MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
76, 1–2 (2024), 105–117
March 2024

research paper
оригинални научни рад
DOI: 10.57016/MV-MsnU3893

GEODESICS OF RIEMANNIAN COMPLEX HYPERBOLIC PLANE

Marijana Babić

Abstract. The complex hyperbolic plane is a symmetric space of negative sectional curvature; hence, it has the structure of a 4-dimensional connected solvable real Lie group with a left-invariant metric. We consider all non-isometric left-invariant Riemannian metrics on this group, denoted by $\mathcal{CH}^2$, and search for real geodesics corresponding to them. Using Euler-Arnold equations, one can translate the second-order differential equations of the geodesics on the group into the first-order equations on its Lie algebra. In the Kähler case we solve these equations on the Lie algebra of $\mathcal{CH}^2$, i.e. we explicitly find curves on algebra corresponding to the geodesics of the standard Einstein metric. Numerical solutions are used to visualize geodesic lines and geodesic spheres of various left-invariant Riemannian metrics.

1. Introduction

The complex hyperbolic plane, a locally unique Kähler manifold of constant negative holomorphic sectional curvature, is well researched and has a rich geometry. For a comprehensive overview, we recommend reading Parker [14] and Goldman [9]. In this article, we focus on left-invariant geometry in the context of Lie groups. Indeed, any connected homogeneous manifold of nonpositive curvature can be represented as a connected solvable Lie group with left-invariant metric [11]. Since the complex hyperbolic plane is a symmetric space of negative sectional curvature, it can be represented by a real solvable Lie group $\mathcal{CH}^2$.

The left-invariant complex geometry of this group has been studied in detail in a broader context of 4-dimensional Lie groups and algebras. Classifications of complex structures [12], hypercomplex structures [4], para-hypercomplex structures [5] and symplectic structures [13] were made with respect to the standard Einstein metric. Hermitian structures of all left-invariant Riemannian metrics on $\mathcal{CH}^2$ were classified in [20]. Automorphisms and all possible left-invariant metrics on this Lie group are known for all three types of signatures: Riemannian [7], Lorentzian [6] and neutral [16]. In addition, left-invariant Riemannian metrics of the n-dimensional complex

2020 Mathematics Subject Classification: 53C22, 22E60
Keywords and phrases: Complex hyperbolic plane; left-invariant metric; Euler-Arnold equations; geodesic lines; geodesic spheres.

hyperbolic space have recently been classified [8]. In [21], the authors classify all left-invariant metrics of arbitrary signature on the n-dimensional real hyperbolic space and determine all geodesically complete ones.

When exploring various left-invariant metrics and complex structures, some interesting questions arise: What does this Lie group look like? Can it be visualized? If so, what should we show? Geodesic spheres are compelling objects for visualization, so we focused our interest on geodesic lines. While complex geodesics of the Bergman metric on $\mathbb{C}H^2$ are well-known, the real geodesics of various left-invariant metrics on the Lie group $\mathcal{CH}^2$ have not been studied so far.

Geodesics on Lie groups can be determined efficiently using a specific method. Arnold [1] demonstrated that Euler's equations of motion for a rigid body can be interpreted as geodesic flows on a Lie group equipped with an invariant metric. This allows the calculation of geodesics in Lie groups by the Euler-Arnold equation, which transforms the system of second-order differential equations in the Lie group into a system of first-order differential equations in its Lie algebra. A comprehensive overview of the Euler-Arnold formalism can be found in Terence Tao's book [17].

In this paper, we use the paraboloid model and horospherical coordinates [10] of the complex hyperbolic space due to their compatibility with the Lie group $\mathcal{CH}^2$ structure. We present two main results. In Theorem 3.2 we present the explicit formulation of the Euler-Arnold equations for any left-invariant Riemannian metric on $\mathcal{CH}^2$, which allows us to visualize geodesic lines and spheres for various metrics. In Theorem 3.3 we determine curves in the Lie algebra ch_2 that correspond to the geodesics of the Kähler metric on $\mathcal{CH}^2$. Finally, we set up equations in Corollary 3.4 that describe geodesics of the same metric on $\mathcal{CH}^2$. Since all considered metrics are Riemannian, each of them is geodesically complete.

There is a wonderful feedback between geometry and computer-aided visualizations whereby studying geometry enables us to create visual representations that in turn deepen our understanding of geometric objects and inspire further research. For example, Goldman's book *Complex Hyperbolic Geometry* – a comprehensive guide and currently the main reference work on the geometry of $\mathbb{C}H^n$ – began "as the twin sibling of a computer program" (see [9, Preface]). In this paper we visualize geodesic lines and spheres of non-isometric left-invariant Riemannian metrics on the Lie group $\mathcal{CH}^2$. Since these objects live in four-dimensional real space $\mathcal{CH}^2$, one has to choose a suitable method for their representation in three-dimensional space. We have chosen to represent the complex component of the product $\mathcal{CH}^2 = \mathbb{R}^+ \ltimes (\mathbb{C} \ltimes \mathbb{R})$ as the modulus of a complex number. This approach effectively reduces the dimension to three and enables the visualization of objects of the four-dimensional $\mathcal{CH}^2$. Wolfram Mathematica's package *Heisenberg* [15], which inspired Goldman's book [9], uses a different method. The authors use a geodesic perspective from an ideal point to project various objects from $\mathbb{C}H^2$ onto the absolute of the Siegel model, i.e. onto the Heisenberg group. This projection is particularly interesting because the absolute is isomorphic to a horosphere, a surface modeling the three-dimensional Euclidean space in $\mathbb{C}H^2$. In other words, it allows the observer to view the complex hyperbolic space with *Euclidean eyes*. A similar approach, the visualization of real hyperbolic space by central projection onto a horosphere – i.e. projection onto a flat computer screen embedded in $\mathbb{R}H^3$ – has used in [19], the Mathematica package [3] as well as in the author's master thesis [2].

This paper is structured as follows: Section 2 considers the complex hyperbolic space in the context of Lie groups. Here we introduce the multiplication law and horospherical coordinates of the group $\mathcal{CH}^2$, the commutators of its Lie algebra ch_2, and describe all non-isometric left-invariant Riemannian metrics. We compute the relation between the left-invariant basis (e_1, e_2, e_3, e_4) and the coordinate basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \frac{\partial}{\partial w})$ of the Lie algebra ch_2. This result is used later to find the geodesics.

The Section 3 is the main part of this paper. Theorem 3.2 gives Euler-Arnold equations for the curve $\gamma(t)$ on the Lie algebra ch_2, which corresponds to the geodesic $c(t)$ of any left-invariant Riemannian metric on the Lie group $\mathcal{CH}^2$. In Theorem 3.3 we explicitly find the curves on the algebra ch_2 corresponding to the geodesic of the Kähler metric on $\mathcal{CH}^2$. In Corollary 3.4 we write the reconstruction problem for finding geodesics of the same metric on $\mathcal{CH}^2$.

In Section 4 we visualize geodesic lines passing through the identity element of the Lie group $\mathcal{CH}^2$ as well as geodesic spheres centered at the identity.

2. Preliminaries

The complex hyperbolic plane is a symmetric space of negative sectional curvature

$$\mathbb{C}H^2 = SU(1,2)/S(U(1) \times U(2)).$$

The group $SU(1,2)$ is its isometry group. In the Iwasawa decomposition, $SU(1,2) = KAN$, the compact part K is the unitary group $U(2)$, the nilpotent part N is the Heisenberg group H^3 and A is a 1-dimensional abelian group. The completely solvable group $\mathcal{CH}^2 = AN$ acts simply transitively on the complex hyperbolic plane, giving it a Lie group structure.

This structure is well-represented in horospherical coordinates of the Siegel paraboloid model of $\mathbb{C}H^2$, first introduced by Goldman and Parker in [10]. The root space decomposition of the isometry group $SU(1,2)$ provides coordinates on the absolute, transforming it into the Heisenberg group, as shown in [9]. They considered a foliation of the complex hyperbolic space by horospheres centered in a fixed point q_∞ on the absolute. The geodesic perspective induced by the parabolic pencil of the real geodesics passing through q_∞ projects the structure of the Heisenberg group onto the horospheres. As a result, each horosphere can be endowed with coordinates from the Heisenberg group, along with a height function to form a system of horospherical coordinates [10].

$$\mathcal{CH}^2 = \mathbb{R}^+ \ltimes H^3 = \mathbb{R}^+ \ltimes (\mathbb{C} \ltimes \mathbb{R}) = \{(x, \zeta, w) \mid x \in \mathbb{R}^+, \zeta \in \mathbb{C}, w \in \mathbb{R}\},$$

with the identification $\zeta = y + iz$ is

$$\mathcal{CH}^2 = \mathbb{R}^+ \ltimes H^3 = \{(x, y, z, w) \mid x \in \mathbb{R}^+, (y, z, w) \in H^3\}.$$

Horospheres centered in q_∞ are level sets given by the equation $H_u : x = u$. The absolute is H_0.

$\mathcal{CH}^2$ belongs to a broader class of harmonic NA groups (one-dimensional extensions of H-type groups [18]). The multiplication law [18] is given in horospherical coordinates

$$(x,y,z,w)\cdot(x',y',z',w') = \left(xx',\ y+\sqrt{x}y',\ z+\sqrt{x}z',\ w+xw'+\frac{1}{2}\sqrt{x}(zy'-yz')\right),$$

with inverse $(x,y,z,w)^{-1} = (\frac{1}{x}, -\frac{1}{\sqrt{x}}y, -\frac{1}{\sqrt{x}}z, -\frac{1}{x}w)$, and the identity element $(1,0,0,0)$.

From the multiplication law, one can derive the commutators of the Lie algebra ch_2 and the relation between coordinate and left-invariant basis.

Lemma 2.1. *The relation between the coordinate* $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \frac{\partial}{\partial w})$ *and the left-invariant basis* (e_1, e_2, e_3, e_4) *of the Lie algebra* ch_2 *is*

$$e_1 = x\frac{\partial}{\partial x},\quad e_2 = \sqrt{x}(\frac{\partial}{\partial y}+\frac{z}{2}\frac{\partial}{\partial w}),\quad e_3 = \sqrt{x}(\frac{\partial}{\partial z}-\frac{y}{2}\frac{\partial}{\partial w}),\quad e_4 = x\frac{\partial}{\partial w}. \tag{1}$$

Proof. Left-invariant vector fields are obtained by left translations of coordinate vector fields. If $\alpha(t) = (1+t,0,0,0)$ is a parameterized curve through the identity $e = (1,0,0,0)$ in the direction of the coordinate vector $\frac{\partial}{\partial x}$ and $q = (x,y,z,w)$ is an arbitrary element of $\mathcal{CH}^2$, then

$$L_q(\alpha(t))) = q\cdot\alpha(t) = (x,y,z,w)\cdot(1+t,0,0,0) = (x(1+t),y,z,w).$$

It follows that

$$\frac{d}{dt}(L_q(\alpha(t))) = (x,0,0,0) = x\frac{\partial}{\partial x} = e_1.$$

Similarly, consider the curve $\beta(t) = (1,t,0,0)$ through the identity in the direction of $\frac{\partial}{\partial y}$

$$L_g(\beta(t))) = g\cdot\beta(t) = (x,y,z,w)\cdot(1,t,0,0) = (x,y+\sqrt{x}t,z,w+\frac{1}{2}\sqrt{x}zt),$$

$$\frac{d}{dt}(L_g(\beta(t))) = (0,\sqrt{x},0,\frac{1}{2}\sqrt{x}z) = \sqrt{x}\frac{\partial}{\partial y}+\frac{1}{2}\sqrt{x}z\frac{\partial}{\partial w} = e_2.$$

We get all other left-invariant fields in the same way. □

Transition matrices between the left-invariant and the coordinate basis are

$$dL_q = C = \begin{pmatrix} x & 0 & 0 & 0\\ 0 & \sqrt{x} & 0 & 0\\ 0 & 0 & \sqrt{x} & 0\\ 0 & \frac{\sqrt{x}}{2}z & -\frac{\sqrt{x}}{2}y & x\end{pmatrix},\qquad dL_q^{-1} = C^{-1} = \begin{pmatrix} \frac{1}{x} & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{x}} & 0 & 0\\ 0 & 0 & \frac{1}{\sqrt{x}} & 0\\ 0 & -\frac{z}{2x} & \frac{y}{2x} & \frac{1}{x}\end{pmatrix}. \tag{2}$$

The nonzero commutators of the Lie algebra ch_2 are

$$ch_2:\quad [e_1,e_2] = \frac{1}{2}e_2,\quad [e_1,e_3] = \frac{1}{2}e_3,\quad [e_1,e_4] = e_4,\quad [e_3,e_2] = e_4. \tag{3}$$

Every positive-definite inner product on a Lie algebra determines a left-invariant Riemannian metric on the corresponding Lie group by left translations. Thus the following theorem describes all non-isometric left-invariant Riemannian metrics on $\mathcal{CH}^2$.

Theorem 2.2 ([20]). *All non-isometric positive definite inner products on ch_2, in some basis with Lie algebra commutators* (3), *are represented by*

$$S(p,b,\beta) = \begin{pmatrix} p & b & 0 & 0 \\ b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \tag{4}$$

where $b \geq 0,\ \beta > 0,\ p - b^2 > 0$.

We denote by $g(p,b,\beta)$ a left-invariant metric defined by inner product $S(p,b,\beta)$. When $b = 0$, $p\beta = 1$, the left-invariant metric is Einstein, and $\mathcal{CH}^2$ equipped with this metric is a Kähler manifold with constant holomorphic sectional curvature $-\beta$ (see [20]).

3. Geodesics

The standard method for finding geodesics in a Lie group with a left-invariant metric is to reduce the system of second-order differential equations of geodesics in the group to the first-order differential equations in its Lie algebra, as described below.

To each curve $c(t)$ in a Lie group G we associate a curve $\gamma(t)$ in its Lie algebra $\mathfrak{g}$

$$\gamma(t) = dL^{-1}_{c(t)}\dot{c}(t). \tag{5}$$

The curve $\gamma(t)$ in the Lie algebra $\mathfrak{g}$ corresponding to a *geodesic* $c(t)$ in G satisfies the Euler-Arnold equation

$$\dot{\gamma} = ad^*_\gamma \gamma, \tag{6}$$

where ad^*_γ is the transpose operator of the operator ad_γ with respect to a left-invariant metric $\langle ad^*_\gamma X, Y\rangle = \langle X, ad_\gamma Y\rangle$.

For proof that the curves obtained this way are truly geodesics, see Arnold's seminal paper [1] or a comprehensive overview of the subject in [17]. To visualize the geodesic spheres, the geodesics obtained from equation (6) must be naturally parameterized. Although the next statement is well-known, we presented a proof for the completeness of the exposition.

Lemma 3.1. *A geodesic $c(t)$ in a Lie group whose corresponding curve $\gamma(t) = dL^{-1}_{c(t)}\dot{c}(t)$ satisfies* (6) *is parameterized by the arclength.*

Proof. Koszul's formula for a left-invariant metric g

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X), \end{aligned}$$

applied to any left-invariant vector field Z and the geodesic $c(t)$ yields

$$0 = 2g(\nabla_{\dot{c}}\dot{c}, Z) = 2\dot{c}g(\dot{c}, Z) - Zg(\dot{c},\dot{c}) - 2g(\dot{c}, [\dot{c}, Z]). \tag{7}$$

Using Leibniz's rule and the left invariance of the vector field Z, we compute

$$\dot{c}g(\dot{c}, Z) = g(\dot{c}(\dot{c}), Z) + g(\dot{c}, \dot{c}(Z)) = g(\ddot{c}, Z)$$

and
$$g(\dot{c}, [\dot{c}, Z]) = g(\dot{c}, ad_{\dot{c}}Z) = g(ad^*_{\dot{c}}\dot{c}, Z).$$

Therefore, the equation (7) becomes

$$2g(\ddot{c}, Z) - Zg(\dot{c}, \dot{c}) - 2g(ad^*_{\dot{c}}\dot{c}, Z) = 0.$$

From (6) we have $g(\dot{\gamma}, Z) = g(ad^*_{\gamma}\gamma, Z)$. Since vector field Z and metric g are left-invariant, by left translation we get $g(\ddot{c}, Z) = g(ad^*_{\dot{c}}\dot{c}, Z)$. It follows that $Zg(\dot{c}, \dot{c}) = 0$, i.e. the geodesic c has a constant speed. □

The following theorem describes curves on the Lie algebra ch_2 corresponding to geodesics of the Lie group $\mathcal{CH}^2$ with respect to all possible non-isometric left-invariant Riemannian metrics.

Theorem 3.2. *Let $g(p, b, \beta)$ be the metric given by the scalar product of the form* (4). *The corresponding Euler-Arnold equations on the Lie algebra ch_2 in basis* (1) *are*

$$\begin{aligned}
\dot{\gamma}^1 &= -\frac{\left(b\gamma^1 + \gamma^2\right)^2 + \left(\gamma^3\right)^2 + 2b\beta\gamma^3\gamma^4 + 2\beta\left(\gamma^4\right)^2}{2\left(p - b^2\right)},\\
\dot{\gamma}^2 &= \frac{pb\left(\gamma^1\right)^2 + \left(p + b^2\right)\gamma^1\gamma^2 + b\left(\gamma^2\right)^2 + b\left(\gamma^3\right)^2 + 2\beta\gamma^4\left(p\gamma^3 + b\gamma^4\right)}{2\left(p - b^2\right)},\\
\dot{\gamma}^3 &= \frac{1}{2}\gamma^1\gamma^3 - \beta\gamma^2\gamma^4,\\
\dot{\gamma}^4 &= \gamma^1\gamma^4.
\end{aligned} \tag{8}$$

Proof. If we denote $\dot{\gamma}$ by $v = v^i e_i$, our goal is to find coordinates v^i in left-invariant basis (1) with Lie algebra commutators (3). From the Euler-Arnold equation (6)

$$v = ad^*_{\gamma}\gamma,$$

we have $\langle v, e_i\rangle = \langle ad^*_{\gamma}\gamma, e_i\rangle = \langle \gamma, ad_{\gamma}e_i\rangle = \langle \gamma, [\gamma, e_i]\rangle$. Applying Einstein's summation convention, the last equation can be rewritten as $\langle v^j e_j, e_i\rangle = \langle \gamma^k e_k, [\gamma^l e_l, e_i]\rangle$, giving $v^j\langle e_j, e_i\rangle = \gamma^k\gamma^l\langle e_k, [e_l, e_i]\rangle$. If we denote by $g_{ji} = \langle e_j, e_i\rangle = g_{ij}$ the coefficients of the left-invariant metric g defined by the scalar product (4), then the last equation becomes $v^j g_{ji} = \gamma^k\gamma^l\langle e_k, [e_l, e_i]\rangle$. Multiplying by the inverse metric, we get

$$v^j = g^{ji}\gamma^k\gamma^l\langle e_k, [e_l, e_i]\rangle = g^{ij}\gamma^k\gamma^l\langle e_k, c^m_{li}e_m\rangle = g^{ij}\gamma^k\gamma^l c^m_{li}\langle e_k, e_m\rangle = g^{ij}g_{km}c^m_{li}\gamma^k\gamma^l,$$

where c^m_{li}, are the structure constants of the Lie algebra ch_2 . The system (8) now follows by direct calculation. □

In the special case of metric being Kähler, the explicit solution of the system (8) is obtained in the following theorem.

Theorem 3.3. *In the case of the Kähler metric, i.e. $p\beta = 1$, $b = 0$, the curve $\gamma(t) = \gamma^i(t)e_i$ in basis* (1) *satisfying Euler-Arnold equations on the Lie algebra ch_2 is given as follows*

- *in the plane* $\Sigma = Span\{e_1, e_4\}$

$$\gamma(t) = \left(-2c_1 \tanh(2c_1 t + c_2), 0, 0, \pm\frac{2c_1}{\beta\cosh(2c_1 t + c_2)}\right), \quad c_2 \in \mathbb{R}, \ c_1 > 0, \tag{9}$$

or $\quad \gamma(t) = (1, 0, 0, 0).$

- *in the planes* $\Pi_\phi = Span\{e_1, e_\phi\}, \quad e_\phi = \cos\phi\, e_2 + \sin\phi\, e_3, \quad \phi \in [0, \pi)$

$$\gamma(t) = \left(-2c_1 \tanh(c_1 t + c_2), \frac{2c_1\cos\phi}{\sqrt{\beta}\cosh(c_1 t + c_2)}, \frac{2c_1\sin\phi}{\sqrt{\beta}\cosh(c_1 t + c_2)}, 0\right), \tag{10}$$

$c_2 \in \mathbb{R}, \quad c_1 > 0.$

- *in* $ch_2 \backslash \left(\bigcup_{\phi\in[0,\pi)} \Pi_\phi\right) \backslash \Sigma$

$$\begin{aligned} \gamma^1 &= \frac{2c_2\left(e^{4c_2(c_4-t)} - 16c_1^2c_2^2 - 4\right)}{\left(2 + e^{2c_2(c_4-t)}\right)^2 + 16c_1^2c_2^2}, \\ \gamma^2 &= \sqrt{\frac{2}{\beta}}r\cos\varphi, \\ \gamma^3 &= \sqrt{\frac{2}{\beta}}r\sin\varphi, \\ \gamma^4 &= \pm\frac{c_1}{\beta}r^2, \end{aligned} \tag{11}$$

where

$$r = \frac{4c_2e^{c_2(c_4-t)}}{\sqrt{4 + 16c_1^2c_2^2 + 4e^{2c_2(c_4-t)} + e^{4c_2(c_4-t)}}},$$

$$\varphi = s\arcsin\left(\frac{1 + c_1^2r^2}{\sqrt{1 + 4c_1^2c_2^2}}\right) + c_3, \qquad s = -\operatorname{sgn}\gamma^1\operatorname{sgn}\gamma^4,$$

$c_3, c_4 \in \mathbb{R}, \quad c_1, c_2 > 0.$

Proof. In the Kähler case, the system (8) is

$$\begin{aligned} \dot\gamma^1 &= -\frac{\beta}{2}\left(\left(\gamma^2\right)^2 + \left(\gamma^3\right)^2 + 2\beta\left(\gamma^4\right)^2\right), \\ \dot\gamma^2 &= \frac{1}{2}\gamma^1\gamma^2 + \beta\gamma^3\gamma^4, \\ \dot\gamma^3 &= \frac{1}{2}\gamma^1\gamma^3 - \beta\gamma^2\gamma^4, \\ \dot\gamma^4 &= \gamma^1\gamma^4. \end{aligned}$$

Linear change of coordinates

$$\gamma^1 = 2\alpha_1, \qquad \gamma^2 = \sqrt{\frac{2}{\beta}}\alpha_2, \qquad \gamma^3 = \sqrt{\frac{2}{\beta}}\alpha_3, \qquad \gamma^4 = \frac{1}{\beta}\alpha_4, \tag{12}$$

transforms the previous system to

$$\begin{aligned}\dot{\alpha}_1 &= -\frac{1}{2}\left(\alpha_2^2+\alpha_3^2+\alpha_4^2\right),\\ \dot{\alpha}_2 &= \alpha_1\alpha_2+\alpha_3\alpha_4,\\ \dot{\alpha}_3 &= \alpha_1\alpha_3-\alpha_2\alpha_4,\\ \dot{\alpha}_4 &= 2\alpha_1\alpha_4.\end{aligned} \tag{13}$$

Note that plane $\Sigma : \alpha_2 = \alpha_3 = 0$ is an invariant manifold of the system (13). The system restricted to this plane is

$$\dot{\alpha}_1 = -\frac{1}{2}\alpha_4^2, \quad \dot{\alpha}_4 = 2\alpha_1\alpha_4,$$

and its solution is

$$\alpha_1 = -c_1\tanh(2c_1t+c_2), \quad \alpha_4 = \pm\frac{2c_1}{\cosh(2c_1t+c_2)}, \quad c_1, c_2 \in \mathbb{R}, \quad c_1 > 0.$$

Returning to the linear change of coordinates (12) we get the solution (9) from the theorem statement.

Now we consider $ch_2\backslash\Sigma$. When we introduce the cylindrical coordinates

$$\alpha_2 = r\cos\varphi, \quad \alpha_3 = r\sin\varphi,$$

system (13) becomes

$$\dot{\alpha}_1 = -\frac{1}{2}\left(r^2+\alpha_4^2\right), \tag{14}$$

$$\dot{r} = \alpha_1 r, \tag{15}$$

$$\dot{\varphi} = -\alpha_4, \tag{16}$$

$$\dot{\alpha}_4 = 2\alpha_1\alpha_4. \tag{17}$$

Combining (15) and (17) gives

$$\alpha_1 = \frac{\dot{r}}{r}, \quad \dot{\alpha}_4 = 2\alpha_1\alpha_4 = 2\frac{\dot{r}}{r}\alpha_4 \tag{18}$$

If $\alpha_4 = 0$, then from the equation (16) φ is constant, say $\varphi = \phi$, $\phi \in [0, \pi)$, and the plane $\Pi_\phi = Span\{e_1, e_\phi\}$, $e_\phi = \cos\phi\, e_2 + \sin\phi\, e_3$, is an invariant plane of the system. Restriction to this plane gives

$$\dot{\alpha}_1 = -\frac{1}{2}r^2, \quad \dot{r} = \alpha_1 r.$$

The solution is

$$\alpha_1 = -c_1\tanh(c_1t+c_2), \quad r = \frac{c_1\sqrt{2}}{\cosh(c_1t+c_2)}, \quad c_1, c_2 \in \mathbb{R}, \quad c_1 > 0,$$

or equivalently

$$\alpha_1 = -c_1\tanh(c_1t+c_2), \quad \alpha_2 = r\cos\phi = \frac{c_1\sqrt{2}\cos\phi}{\cosh(c_1t+c_2)}, \quad \alpha_3 = r\sin\phi = \frac{c_1\sqrt{2}\sin\phi}{\cosh(c_1t+c_2)}.$$

Returning to the linear change of coordinates (12) we get the solution (10) from the theorem statement.

If $\alpha_4 \neq 0$, then from (18) we get $\frac{\dot{\alpha}_4}{\alpha_4} = 2\frac{\dot{r}}{r}$. Integrate the last equation

$$\ln|\alpha_4| = 2\ln r + \bar{c}, \quad |\alpha_4| = r^2 e^{\bar{c}},$$

$$\alpha_4 = \pm c_1 r^2, \quad c_1 > 0. \tag{19}$$

From (14), (15) and (19) we have

$$-\frac{1}{2}\left(r^2 + c_1^2 r^4\right) = -\frac{1}{2}\left(r^2 + \alpha_4^2\right) = \dot{\alpha}_1 = \frac{d\alpha_1}{dt} = \frac{d\alpha_1}{dr}\frac{dr}{dt} = \frac{d\alpha_1}{dr}\dot{r} = \frac{d\alpha_1}{dr}\alpha_1 r,$$

$$-\frac{1}{2}\left(r + c_1^2 r^3\right)dr = \alpha_1 d\alpha_1,$$

Integrate the previous equation

$$-\frac{1}{2}\left(\frac{r^2}{2} + \frac{c_1^2 r^4}{4}\right) + c = \frac{\alpha_1^2}{2}, \quad c \geq 0, \quad \text{thus,} \quad \alpha_1^2 = -\frac{r^2}{2}\left(1 + \frac{c_1^2 r^2}{2}\right) + 2c.$$

Rename $c_2^2 = 2c$, and we get

$$\alpha_1 = \pm\sqrt{-\frac{r^2}{2}\left(1 + \frac{c_1^2 r^2}{2}\right) + c_2^2}, \tag{20}$$

with $c_2^2 > \frac{r}{2}\left(1 + \frac{c_1^2 r^2}{2}\right)$ or equivalently

$$4c_2^2 - 2r^2 - c_1^2 r^4 > 0. \tag{21}$$

From (15) and (20) we have

$$\frac{dr}{dt} = \dot{r} = \alpha_1 r = \pm r\sqrt{-\frac{r^2}{2}\left(1 + \frac{c_1^2 r^2}{4}\right) + c_2^2} \quad \text{and} \quad dt = \pm\frac{dr}{r\sqrt{-\frac{r^2}{2}\left(1 + \frac{c_1^2 r^2}{4}\right) + c_2^2}}.$$

By integrating the last equation we get

$$t = \pm\left(\frac{1}{2c_2}\left(\ln\left(8c_2^2 - 2r^2 + 4c_2\sqrt{4c_2^2 - 2r^2 - c_1^2 r^4}\right) - 2\ln r\right) + c_4\right), \tag{22}$$

and
$$r = \frac{4c_2 e^{c_2(c_4 - t)}}{\sqrt{4 + 16c_1^2 c_2^2 + 4e^{2c_2(c_4 - t)} + e^{4c_2(c_4 - t)}}}. \tag{23}$$

If we return (23) in (21) we see that the condition for constants is fulfilled for every t

$$4c_2^2 - 2r^2 - c_1^2 r^4 > 0, \quad \frac{4c_2^2\left(e^{4c_2(c_4 - t)} - 16c_1^2 c_2^2 - 4\right)^2}{\left(\left(e^{2c_2(c_4 - t)} + 2\right)^2 + 16c_1^2 c_2^2\right)^2} > 0.$$

From (16) and (19) $\dot{\varphi} = -\alpha_4 = \mp c_1 r^2$. From the previous equation and (15)

$$-\operatorname{sgn}(\alpha_4) c_1 r^2 = \dot{\varphi} = \frac{d\varphi}{dt} = \frac{d\varphi}{dr}\frac{dr}{dt} = \frac{d\varphi}{dr}\alpha_1 r.$$

From the previous equation and (20)

$$-\operatorname{sgn}(\alpha_4) c_1 r = \operatorname{sgn}(\alpha_1)\frac{d\varphi}{dr}\sqrt{-\frac{r^2}{2}\left(1 + \frac{c_1^2 r^2}{4}\right) + c_2^2},$$

$$d\varphi = -\operatorname{sgn}(\alpha_1)\operatorname{sgn}(\alpha_4)\frac{c_1 r dr}{\sqrt{-\frac{r^2}{2}\left(1+\frac{c_1^2 r^2}{4}\right)+c_2^2}},$$

$$\varphi = -\operatorname{sgn}(\alpha_1)\operatorname{sgn}(\alpha_4)\arcsin\left(\frac{1+c_1^2 r^2}{\sqrt{1+4c_1 c_2}}\right)+c_3. \tag{24}$$

The condition for constants $\frac{1+c_1^2r^2}{\sqrt{1+4c_1^2c_2^2}} \in (-1,1)$, or equivalently $4c_2^2 > 2r^2 + c_1^2 r^4$ is the same as we already got in (21) and it is always fulfilled.

From (15) and (23) we have

$$\alpha_1 = \frac{\dot r}{r} = \frac{c_2\left(e^{4c_2(c_4-t)} - 16c_1^2c_2^2 - 4\right)}{\left(e^{2c_2(c_4-t)}+2\right)^2 + 16c_1^2c_2^2}. \tag{25}$$

Finally when we return the solutions (25), (23), (24), (19) into the linear change of coordinates (12), we obtain the solution (11) from the theorem statement. □

Once the solution of the Euler-Arnold equation has been found in the algebra, one should solve the reconstruction problem given by (5) to find the geodesic on the group. We explicitly write this problem for the Kähler metric.

Corollary 3.4. *When the metric is Kähler, i.e.* $p\beta = 1$, $b = 0$, *the geodesic* $c(t) = (x, y, z, w)$ *on the Lie group* $\mathcal{CH}^2$ *through the identity satisfies*

$$\dot x = x\gamma^1, \quad \dot y = \sqrt{x}\gamma^2, \quad \dot z = \sqrt{x}\gamma^3, \quad \dot w = \frac{1}{2}z\sqrt{x}\gamma^2 - \frac{1}{2}y\sqrt{x}\gamma^3 + x\gamma^4, \tag{26}$$

with the initial conditions $c(0) = (1, 0, 0, 0)$, $\dot c(0) = \gamma(0)$, *where the curve* γ *is given by Theorem 3.3. Specifically,*

- *In* $\sigma : y = z = 0$, *the integral plane through the identity of the distribution* $\Sigma = Span\{e_1, e_4\}$, *the geodesics are*
$$c(t) = \left(\frac{\cosh c_2}{\cosh(2c_1 t + c_2)}, 0, 0, \pm\left(\frac{\cosh c_2}{\beta}\tanh(2c_1t+c_2) - \frac{\sinh c_2}{\beta}\right)\right),$$
$c_1, c_2 \in \mathbb{R}$, $c_1 > 0$, *and* $\quad c(t) = (e^t, 0, 0, 0), \quad t \in \mathbb{R}$.

- *In the family* $\pi_\phi : w = 0,\ \cos\phi\, z - \sin\phi\, y = 0$, *the integral planes through the identity of the distributions* $\Pi_\phi = Span\{e_1, e_\phi\}$, $e_\phi = \cos\phi\, e_2 + \sin\phi\, e_3$, $\phi \in [0, \pi)$, *the geodesic* $c(t)$ *is given by*
$$x(t) = \frac{\cosh^2 c_2}{\cosh^2(c_1t + c_2)},$$
$$y(t) = \frac{2}{\sqrt{\beta}}\cos(\phi)\left(\cosh c_2\ \tanh(c_1t+c_2) - \sinh(c_2)\right),$$
$$z(t) = \frac{2}{\sqrt{\beta}}\sin(\phi)\left(\cosh c_2\ \tanh(c_1t+c_2) - \sinh(c_2)\right),$$
$$w(t) = 0, \quad c_1, c_2 \in \mathbb{R}, \quad c_1 > 0 \quad t \in \mathbb{R}.$$

Proof. If we substitute the transition matrix (2) in (5) we get the equations (26) directly. Straightforward calculation using special case (9) and (10) from Theorem 3.3 gives geodesics of planes σ and π_ϕ respectively. □

Remark 3.5. Besides geodesic lines the only proper totally geodesic subspaces of $\mathbb{C}H^2$ are complex lines and totally real Lagrangian planes [14]. These spaces are related to invariant manifolds of the system (13) from Theorem 3.3. Namely the plane $\Sigma = Span\{e_1, e_4\}$ has Lie algebra commutators $[e_1, e_4] = e_4$, and each plane of the family

$$\Pi_\phi = Span\{e_1, e_\phi\}, \quad e_\phi = \cos\phi\, e_2 + \sin\phi\, e_3, \quad \phi \in [0, \pi)$$

has commutators $[e_1, e_\phi] = \frac{1}{2} e_\phi$. As Lie algebras, all of them are isomorphic to the real hyperbolic plane. Using curvature formulas from [20] one can calculate that Σ has constant sectional curvature $-\beta$ with respect to Kähler metric $g(\frac{1}{\beta}, 0, \beta)$ and it is isometric to complex line. Planes Π_ϕ have constant sectional curvature $-\frac{1}{4}\beta$ and they correspond to totally real Lagrangian planes of the complex hyperbolic plane. Their integral planes σ and π_ϕ are the half-plane models of the real hyperbolic plane.

4. Visualization

Now we use horospherical coordinates to visualize geodesics through the identity element of the Lie group $\mathcal{CH}^2 = \mathbb{R}^+ \ltimes H^3 = \mathbb{R}^+ \ltimes (\mathbb{C} \ltimes \mathbb{R}) = \{(x, \zeta, w) \mid x \in \mathbb{R}^+, \zeta \in \mathbb{C}, w \in \mathbb{R}\}$ equipped with non-isometric left-invariant Riemannian metrics. In order to visualize objects of 4-dimensional real space we show the modulus of complex number $\zeta = y + iz$, i.e. the plot coordinates are $(x, \|\zeta\|, w)$. For creating the below images we used numerical solutions of the equations of geodesics from Theorem 3.2 with different parameters of metrics $g(p, x, \beta)$ defined by inner products (4). Thanks to Lemma 3.1 we know that the geodesics are of constant speed, which allows us to visualize the geodesic spheres.

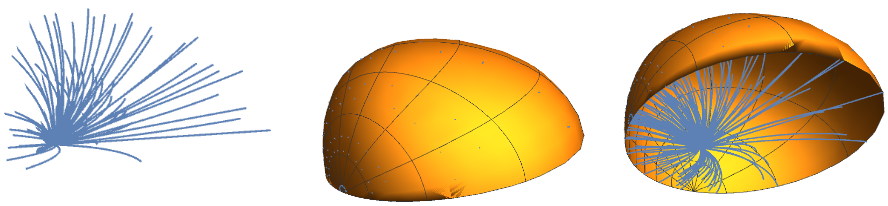

Figure 1: Standard Kähler metric of $\mathbb{C}H^2$: $p = 1, \beta = 1, b = 0$

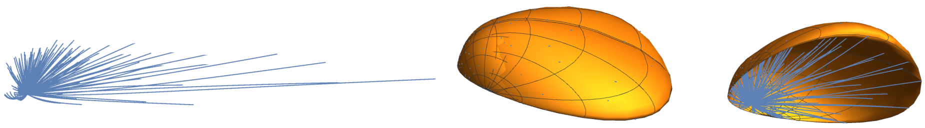

Figure 2: $p = 1, \beta = \frac{1}{2}, b = 2$

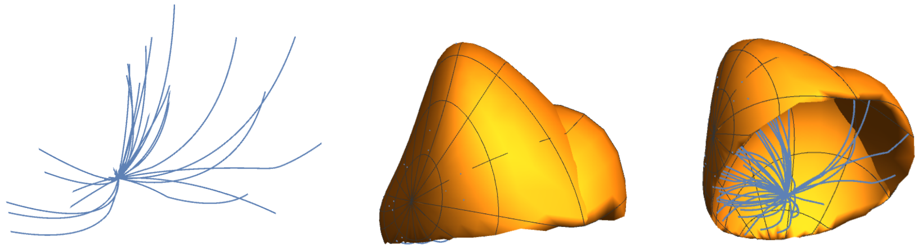

Figure 3: $p = 1, \beta = 2, b = 1$

Acknowledgement. This research has been supported by the Ministry of Science and Technological Development of Serbia, through Mathematical Institute of the Serbian Academy of Sciences and Arts.

References

[1] V. I. Arnold, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits*, Annales de l'Institut Fourier, **16(1)** (1966), 319–361.

[2] M. Babić, *Visualization of Lobachevsky space by central projection onto a Horosphere*, Master's Thesis, University of Belgrade, Faculty of Mathematics, 2010.

[3] M. Babić, S. Vukmirović, *L3toHorosphere - central projection of hyperbolic space onto horosphere*, Mathematica package 8335, Mathsource, 2012. (https://library.wolfram.com/infocenter/MathSource/8335/)

[4] M. L. Barberis, *Hypercomplex structures on four-dimensional Lie groups*, Proc. Am. Math. Soc., **125(4)** (1997), 1043–1054.

[5] N. Blažić, S. Vukmirović, *Para-hypercomplex structures on a four-dimensional Lie group*, Contemporary Geometry and Related Topics, Proceedings of the Workshop Hardcover, 2004, 41–56.

[6] G. Calvaruso, A. Zaeim, *Four-dimensional Lorentzian Lie groups*, Differ. Geom. Appl., **31(4)** (2013), 496–509.

[7] T. Christodoulakis, G. O. Papadopoulos, A. Dimakis, *Automorphisms of real four-dimensional Lie algebras and the invariant characterization of homogeneous 4-spaces*, J. Phys. A, Math. Gen., **36(2)** (2002), 427–441.

[8] A. Dekić, M. Babić, S. Vukmirović *Classification of left invariant riemannian metrics on complex hyperbolic space*, Mediterr. J. Math., **19** (2022), Article 232.

[9] W. Goldman, *Complex Hyperbolic Geometry*, Clarendon Press, Oxford, 1999.

[10] W. Goldman, J. Parker, *Dirichlet polyhedra for dihedral groups acting on complex hyperbolic space*, J. Geom. Anal., **2(6)** (1992), 517–554.

[11] E. Heintze, *On homogeneous manifolds of negative curvature*, Math. Annalen, **211** (1974), 23–34.

[12] G. Ovando, *Complex structures on* $\mathbb{R}H^4$ *and* $\mathbb{C}H^2$, Mat. Contemp., **17** (1999), 37–49.

[13] G. Ovando, *Four dimensional symplectic Lie algebras*, Beitr. Algebra Geom., **47(2)** (2006), 419–434.

[14] J. Parker, *Notes on Complex Hyperbolic Geometry*, preprint, University of Durham, 2003.

[15] M. Phillips, W. Goldman, R. Miner, *Heisenberg*, Mathematica package 1352, Mathsource, 1995. (https://library.wolfram.com/infocenter/MathSource/1352/)

[16] T. Šukilović, *Classification of left invariant metrics on 4-dimensional solvable Lie groups*, Theor. Appl. Mech., **47(2)** (2020), 181–204.

[17] T. Tao, *Compactness and contradiction*, American Mathematical Society, 2013.

[18] S. Thangavelu, *An Introduction to the Uncertainty Principle, Hardy's Theorem on Lie Groups*, Springer Science+Business Media 2004.

[19] S. Vukmirović, M. Babić, *Central projection of hyperbolic space onto a horosphere*, KoG, **16** (2012), 27–35.

[20] S. Vukmirović, M. Babić, A. Dekić, *Classification of left-invariant Hermitian structures on 4-dimensional non-compact rank one symmetric spaces*, Rev. Unión Mat. Argent., **60(2)** (2019), 343–358.

[21] S. Vukmirović, T. Šukilović, *Geodesic completeness of the left-invariant metrics on* $\mathbb{R}H^n$, Ukr. Math. J., **72(5)** (2020), 702–711.

(received 12.07.2023; in revised form 31.01.2024; available online 19.02.2024)

Mathematical Insitute of the Serbian Academy of Science and Arts, Kneza Mihaila 36, Belgrade 11000, Serbia

E-mail: marijana@mi.sanu.ac.rs