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MULTIVALUED COUPLED COINCIDENCE POINT RESULTS IN METRIC SPACES

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Abstract. In this paper, we use an inequality involving a coupled multivalued mapping and a singlevalued mapping to obtain a coupled coincidence point theorem. We discuss special conditions under which coupled common fixed point theorems are obtained. The result combines several ideas prevalent in fixed point theory studies. There are several corollaries and illustrative examples. The Hausdorff-Pompeiu metric between sets is used. The work is in the context of metric spaces and is a part of set-valued analysis with the singlevalued consequences.

1. Introduction

In this paper we establish a coupled coincidence point result for a singlevalued mapping and a coupled multivalued mapping defined on a metric space. We also discuss additional conditions under which the coupled coincidence point is a coupled common fixed point. We achieve our goal by merging several ideas, which are discussed below. The famous Banach principle of contraction mappings was extended to the field of set-valued analysis by Nadler [9] in 1969, where he proved the multi-valued contraction mapping theorem using the Hausdorff-Pompeiu metric, which is a metric on the set of nonempty closed and bounded subsets of a metric space. This work was followed by several other papers forming the metric fixed point theory of multivalued functions, which remains an active area of research in mathematics to this day.

The concept of coupled fixed point was first introduced by Guo et al. [6]. Only after the publication of the work of Bhaskar et al. [4] in 2006 coupled fixed points and related results for both singlevalued and multivalued coupled mappings have appeared in large numbers. Some recent work in this area includes $[1, 2, 10, 11]$.

In the development of metric fixed point theory, contractive inequalities with rational expressions first appeared in the work of Dass and Gupta [5]. Subsequently, several results have appeared in this area for mappings satisfying rational inequalities.

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The use of a class of functions known as MT-functions to generalize Banach's theorem was made by Mizoguchi and Takahashi [8]. Subsequently, this class of functions was used to obtain a new class of contractions whose fixed point properties have been studied in several papers.

Bringing together the above research trends, we establish here a new coincidence point theorem in metric spaces. Several consequences and supporting examples are discussed.

In the following, we discuss some concepts from the set-valued analysis. Let (X, d) be a metric space. We use the following notations in our paper.

$N(X)$ = the collection of all nonempty subsets of X,

$CB(X)$ = the collection of all nonempty closed and bounded subsets of X, and

$C(X)$ = the collection of all nonempty compact subsets of X.

For $A, B \in CB(X)$, $H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}$, where $D(x, B) = \inf\{d(x, b) : b \in B\}$ and $D(y, A) = \inf\{d(y, a) : a \in A\}$. H is known as the Hausdorff-Pompeiu metric induced by d on $CB(X)$ [9]. Furthermore, if (X, d) is complete, then $(CB(X), H)$ is also complete. Nadler [9] established the following lemma.

Lemma 1.1 ([9]). *Let $A, B \in C(X)$ and $k \geq 1$. For every $x \in A$ there exists $y \in B$ such that $d(x, y) \leq k\ H(A, B)$.*

Definition 1.2 ([11]). Let X be a nonempty set and $F : X \times X \to N(X)$ be a coupled multivalued mapping. Then $(x, y) \in X \times X$ is called a coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$.

Definition 1.3 ([7]). Let X be a nonempty set, $F : X \times X \to C(X)$ be a coupled multivalued mapping, and $f : X \to X$ be a singlevalued mapping. Then $(x, y) \in X \times X$ is called a coupled coincidence point of F and f if $fx \in F(x, y)$ and $fy \in F(y, x)$.

We denote the set of all coupled coincidence points of F and f by $C(F, f)$. Note that if $(x, y) \in C(F, f)$, then (y, x) is also in $C(F, f)$.

Definition 1.4 ([1]). Let X be a nonempty set, $F : X \times X \to N(X)$ be a coupled multivalued mapping, and $f : X \to X$ be a singlevalued mapping. Then $(x, y) \in X \times X$ is called a coupled common fixed point of F and f if $x = fx \in F(x, y)$ and $y = fy \in F(y, x)$.

Definition 1.5 ([1]). Let X be a nonempty set, $F : X \times X \to N(X)$ be a coupled multivalued mapping, and $f : X \to X$ be a singlevalued mapping. The pair (F, f) is called w-compatible if $fF(x, y) \subseteq F(fx,\ fy)$, whenever $(x, y) \in C(F, f)$.

Definition 1.6 ([1]). Let X be a nonempty set, $F : X \times X \to N(X)$ be a coupled multivalued mapping, and $f : X \to X$ be a singlevalued mapping. Then the pair (F, f) is called weakly commutative at some point $(x, y) \in X \times X$ if $ffx \in F(fx, fy)$ and $ffy \in F(fy, fx)$.

We use the following class of functions in our theorems.

Definition 1.7 ([8]). A function $\varphi : [0, +\infty) \to [0, 1)$ is called an MT-function (or R-function) if it satisfies the Mizoguchi-Takahashi condition, that is, $\limsup_{s\to t^+} \varphi(s) < 1$ for all $t \in [0, +\infty)$.

Denote by Ψ the set of all MT-functions and Θ the family of all functions $\theta : [0, +\infty)^6 \to [0, +\infty)$ such that θ is nondecreasing and continuous in each coordinate; $\theta(t, t, t, t, t, t) \le t$ for all $t \ge 0$ and $\theta(t_1, t_2, t_3, t_4, t_5, t_6) = 0$ for $t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = 0$.

2. Main results

Let (X, d) be a metric space and $F : X \times X \to N(X)$ be a coupled multivalued mapping, $g : X \to X$ be a self mapping and $\theta \in \Theta$. In the following we define $N(x, y, u, v)$ and $M(x, y, u, v)$, which we use in our main theorem.

$$N(x, y, u, v) = \max\{d(gx, gu), d(gy, gv)\}$$

$$\begin{aligned} M(x, y, u, v) = \theta\Big(&d(gx, gu), d(gy, gv), \frac{[1 + D(gx, F(x, y))]D(gx, F(x, y))}{1 + d(gx, gu)},\\ &\frac{[1 + D(gu, F(u, v))]D(gu, F(x, y))}{1 + d(gx, gu)}, \frac{[1 + D(gy, F(y, x))]D(gy, F(y, x))}{1 + d(gy, gv)},\\ &\frac{[1 + D(gv, F(v, u))]D(gv, F(y, x))}{1 + d(gy, gv)}\Big) \end{aligned}$$

Theorem 2.1. *Let (X, d) be a complete metric space, $F : X\times X \to C(X)$ be a coupled multivalued mapping and $g : X \to X$ be a singlevalued mapping. Suppose there exists $\psi \in \Psi$ such that for all $x, y, u, v \in X$ the following inequality holds*

$$H(F(x, y), F(u, v)) \le \psi(N(x, y, u, v))M(x, y, u, v). \tag{1}$$

Suppose further that $F(x, y) \subseteq g(X)$ for all $(x, y) \in X \times X$ and $g(X)$ is a closed subset of X. Then F and g have a coupled coincidence point.

Moreover, F and g will have a coupled common fixed point if one of the following conditions holds:

(a) The pair (F, g) is w-compatible, and there exist $(x, y) \in C(F, g), u, v \in X$ such that $\lim_{n\to+\infty} g^n x = u$ and $\lim_{n\to+\infty} g^n y = v$, and also g is continuous at u and v.

(b) There exists $(x, y) \in C(F, g)$ such that the pair (F, g) is weakly commuting at (x, y) and gx, gy are fixed points of g.

(c) There exists $(x, y) \in C(F,\ g)$, $u,\ v \in X$ such that $\lim_{n\to+\infty} g^n u = x$ and $\lim_{n\to+\infty} g^n v = y$, and also g is continuous at x and y.

Proof. Let $x_0, y_0 \in X$ be arbitrary. Since $F(x, y) \subseteq g(X)$, for every $u \in F(x, y)$ there exists $p \in X$ such that $u = gp$. By the condition $F(x_0, y_0) \subseteq g(X)$ and $F(y_0,\ x_0) \subseteq g(X)$. Then we can choose $x_1, y_1 \in X$ such that $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0,\ x_0)$.

Since $F(x_0, y_0),\ F(x_1, y_1) \in C(X)$ and $gx_1 \in F(x_0, y_0)$, by Lemma 1.1 there exists $x_2 \in X$ such that $gx_2 \in F(x_1, y_1)$ and $d(gx_1, gx_2) \le H(F(x_0, y_0), F(x_1, y_1))$. Since $F(y_0, x_0), F(y_1, x_1) \in C(X)$ and $gy_1 \in F(y_0, x_0)$, by Lemma 1.1, there exists $y_2 \in X$ such that $gy_2 \in F(y_1, x_1)$ and $d(gy_1, gy_2) \le H(F(y_0, x_0), F(y_1,\ x_1))$. Again, $F(x_1, y_1), F(x_2, y_2) \in C(X)$ and $gx_2 \in F(x_1, y_1)$, by Lemma 1.1, there exists $x_3 \in X$ such that $gx_3 \in F(x_2, y_2)$ and $d(gx_2, gx_3) \le H(F(x_1, y_1), F(x_2, y_2))$. Similarly, since $F(y_1, x_1), F(y_2, x_2) \in C(X)$ and $gy_2 \in F(y_1, x_1)$, by Lemma 1.1, there exists $y_3 \in X$ such that $gy_3 \in F(y_2, x_2)$ and $d(gy_2, gy_3) \le H(F(y_1, x_1), F(y_2,\ x_2))$.

Continuing this process we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \ge 1$, $gx_n \in F(x_{n-1}, y_{n-1})$ and $gy_n \in F(y_{n-1}, x_{n-1})$,

$$d(gx_n, gx_{n+1}) \le H(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \tag{2}$$

and

$$d(gy_n, gy_{n+1}) \le H(F(y_{n-1}, x_{n-1}), F(y_n, x_n)).$$

We first prove that the sequences $\{d(gx_n, gx_{n+1})\}$ and $\{d(gy_n, gy_{n+1})\}$ are Cauchy sequences. Using (1) and (2), we have

$$\begin{aligned} d(gx_n,\ gx_{n+1}) &\le H(F(x_{n-1}, y_{n-1}),\ F(x_n, y_n)) \\ &\le \psi(N(x_{n-1}, y_{n-1},\ x_n, y_n))\ M(x_{n-1}, y_{n-1},\ x_n, y_n). \end{aligned} \tag{3}$$

By the definition of $N(x, y,\ u,\ v)$ and $M(x, y,\ u,\ v)$ and the property of θ, we have

$$N(x_{n-1}, y_{n-1}, x_n, y_n) = \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}$$

and

$$\begin{aligned} M(x_{n-1}, y_{n-1}, x_n, y_n) &= \theta(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \\ &\frac{[1 + D(gx_{n-1}, F(x_{n-1}, y_{n-1}))]D(gx_{n-1}, F(x_{n-1}, y_{n-1}))}{1 + d(gx_{n-1}, gx_n)}, \\ &\frac{[1 + D(gx_n, F(x_n, y_n))]D(gx_n, F(x_{n-1}, y_{n-1}))}{1 + d(gx_{n-1}, gx_n)}, \\ &\frac{[1 + D(gy_{n-1}, F(y_{n-1}, x_{n-1}))]D(gy_{n-1}, F(y_{n-1}, x_{n-1}))}{1 + d(gy_{n-1}, gy_n)}, \\ &\frac{[1 + D(gy_n, F(y_n, x_n))]D(gy_n, F(y_{n-1}, x_{n-1}))}{1 + d(gy_{n-1}, gy_n)}) \\ &\le \theta(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), \frac{[1 + d(gx_{n-1}, gx_n)]d(gx_{n-1}, gx_n)}{1 + d(gx_{n-1}, gx_n)}, \\ &\frac{[1 + d(gx_n, gx_{n+1})]d(gx_n, gx_n)}{1 + d(gx_{n-1}, gx_n)}, \frac{[1 + d(gy_{n-1}, gy_n)]d(gy_{n-1}, gy_n)}{1 + d(gy_{n-1}, gy_n)}, \\ &\frac{[1 + d(gy_n, gy_{n+1})]d(gy_n, gy_n)}{1 + d(gy_{n-1}, gy_n)}) \\ &\le \theta(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), 0, d(gy_{n-1}, gy_n), 0) \\ &\le \theta(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}, \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}, \\ &\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}, \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}, \\ &\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}, \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) \\ &\le \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}. \end{aligned}$$

From (3), we have

$$d(gx_n, gx_{n+1}) \leq \psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}. \tag{4}$$

Using the fact $\psi(t) \leq 1$, we have

$$d(gx_n, gx_{n+1}) \leq \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}. \tag{5}$$

Similarly, we can prove that

$$d(gy_n, gy_{n+1}) \leq \max\{d(gx_{n-1},\ gx_n), d(gy_{n-1}, gy_n)\}. \tag{6}$$

From (5) and (6), we have

$$\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} \leq \max\{d(gx_{n-1}, gx_n), d(gy_{n-1},\ gy_n)\}, \tag{7}$$

which implies that $\{\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\}$ is a nondecreasing sequence of real number. Hence there exists a real number $r \geq 0$ such that

$$\lim_{n \to +\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} = r. \tag{8}$$

Since $\psi \in \Psi$, we have $\limsup_{x \to r^+} \psi(x) < 1$ and $\psi(r) < 1$. Then there exists $\alpha \in [0,\ 1)$ and $\delta > 0$ such that

$$\psi(t) \leq \alpha, \text{ for all } t \in [r,\ r + \delta). \tag{9}$$

From (7) and (8), there exists $n_0 \in \mathbb{N}$ such that

$$r \leq \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\} \leq r + \delta, \text{ for all } n \geq n_0. \tag{10}$$

Thus for all $n \geq n_0$, it follows from (4), (9) and (10) that

$$d(gx_n, gx_{n+1}) \leq \alpha \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}. \tag{11}$$

Similarly, for all $n \geq n_0$, we have

$$d(gy_n, gy_{n+1}) \leq \alpha \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}. \tag{12}$$

Combining (11) and (12), we have for all $n \geq n_0$,

$$\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} \leq \alpha \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}. \tag{13}$$

Letting $n \to +\infty$ in the above inequality and using (8), we obtain $r \leq \alpha\ r$, where $\alpha \in [0, 1)$, which is a contradiction unless $r = 0$. Hence

$$\lim_{n \to +\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n,\ gy_{n+1})\} = 0.$$

If there exists a positive integer m such that $\max\{d(gx_m, gx_{m+1}), d(gy_m, gy_{m+1})\} = 0$, then $d(gx_m, gx_{m+1}) = 0$ and $d(gy_m, gy_{m+1}) = 0$, which imply that $gx_m = gx_{m+1} \in F(x_m, y_m)$ and $gy_m = gy_{m+1} \in F(y_m, x_m)$, that is, (x_m, y_m) is a coupled coincidence point of F and g. Hence we shall assume that

$$a_n = \max\{d(gx_n, gx_{n+1}), d(gy_n,\ gy_{n+1})\} \neq 0, \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

From (13) and (14), we have $0 < a_n \leq \alpha\ a_{n-1}$, for all $n \geq n_0$. From here, we have

$$\sum_{n=0}^{+\infty} a_n \leq \sum_{n=0}^{n_0-1} a_n + \sum_{n=n_0}^{+\infty} a_n \leq \sum_{n=0}^{n_0-1} a_n + a_{n_0-1} \sum_{k=1}^{+\infty} \alpha^k. \tag{15}$$

Since $\alpha \in [0,\ 1)$, $\sum_{k=1}^{+\infty} \alpha^k$ is convergent and hence $\sum_{n=0}^{+\infty} a_n$ is convergent, that is,

$\sum_{n=0}^{\infty} a_n < +\infty$. Thus by (15), we have

$$\sum \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} < +\infty. \tag{16}$$

This implies that $\{gx_n\}_{n=0}^{+\infty}$ and $\{gy_n\}_{n=0}^{+\infty}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is closed in X and X is complete, there exist $x, y \in X$ such that

$$\lim_{n\to+\infty} gx_n = gx \quad \text{and} \quad \lim_{n\to+\infty} gy_n = gy. \tag{17}$$

Since $gx_{n+1} \in F(x_n, y_n)$ and $gy_{n+1} \in F(y_n,\ x_n)$, using (1) and the property of ψ, we have

$$\begin{aligned} D(gx_{n+1}, F(x,y)) &\le H(F(x_n, y_n), F(x,y)) \\ &\le \psi(N(x_n, y_n, x, y))M(x_n, y_n, x, y) \le M(x_n, y_n, x, y). \end{aligned} \tag{18}$$

Now,

$$\begin{aligned} M(x_n, y_n, x, y) = {} & \theta(d(gx_n, gx), d(gy_n, gy), \frac{[1 + D(gx_n, F(x_n, y_n))]D(gx_n, F(x_n, y_n))}{1 + d(gx_n, gx)}, \\ & \frac{[1 + D(gx, F(x,y))]D(gx, F(x_n, y_n))}{1 + d(gx_n, gx)}, \\ & \frac{[1 + D(gy_n, F(y_n, x_n))]D(gy_n, F(y_n, x_n))}{1 + d(gy_n, gy)}, \\ & \frac{[1 + D(gy, F(y,x))]D(gy, F(y_n, x_n))}{1 + d(gy_n, gy)}) \\ \le {} & \theta(d(gx_n, gx), d(gy_n, gy), \frac{[1 + d(gx_n, gx_{n+1})]d(gx_n, gx_{n+1})}{1 + d(gx_n, gx)}, \\ & \frac{[1 + D(gx, F(x,y))]d(gx, gx_{n+1})}{1 + d(gx_n, gx)}, \frac{[1 + d(gy_n, gy_{n+1})]d(gy_n, gy_{n+1})}{1 + d(gy_n, gy)}, \\ & \frac{[1 + D(gy,\ F(y,\ x))]d(gy,\ gy_{n+1})}{1 + d(gy_n,\ gy)}). \end{aligned}$$

Using the property of θ, we have

$$\lim_{n\to+\infty} M(x_n, y_n, x, y) \le \theta(0, 0, 0, 0, 0, 0) = 0. \tag{19}$$

Taking limit as $n \to +\infty$ in (18) and using (19), we obtain $D(gx,\ F(x,y)) \le 0$, which implies that $D(gx,\ F(x,y)) = 0$, i.e., $gx \in \overline{F(x,y)} = F(x,y)$. Similarly, we can prove that $gy \in F(y,\ x)$. Therefore, we conclude that (x,y) is a coupled coincidence point of F and g and hence $C(F,\ g)$ is nonempty.

Suppose that the condition (a) holds. Therefore, there exist $(x,y) \in C(F,g)$ and $u, v \in X$, such that

$$\lim_{n\to+\infty} g^n x = u \quad \text{and} \quad \lim_{n\to+\infty} g^n y = v, \tag{20}$$

and also g is continuous at u and v.

As g is continuous at u and v, we have $u = \lim_{n\to+\infty} g^{n+1}x = \lim_{n\to+\infty} g(g^n x) = gu$ and $v = \lim_{n\to+\infty} g^{n+1}y = \lim_{n\to+\infty} g(g^n y) = gv$. Here u and v are fixed points of g. Now $(x,y) \in C(F,g)$ implies that $gx \in F(x,y)$ and $gy \in F(y,x)$. Applying the w-compatibility of the pair (F, g), we have $ggx = g^2x \in gF(x, y) \subseteq F(gx, gy)$. Similarly, $ggy = g^2y \in gF(y, x) \subseteq F(gy, gx)$. So $(gx, gy) \in C(F, g)$. Continuing this process, we can show by the mathematical induction that $(g^{n-1}x, g^{n-1}y) \in C(F, g)$, for all $n \geq 1$. Hence $g(g^{n-1}x) \in F(g^{n-1}x, g^{n-1}y)$ and $g(g^{n-1}y) \in F(g^{n-1}y, g^{n-1}x)$, that is, $g^nx \in F(g^{n-1}x, g^{n-1}y)$ and $g^ny \in F(g^{n-1}y, g^{n-1}x)$, for all $n \geq 1$.

By using (1) and the property of ψ, we obtain

$$\begin{aligned} &D(g^nx, F(u, v)) \leq H(F(g^{n-1}x, g^{n-1}y), F(u,\ v)) \\ &\leq \psi(N(g^{n-1}x, g^{n-1}y, u, v))M(g^{n-1}x, g^{n-1}y, u, v) < M(g^{n-1}x, g^{n-1}y, u, v), \end{aligned} \tag{21}$$

where,

$$\begin{aligned} M(g^{n-1}x, g^{n-1}y, u, v) = &\theta(d(g^nx, gu), d(g^ny, gv), \\ &\frac{[1 + D(g^nx, F(g^{n-1}x, g^{n-1}y))]D(g^nx, F(g^{n-1}x, g^{n-1}y))}{1 + d(g^nx, gu)}, \\ &\frac{[1 + D(gu, F(u, v))]D(gu, F(g^{n-1}x, g^{n-1}y))}{1 + d(g^nx, gu)}, \\ &\frac{[1 + D(g^ny, F(g^{n-1}y, g^{n-1}x))]D(g^ny, F(g^{n-1}y, g^{n-1}x))}{1 + d(g^ny, gv)}, \\ &\frac{[1 + D(gv, F(v, u))]D(gv, F(g^{n-1}y, g^{n-1}x))}{1 + d(g^ny, gv)}) \\ \leq &\theta(d(g^nx, u), d(g^ny, v), \frac{[1 + d(g^nx, g^nx)]d(g^nx, g^nx)}{1 + d(g^nx, u)}, \\ &\frac{[1 + D(u, F(u, v))]d(u, g^nx)}{1 + d(g^nx, u)}, \frac{[1 + d(g^ny, g^ny))]d(g^ny, g^ny)}{1 + d(g^ny, v)}, \\ &\frac{[1 + D(v, F(v, u))]d(v, g^ny)}{1 + d(g^ny, v)}) \\ = &\theta\Big(d(g^nx, u), d(g^ny, v), 0, \frac{[1 + D(u, F(u, v))]d(u, g^nx)}{1 + d(g^nx, u)}, \\ &0, \frac{[1 + D(v, F(v, u))]d(v, g^ny)}{1 + d(g^ny, v)}\Big). \end{aligned}$$

Using (20) and the property of θ, we have $\lim_{n\to+\infty} M(g^{n-1}x, g^{n-1}y, u, v) \leq \theta(0, 0, 0, 0, 0, 0) = 0$. Taking limit as $n \to +\infty$ in (21), we have $D(u, F(u, v)) \leq 0$ which implies that $D(u, F(u, v)) = 0$. Now, $D(u, F(u, v)) = 0$ implies that $u \in \overline{F(u, v)} = F(u, v)$. Similarly, we can prove that $v \in F(v,\ u)$. Therefore, (u, v) is a coupled common fixed point of F and g.

Suppose that the condition (b) holds. Therefore, there exists $(x, y) \in C(F, g)$ such that the pair (F, g) is weakly commuting at (x, y) and also gx and gy are fixed points of g. As gx and gy are fixed points of g, we have $gx = ggx = g^2x$ and $gy = ggy = g^2y$. As (F, g) is weakly commuting at (x, y), we have $gx = g^2x = ggx \in F(gx, gy)$ and $gy = g^2y = ggy \in F(gy, gx)$, i.e., (gx, gy) is a coupled common fixed point of F and g.

Suppose that the condition (c) holds. Therefore, there exists $(x, y) \in C(F, g)$ and

$u, v \in X$ such that $\lim_{n\to+\infty} g^n u = x$ and $\lim_{n\to+\infty} g^n v = y$, and also g is continuous at x and y. Since g is continuous at x and y, we have $x = \lim_{n\to+\infty} g^{n+1}u = \lim_{n\to+\infty} g(g^n u) = gx$ and $y = \lim_{n\to+\infty} g^{n+1}v = \lim_{n\to+\infty} g(g^n v) = gy$. Then x and y are fixed points of g. Therefore, we have $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$, i.e., (x, y) is a coupled common fixed point of F and g. $\square$

3. Consequences and examples

In Theorem 2.1, consider $\theta(x_1, x_2, x_3, x_4, x_5, x_6) = \max\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\psi(t) = k$, where $k \in [0, 1)$; we obtain the following corollary.

Corollary 3.1. *Let (X, d) be a complete metric space, $F : X \times X \to C(X)$ be a coupled multivalued mapping and $g : X \to X$ be a singlevalued mapping. Suppose there exists $k \in [0, 1)$ such that for all $x, y, u, v \in X$ the following inequality holds*

$$\begin{aligned} H(F(x,y),F(u,v)) \leq k \max\{&d(gx,gu), d(gy,gv), \\ &\frac{[1 + D(gx,F(x,y))]D(gx,F(x,y))}{1 + d(gx,gu)}, \frac{[1 + D(gu,F(u,v))]D(gu,F(x,y))}{1 + d(gx,gu)}, \\ &\frac{[1 + D(gy,F(y,x))]D(gy,F(y,x))}{1 + d(gy,gv)}, \frac{[1 + D(gv,F(v,u))]D(gv,F(y,x))}{1 + d(gy,gv)}\}. \end{aligned}$$

Suppose further that $F(x, y) \subseteq g(X)$ for all $(x, y) \in X \times X$ and $g(X)$ is a closed subset of X. Then F and g have a coupled coincidence point. Moreover, F and g will have a coupled common fixed point under each of the conditions as mentioned in Theorem 2.1.

In Theorem 2.1, given $\theta(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{2}(x_1 + x_2)$ and $\psi(t) = k$, where $k \in [0, 1)$, we have the following corollary.

Corollary 3.2. *Let (X, d) be a complete metric space, $F : X \times X \to C(X)$ be a coupled multivalued mapping and $g : X \to X$ be a singlevalued mapping. Suppose there exists $k \in [0, 1)$ such that for all $x, y, u, v \in X$ the following inequality holds $H(F(x, y), F(u, v)) \leq \frac{k}{2}\,[d(gx, gu) + d(gy, gv)]$. Suppose that $F(x, y) \subseteq g(X)$ for all $(x, y) \in X \times X$ and $g(X)$ is a closed subset of X. Then F and g have a coupled coincidence point. Moreover, F and g will have a coupled common fixed point under each of the conditions as mentioned in Theorem 2.1.*

In Theorem 2.1, considering $\theta(x_1, x_2, x_3, x_4, x_5, x_6) = \max\{x_1, x_2\}$, we have the following corollary.

Corollary 3.3. *Let (X, d) be a complete metric space, $F : X \times X \to C(X)$ be a coupled multivalued mapping and $g : X \to X$ be a singlevalued mapping. Suppose there exists $\psi \in \Psi$ such that for all $x, y, u, v \in X$ the following inequality holds*

$$H(F(x, y), F(u, v)) \leq \psi(\max\{d(gx, gu), d(gy, gv)\}) \max\{d(gx, gu), d(gy, gv)\}.$$

Suppose further that $F(x,y) \subseteq g(X)$ for all $(x,y) \in X \times X$ and $g(X)$ is a closed subset of X. Then F and g have a coupled coincidence point. Moreover, F and g will have a coupled common fixed point under each of the conditions as mentioned in Theorem 2.1.

EXAMPLE 3.4. Let $X = [0,1]$ and d be the usual metric on X. Let $F : X \times X \to C(X)$ and $g : X \to X$ be defined respectively as follows: $F(x,y) = \left[0, \frac{x^2}{4}\right]$ for $x, y \in X$ and $gx = \frac{x^2}{2}$ for $x \in X$. Let $\theta : [0,\ +\infty)^6 \to [0,\ +\infty)$ be defined as follows:

$$\theta(x_1, x_2, x_3, x_4, x_5, x_6) = \max\{x_1, x_2, x_3, x_4, x_5, x_6\}.$$

Let $\psi : [0, +\infty) \to [0, 1)$ be defined as follows:

$$\psi(t) = \begin{cases} \dfrac{1}{2} + \dfrac{t^2}{4}, & \text{if } t \text{ is rational and } 0 \leq t \leq 1, \\ \\ \dfrac{3}{4}, & \text{otherwise.} \end{cases}$$

All the conditions of Theorem 2.1 are satisfied and $(0,0)$ is a coupled coincidence point and also a coupled common fixed point of F and g.

EXAMPLE 3.5. Take the metric space (X, d) as considered in Example 3.4. Let $F : X \times X \to C(X)$ and $g : X \to X$ be defined respectively as follows:

$$F(x,\ y) = \left[0,\ \frac{x+y}{24}\right] \quad \text{and} \quad gx = \frac{x}{2} \quad \text{for } x \in X.$$

Let $\theta : [0, +\infty)^6 \to [0, +\infty)$ be defined as follows:

$$\theta(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}{6}.$$

We take the function $\psi \in \Psi$ as taken in Example 3.4. All the conditions of the Theorem 2.1 are satisfied and $(0,0)$ is a coupled coincidence point and also a coupled common fixed point of F and g.

EXAMPLE 3.6. Take the metric space (X, d) as considered in Example 3.4. Let $F : X \times X \to C(X)$ and $g : X \to X$ be defined respectively as follows:

$$F(x,y) = \left\{\frac{1}{2}\right\} \quad \text{for} \quad x, y \in X \quad \text{and} \quad gx = 1 - \frac{x}{2} \quad \text{for} \quad x \in X.$$

We take the functions $\theta \in \Theta$ and $\psi \in \Psi$ as taken in Example 3.5. Except the conditions (a), (b) and (c), all the conditions of the Theorem 2.1 are satisfied. Here $(1,1)$ is a coupled coincidence point of F and g but they have no coupled common fixed point.

4. Conclusions

The main result of this paper is a coupled coincidence point theorem of a singlevalued and a coupled multivalued mapping. The result is derived on a metric space and obtained by combining several ideas. For a recent survey of various aspects of metric fixed point theory, we refer to [3]. Hybrid results may continue to be obtained in fixed point theory and related studies, combining several individual research trends, as in the present case. Efforts in this direction should generate interest.

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