfaces (2) contains the hypersurfaces of the form

$$c_{2N-2}z_1z_N = c_{1N}z_2z_{N-2}, \ldots, c_{N\frac{N}{2}+1}z_1z_N = c_{1N}z_{\frac{N}{2}}z_{\frac{N}{2}+1}. \tag{3}$$

Proposition 2.1 implies:

COROLLARY 2.3. *The toric manifold in* $\mathbb{C}P^{N-1}$ *obtained as the closure of a principal orbit for the* $\theta_{2n,n}$*- action on* $\mathbb{C}P^{N-1}$ *is the intersection of the family of the following hypersurfaces in* $\mathbb{C}P^{N-1}$*:*

$$c_{kl}z_iz_j = c_{ij}z_kz_l \quad \textit{iff} \quad \rho_i\rho_j = \rho_k\rho_l, \tag{4}$$

with $z_i \in \mathbb{C}$, $1 \leq i \leq N$, *then* $c_{ij} = a_ia_j$, $c_{kl} = a_ka_l$ *and* $1 \leq i,j,k,l \leq 2n$, $i \neq j, k \neq l$.

It is classically known that the boundary of the closure of this orbit consists of the orbits of smaller dimension. We will describe the families of hypersurfaces that define these orbits.

PROPOSITION 2.4. *A* $\theta_{2n,n}$*- orbit is contained in the boundary of a principal orbit* $\emptyset(\mathbf{a})$ *if and only if it is given by the intersection of the following family of hypersurfaces:*

$$\begin{cases} z_{i_1} = \ldots = z_{i_s} = 0, \\ c_{kl}z_iz_j = c_{ij}z_kz_l \qquad \textit{iff} \quad \rho_i\rho_j = \rho_k\rho_l, \end{cases} \tag{5}$$

so that for any i_j *it holds that whenever* $\rho_{i_j}\rho_m = \rho_p\rho_q$ *then* $z_p = 0$ *or* $z_q = 0$. *Here* $m,p,q \neq i_j$ *and* $i,j,k,l \neq i_1,\ldots,i_s$.

Proof. We obtain the boundry of a principal orbit $\emptyset(\mathbf{a})$ if we set some coordinates z_i equal to zero in (2). Let us assume that $z_{i_1} = \ldots = z_{i_p} = 0$. Then it follows from the equations of the hypersurfaces in (2), given by $c_{kl}z_{i_j}z_m = c_{i_jm}z_kz_l$, where $\rho_{i_j}\rho_m = \rho_k\rho_l$, that $z_k = 0$ or $z_l = 0$. We use $z_{i_1},\ldots z_{i_s}$ to denote all z_i's that we obtain as equal to zero in this way. This leaves us with the system of equations

$$c_{kl}z_iz_j = c_{ij}z_kz_l, \quad \text{iff} \quad \rho_i\rho_j = \rho_k\rho_l,$$

where $i,j,k,l \neq i_1,\ldots,i_s$. ☐

We describe the orbits of the maximal dimension, which are contained in the boundary of the principal orbits, in more detail.

THEOREM 2.5. *There are* $4n$ *boundary orbits of a principal orbit* $\emptyset(\mathbf{a})$ *whose dimension is maximal. These orbits are given by one of the following systems of equations:*

$$\begin{cases} z_{i_1} = z_{i_2} = \ldots = z_{i_{\frac{N}{2}}} = 0, & t_i \in \rho_{i_j},\ 1 \le j \le \frac{N}{2} \\ c_{kl} z_p z_q = c_{pq} z_k z_l & \textit{iff} \quad \rho_p \rho_q = \rho_k \rho_l \end{cases} \tag{6}$$

$$\begin{cases} z_{i_1} = z_{i_2} = \ldots = z_{i_{\frac{N}{2}}} = 0, & t_i \notin \rho_{i_j},\ 1 \le j \le \frac{N}{2} \\ c_{kl} z_p z_q = c_{pq} z_k z_l & \textit{iff} \quad \rho_p \rho_q = \rho_k \rho_l \end{cases} \tag{7}$$

where $1 \le i \le 2n$ *and* $k, l, p, q \ne i_1, \ldots, i_{\frac{N}{2}}$.

Proof. A boundary orbit of the maximal dimension is obtained if we assume that a coordinate z_i in (2) is zero. Without loss of generality, we can assume that $z_1 = 0$. Then it follows from (3) that $c_{iN-i+1} z_1 z_N = z_{1N} z_i z_{N-i+1}$, which implies $z_i = 0$ or $z_{N-i+1} = 0$, where $2 \le i \le \frac{N}{2}$. To clarify the representation, we distinguish the following two cases.

1) Assume that (3) implies that $z_2 = z_3 = \ldots = z_{\frac{N}{2}} = 0$. This is the minimal system of z_i's, that must be zero. Indeed, if we consider any other equation in (2) that contains z_1, e.g. $c_{kl} z_1 z_i = c_{1i} z_k z_l$, then since $\rho_1 \rho_i = \rho_k \rho_l$, we must conclude that $1 \le k \le \frac{N}{2}$ or $1 \le l \le \frac{N}{2}$. This is because ρ_j, where $\frac{N}{2} + 1 \le j \le N$, does not contain t_1. This equation therefore does not imply any new conditions for the variables. In the same way, we derive the same for all equations in (2) of the form $c_{kl} z_p z_i = c_{pi} z_k z_l$, where $2 \le p \le \frac{N}{2}$. Therefore, the assumption that $z_1 = 0$ in this case leads to the following boundary orbit of the maximal dimension:

$$\begin{cases} z_1 = z_2 = \ldots = z_{\frac{N}{2}} = 0 \\ c_{kl} z_i z_j = c_{ij} z_k z_l & \text{iff} \quad \rho_i \rho j = \rho_k \rho_l \end{cases}$$

where $i, j, k, l \ne 1, \ldots \frac{N}{2}$.

2) Assume that $z_2 \ne is0$. Note that $\rho_1 = t_1 \cdots t_n$ and $\rho_2 = t_1 \cdots t_{n-1} t_{n+1}$. If then $\rho_1 \rho_k = \rho_2 \rho_l$ we conclude that $t_n \in \rho_l$ and $t_{n+1} \notin \rho_l$, since $t_{n+1}^2 \notin \rho_2 \rho_l = \rho_1 \rho_k$ as $t_{n+1} \notin \rho_1$. Therefore, the equations

$$c_{2l} z_1 z_k = c_{1k} z_2 z_l \quad \text{iff} \quad \rho_1 \rho_k = \rho_2 \rho_l$$

result in that

$$z_l = 0 \quad \text{for } l \text{ so that } t_n \in \rho_l \text{ and } t_{n+1} \notin \rho_l. \tag{8}$$

We claim that one of the following cases must be satisfied:

1. if $t_n \in \rho_k$ then $z_k = 0$,
2. if $t_{n+1} \notin \rho_k$, then $z_k = 0$,

for all such k. To prove this, we assume that the first case is not true, i.e. that there exists k_0 such that $t_n \in \rho_{k_0}$ and $z_{k_0} \ne 0$. Then (8) implies that $t_{n+1} \in \rho_{k_0}$. Let ρ_{l_0} be obtained from ρ_{k_0} by replacing t_{n+1} by some t_i such that $t_i \notin \rho_{k_0}$. Then $t_n \in \rho_{l_0}$, but $t_{n+1} \notin \rho_{l_0}$ and by (8) we conclude that $z_{l_0} = 0$. We consider the equations from (2) of the form

$$c_{pl_0} z_q z_{k_0} = c_{qk_0} z_p z_{l_0} \quad \text{iff} \quad \rho_q \rho_{k_0} = \rho_p \rho_{l_0}.$$

Since $z_{l_0} = 0$ and $z_{k_0} \neq 0$, it follows that $z_q = 0$. Since $t_{n+1} \in \rho_{k_0}$, $t_i \notin \rho_{k_0}$ and $t_{n+1} \notin \rho_{l_0}$, $t_i \in \rho_{l_0}$ we also have that $t_{n+1} \notin \rho_q$ and $t_i \in \rho_q$. Thus, for any q such that $t_{n+1} \notin \rho_q$ and $t_i \in \rho_q$ for some $t_i \notin \rho_{k_0}$, we have that $z_q = 0$. In other words, for any k such that $t_{n+1} \notin \rho_k$, we have that $\rho_k \neq \rho_{k_0}$ such that there exists $t_i \notin \rho_{k_0}$ such that $t_i \in \rho_k$, which then implies that $z_k = 0$. This proves that the second case is satisfied.

The case when $z_1 = \ldots = z_s = 0$, but $z_{s+1} \neq 0$, where $s \leq \frac{N}{2} - 1$, can be considered in the same way, since the characters ρ_s and ρ_{s+1} differ in one parameter t_i. □

The formulas (6) and (7) imply that these boundary orbits are in fact the principal orbits in the complex projective space of the smaller dimension. More precisely we have:

Corollary 2.6. *An orbit of the maximal dimension which is contained in the boundary of a principal orbit* $\emptyset(\mathbf{a})$ *is a principal orbit for* $(\mathbb{C}^*)^{2n-1}$*-action on* $\mathbb{C}P^{\frac{N}{2}-1}$ *given by one of the following representations*

1. *the* n*-th symmetric power representation of* $(\mathbb{C}^*)^{2n-1}$ *in* $(\mathbb{C}^*)^{\frac{N}{2}}$*, or*
2. *the* $(n-1)$*-th symmetric power representation of* $(\mathbb{C}^*)^{2n-1}$ *in* $(\mathbb{C}^*)^{\frac{N}{2}}$*,*

and the standard action of $(\mathbb{C}^*)^{\frac{N}{2}}$ *on* $\mathbb{C}P^{\frac{N}{2}-1}$*.*

Proof. Let a boundary orbit has the form (6). The coordinates $z_{i_1}, \ldots, z_{i_{\frac{N}{2}}}$ are all zero and $t_i \in \rho_l$ if $l = i_1, \ldots, i_{\frac{N}{2}}$ for some $1 \leq i \leq 2n$. It follows that $t_i \notin \rho_l$, $l \neq i_1, \ldots, i_{\frac{N}{2}}$. This implies that the induced action of the torus $(\mathbb{C}^*)^{2n}$ on this orbit is given by n-th symmetric power representation $\rho^{n,2n-1}$ of $(\mathbb{C}^*)^{2n-1}$ in $(\mathbb{C}^*)^{\frac{N}{2}}$, as $\frac{N}{2} = \frac{1}{2}\binom{2n}{n} = \binom{2n-1}{n-1} = \binom{2n-1}{n}$ and the standard action of the torus $(\mathbb{C}^*)^{\frac{N}{2}}$ on $\mathbb{C}P^{\frac{N}{2}-1}$. The representation $\rho^{n,2n-1} : (\mathbb{C}^*)^{2n-1} \to (\mathbb{C}^*)^{\frac{N}{2}}$ is given by the characters $\rho_l^{n,2n-1} = \rho_l$ for $l \neq i_1, \ldots, i_{\frac{N}{2}}$. Therefore this orbit is a principal orbit in $\mathbb{C}P^{\frac{N}{2}-1}$ under the action of $(\mathbb{C}^*)^{2n-1}$ given by the n-th symmetric power representation of $(\mathbb{C}^*)^{2n-1}$ in $(\mathbb{C}^*)^{\frac{N}{2}}$ and the standard action of $(\mathbb{C}^*)^{\frac{N}{2}}$.

If a boundary orbit has the form (7) then the coordinates $z_{i_1}, \ldots, z_{i_{\frac{N}{2}}}$ are all zero where $t_i \notin \rho_l$ for $l \neq i_1, \ldots, i_{\frac{N}{2}}$. This means that $t_i \in \rho_l$ for $l = i_1, \ldots, i_{\frac{N}{2}}$. Thus, the induced action of the torus $(\mathbb{C}^*)^{2n}$ on this orbit is given by the $(n-1)$-th symmetric power representation $\rho^{n-1,2n-1}$ of $(\mathbb{C}^*)^{2n-1}$ in $(\mathbb{C}^*)^{\frac{N}{2}}$, as $\frac{N}{2} = \binom{2n-1}{n-1}$ and the standard action of the torus $(\mathbb{C}^*)^{\frac{N}{2}}$ on $\mathbb{C}P^{\frac{N}{2}-1}$. The representation $\rho^{n-1,2n-1}$: $\mathbb{C}^{2n-1} \to \mathbb{C}^{\frac{N}{2}}$ is given by the characters $\rho_l^{n-1,2n-1} = \rho_l$ for $l \neq i_1, \ldots, i_{\frac{N}{2}}$. Therefore, this orbit is a principal orbit in $\mathbb{C}P^{\frac{N}{2}-1}$ under the action of $(\mathbb{C}^*)^{2n-1}$ given by the $(n-1)$-th symmetric power representation of $(\mathbb{C}^*)^{2n-1}$ in $(\mathbb{C}^*)^{\frac{N}{2}}$ and the standard action of $(\mathbb{C}^*)^{\frac{N}{2}}$. □

Theorem 2.5 has interesting consequences in the theory of polytopes. We have already pointed out that the image of a principal orbit by the moment map is the interior of the hypersimplex $\Delta_{2n,n}$. It is the classical result [1, 12] that the closure of this orbit is a toric manifold that maps through the moment map to the hypersimplex $\Delta_{2n,n}$. Moreover, there is a bijection between the orbits from the boundary of this orbit and the faces of $\Delta_{2n,n}$. It follows that there is a bijection between the boundary orbits of the maximal dimension and the facets of $\Delta_{2n,n}$. Using Theorem 2.5 we provide the new proof for the known result [19].

PROPOSITION 2.7. *The facets of the hypersimplex $\Delta_{2n,n}$ are the hypersimplices $\Delta_{2n-1,n}$ and $\Delta_{2n-1,n-1}$.*

Proof. The facets of the hypersimplex $\Delta_{2n,n}$ correspond to the boundary orbits of the maximal dimension of a principal orbit through the moment map. If a boundary orbit has the form (6), then by Corollary 2.6 it is an orbit of $(\mathbb{C}^*)^{2n-1}$-action on $\mathbb{C}P^{\frac{N}{2}}$ given by the n-th symmetric power representation, which implies that its image by the moment map is the hypersimplex $\Delta_{2n-1,n}$. If a boundary orbit has the form (7), then Corollary 2.6 implies that it is an orbit of the $(\mathbb{C}^*)^{2n-1}$ action on $\mathbb{C}P^{\frac{N}{2}}$ given by the $(n-1)$-th symmetric power representation, which implies that its image by the moment map is the hypersimplex $\Delta_{2n,n-1}$. □

2.2 Non-prinipal orbits

A non-principal orbit is an orbit $\O(\mathbf{a})$ of a point $\mathbf{a} = (a_1 : \ldots a_N) \in \mathbb{C}P^{N-1}$ which contains at least one zero coordinate. Let $a_{p_1} = \ldots = a_{p_m} = 0$. Then $\mu(\O(\mathbf{a}))$ is a convex hull of the weight vectors Λ_i, where $i \neq p_1, \ldots p_m$.

REMARK 2.8. Note that in this way we obtain that any polytope over some of the weight vectors Λ_i can be obtained as the moment polytope for some $(\mathbb{C}^*)^{2n}$-orbit of a point from $\mathbb{C}P^{N-1}$. In other word any subpolytope of the hypersimplex $\Delta_{2n,n}$ can be realized as the moment polytope. We point out that this is not the case in general, as, for example, the $(\mathbb{C}^*)^4$-action on the complex Grassmann manifold $G_{4,2}$ shows [5].

PROPOSITION 2.9. *The orbit $\O(\mathbf{a})$ of a point $\mathbf{a} = (a_1 : \ldots : a_N) \in \mathbb{C}P^N$ where $a_{p_1} = \ldots = a_{p_m} = 0$ is given by the intersection of the family of the following hypersurfaces in $\mathbb{C}P^{N-1}$:*

$$c_{kl} z_i z_j = c_{ij} z_k z_l \quad \text{iff} \quad \rho_i \rho_j = \rho_k \rho_l, \tag{9}$$

where $i, j, k, l \neq p_1, \ldots, p_m$, then $z_i \neq 0$ for $i \neq p_1, \ldots, p_m$ and $c_{ij} = a_i a_j$, $c_{kl} = a_k a_l$, $i \neq j, k \neq l$.

As for the closure of the principal orbits we obtain:

COROLLARY 2.10. *The toric manifold in $\mathbb{C}P^{N-1}$ which is obtained as the closure of $\theta_{2n,n}$-orbit of a point $\mathbf{a} = (a_1 : \ldots : a_N) \in \mathbb{C}P^N$ where $a_{p_1} = \ldots = a_{p_m} = 0$ is given by the intersection of the family of the following hypersurfaces in $\mathbb{C}P^{N-1}$:*

$$c_{kl} z_i z_j = c_{ij} z_k z_l \quad \text{iff} \quad \rho_i \rho_j = \rho_k \rho_l, \tag{10}$$

where $i, j, k, l \neq p_1, \ldots, p_m$, then $z_i \in \mathbb{C}$ and $c_{ij} = a_i a_j$, $c_{kl} = a_k a_l$, $i \neq j, k \neq l$.

Remark 2.11. We can define the action of the permutation group S_{2n} on the set $\mathbf{N} = \{1, \dots, N\}$ in the following way. Order all n- combinations over $2n$ elements as $\mathbf{C} = \{\{1, \dots, n\}, \{1, \dots, n-1, n+1\}, \dots, \{1, \dots, n-1, 2n\}, \{1, \dots, n-2, n, n+1\}, \dots \{1, \dots, n-2, 2n-1, 2n\}, \dots \{n+1, \dots, 2n\}\}$. Then consider the bijection between the sets $\mathbf{N}$ and $\mathbf{C}$ which keeps this order. For $i \in \mathbf{N}$ denote by $i = \{i_1, \dots, i_n\}$ its image under such bijection. We define $\sigma(i) = \{\sigma(i_1), \dots, \sigma(i_n)\}$, $\sigma \in S_n$. Then it is defined the action of S_n on the set of characters ρ_i and the set of variables z_i by $\sigma(\rho_i) = \rho_{\sigma(i)}$, $\sigma(z_i) = z_{\sigma(i)}$.

Remark 2.12. The system of equations (2) which defines a principal orbit is invariant under the action of the permutation group S_{2n}. This fact essentially enabled us to obtain explicit description of the type of boundary orbits in Proposition 2.5. For non-principal orbits this is no more the case, the system (9) is, in general, no more invariant under the action of permutation group S_{2n}, so the boundary orbits of the non-principal orbits may be of different types.

3. Orbits of $(\mathbb{C}^*)^{2n}$-action on $G_{2n,n}(\mathbb{C})$

The action of $(\mathbb{C}^*)^{2n}$ on $G_{2n,n}(\mathbb{C})$ is induced by the standard action of $(\mathbb{C}^*)^{2n}$ on $\mathbb{C}^{2n}$. Consider the standard basis in $\mathbb{C}^{2n}$. Then for $X \in G_{2n,n}(\mathbb{C})$ after we choose the basis for X we can represent it by $2n \times n$ matrix A_X. The $n \times n$ minors of this matrix define the Plücker coordinates for X which are unique up to constant. We denote them by $P^{l_i}(X)$, where l_i runs through all combination of n elements of the set $\{1, \dots, 2n\}$. Thus l_i's are the n-th symmetric powers of the set $\{1, \dots, 2n\}$. The Plücker coordinates satisfy the following relations [14].

Lemma 3.1. *The Plücker coordinates $P^{l_i} = P^{l_i}(X)$ for $X \in G_{2n,n}(\mathbb{C})$ satisfies*

$$P^{l_i} \cdot P^{l_j} = \sum_{l=1}^{n} (-1)^{l+1} P^{l_{i(i_l \to j_1)}} \cdot P^{l_{j(j_1 \to i_l)}}, \tag{11}$$

where $l_i = i_1 \dots i_n$, $l_j = j_1 \dots j_n$ and $l_{i(i_l \to j_1)} = i_1 \dots \hat{i_l}(j_1) \dots i_n$, $l_{j(i_1 \to i_l)} = i_l j_2 \dots j_n$.

Remark 3.2. Note that in (11) it holds that $l_i l_j = l_{i(i_i \to j_1)} l_{j(j_1 \to i_l)}$.

The Plücker coordinates, being unique up to constant, give the embedding of $G_{2n,n}(\mathbb{C})$ into complex projective space $\mathbb{C}P^{N-1}$, where $N = \binom{2n}{n}$. Therefore, from 11 we obtain the following lemma.

Lemma 3.3. *The Plücker embedding of $G_{2n,n}(\mathbb{C})$ in $\mathbb{C}P^{N-1}$ is given by the intersection of the following family of hypersurfaces:*

$$z_i z_j = \sum_{l=1}^{n} (-1)^{l+1} z_{i(i_l \to j_1)} z_{j(j_1 \to i_l)}, \tag{12}$$

where $2 \le i \ne j \le N$.

The number of equations in (12) is quite large comparing to the dimension of $G_{2n,n}(\mathbb{C})$. Therefore, it is of interest to find the minimal number of such equations which define this embedding. The points in $G_{2n,n}$ whose all Pücker coordinates are non-zero form the main stratum W (see [6]).

Corollary 3.4. *The Plücker embedding of the main stratum $W \subset G_{2n,n}(\mathbb{C})$ is given by the following system of quadratic equations:*

$$z_{i^0} z_j = \sum_{l=1}^{n} (-1)^{l+1} z_{(i^0_l \to j_1)} z_{j(j_1 \to i^0_l)}, \tag{13}$$

for any fixed i^0, $1 \leq i^0 \leq N$ and all j such that $|\rho_{i^0} \cap \rho_j| + 2 \leq n$. Here $z_k \neq 0$ for all k.

Proof. For the clearness we verify formula (13) for $i^0 = 1$. It follows from (11) that the $G_{2n.n}(\mathbb{C})$ when embedded in $\mathbb{C}P^{N-1}$ belong to the intersection of the surfaces (12). Since the dimension of $G_{2n,2}(\mathbb{C})$ is n^2 it follows that the intersection (12) has the dimensions $\geq n^2$. We calculate now the number of equations in (13). We find useful to differentiate the following:

1. ρ_1 and ρ_j do not have any common k's, then $j = N$ and there is one equation, that is $\binom{n}{0}^2$.
2. ρ_1 and ρ_j have common exactly one k, $1 \leq k \leq n$. Then there are $\binom{n}{n-1} = n$ choices for ρ_j and n choices for k what together gives that there are $n^2 = \binom{n}{n-1}^2$ equations.
3. ρ_1 and ρ_j have common exactly two parameters k_1, k_2, $1 \leq k_1, k_2 \leq n$. There are $\binom{n}{n-2}$ choices for ρ_j and $\binom{n}{2}$ choices for $\{k_1, k_2\}$ what together gives $\binom{n}{2}^2$ equations.

Thus, in general we obtain that if ρ_1 and ρ_j have common exactly s parameters $k_1, \ldots, k_s$, where $1 \leq k_1, \ldots k_s \leq n$ then there are $\binom{n}{n-s}$ choices for ρ_j and $\binom{n}{s}$ choices for $\{k_1, \ldots, k_s\}$ what together gives $\binom{n}{s}^2$ equations. All together we obtain that the number of equations is: $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n-2}^2$. On the other hand by the Vandemond identity we have that $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}$, what implies that the number of equations in (12) is $\binom{2n}{n} - \binom{n}{n-1}^2 - \binom{n}{n}^2 = \binom{2n}{n} - (n^2+1)$.

It implies that the intersection (12) has $n^2 + 1$ free coordinates. Since the coordinates are homogeneous, it implies that this intersection has the dimension n^2. □

Together with Proposition 2.1 we obtain the description of the principal orbits in $G_{2n,n}(\mathbb{C})$ in homogeneous coordinates:

Corollary 3.5. *The principal $\theta_{2n,n}$-orbit in $G_{2n,n}(\mathbb{C}) \subset \mathbb{C}P^{N-1}$ is given by the intersection of the following family of hypersurfaces:*

$$\begin{cases} z_{i^0} z_j = \sum\limits_{l=1}^{n} (-1)^{l+1} z_{(i^0_l \to j_1)} z_{j(j_1 \to i^0_l)}, \\ c_{kl} z_p z_q = c_{pq} z_k z_l \quad \textit{iff} \quad \rho_p \rho_q = \rho_k \rho_l, \end{cases} \tag{14}$$

for any fixed i^0, $1 \leq i^0 \leq N$, *where* $|\rho_{i^0} \cap \rho_j| + 2 \leq n$ *and* $c_{kl} \in \mathbb{C}^*$. *Here* $z_i \neq 0$ *for all* i.

When finding the closure of a principal in $G_{2n,n}(\mathbb{C})$ we must take into consideration the other equations from (12) as well, since the boundary orbits are of smaller dimensions.

Proposition 3.6. *The toric manifold in* $\mathbb{C}P^{N-1}$ *which is obtained as the closure of the principal* $\theta_{2n,n}$*- orbit in* $G_{2n,n}(\mathbb{C}) \subset \mathbb{C}P^{N-1}$ *is the intersection of the following hypersurfaces*

$$\begin{cases} z_i z_j = \sum_{l=1}^{n} (-1)^{l+1} z_{i(i_l \to j_1)} z_{j(j_1 \to i_l)}, \\ c_{kl} z_i z_j = c_{ij} z_k z_l \quad \textit{iff} \quad \rho_i \rho_j = \rho_k \rho_l, \end{cases} \tag{15}$$

where $z_i \in \mathbb{C}$, $1 \leq i \leq N$.

As in the case of the complex projective space we also have:

Corollary 3.7. *An orbit of the maximal dimension which is contained in a principal orbit* $\emptyset(\mathbf{a})$ *in* $G_{2n,n}(\mathbb{C}) \subset \mathbb{C}P^{N-1}$ *is a principal orbit of a point in* $G_{2n-1,n}(\mathbb{C}) \subset \mathbb{C}P^{\frac{N}{2}-1}$ *or* $G_{2n-1,n-1}(\mathbb{C}) \subset \mathbb{C}P^{\frac{N}{2}-1}$ *under the standard action of* $(\mathbb{C}^*)^{2n-1}$.

Proof. Following Corollary 2.6 we demonstrate the proof in two typical cases. **1.** Consider a boundary orbit such that $z_{\frac{N}{2}+1} = \ldots = z_N = 0$ and $z_1, \ldots, z_{\frac{N}{2}} \neq 0$. Let a point $X \in G_{2n,n}(\mathbb{C})$ belongs to boundary orbit. Then all minors of the point X which do not contain the first row are equal to zero. It implies that $x_{n+11} = x_{n+21} = \ldots = x_{2n1} = 0$, where x_{ij} are the entries of the matrix X. Thus, the first column of X is $(1, 0, \ldots, 0)$ which implies that $X \in G_{2n-1,n-1}(\mathbb{C})$. Moreover, $n \times n$-minors of the matrix X from which the first column is removed are non zero, which implies that X is in a principal orbit of $G_{2n-1,n}(\mathbb{C})$ under the action of $(\mathbb{C}^*)^{2n-1}$.

2. Consider a boundary orbit such that $z_1 = \ldots = z_{\frac{N}{2}} = 0$ and $z_{\frac{N}{2}+1}, \ldots, z_N \neq 0$. Then all minors which contain the first row of a corresponding point X are equal to zero. It further implies that $x_{11} = x_{12} = \ldots = x_{1n} = 0$. Thus the first row of X is zero, which implies that $X \in G_{2n-1,n-1}(\mathbb{C})$ and all its minors when removed the first row are non-zero meaning that X belongs to a principal orbit in $G_{2n-1,n-1}(\mathbb{C})$. $\square$

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University of Montenegro, Faculty of Science and Mathematics, Podgorica, Montenegro
E-mail: sterzic@ucg.ac.me