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SLANT LIGHTLIKE SUBMANIFOLDS OF GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we introduce the notion of slant lightlike submanifold of a golden semi-Riemannian manifold and provide a characterization theorem with some non-trivial examples of such submanifolds. We find necessary and sufficient conditions for integrability of distributions. Finally, we study curvature properties of slant lightlike submanifolds of golden semi-Riemannian manifolds.

1. Introduction

A submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called a lightlike submanifold if the induced metric g on it is degenerate, i.e. there exists a non zero vector field $Y \in \Gamma(TM)$ such that $g(Y, Z) = 0$, for all $Z \in \Gamma(TM)$. In [3], Duggal and Bejancu introduced the notion of lightlike submanifolds of a semi-Riemannian manifold. The geometry of slant lightlike submanifolds of indefinite Hermitian manifolds has been studied in [14]. Many authors have studied on lightlike submanifolds in various spaces [4, 15, 16]. In [16], the authors found some equivalent conditions for integrability of distributions. Golden proportion ψ is the real positive root of the equation $x^2 - x - 1 = 0$ (thus $\psi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$) and was described by Kepler (1571-1630). Inspired by the Golden proportion, Crasmareanu and Hretcanu defined golden structure $\tilde{P}$ which is a tensor field satisfying $\tilde{P}^2 - \tilde{P} - I = 0$ on a manifold $\overline{M}$ [2].

A Riemannian manifold $\overline{M}$ with a golden structure $\tilde{P}$ is called a golden Riemannian manifold and was studied in [2, 6]. In [6], the authors studied invariant submanifolds of a golden Riemannian manifold. In [5], the authors investigated the integrability of golden Riemannian structures. In [10], Poyraz and Yasar studied lightlike hypersurfaces of a golden semi-Riemannian manifold and proved that there

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is no radical anti-invariant lightlike hypersurface of a golden semi-Riemannian manifold. In [10], they also studied screen semi-invariant and screen conformal screen semi-invariant lightlike hypersurfaces of a golden semi-Riemannian manifold. In [11], Poyraz and Yasar studied lightlike submanifolds of a golden semi-Riemannian manifold and proved that there is no radical anti-invariant lightlike submanifold of a golden semi-Riemannian manifolds. In [1], Acet introduced the notion of screen pseudo slant lightlike submanifolds of a golden semi-Riemannian manifold and also found some equivalent conditions for integrability of distributions. In [9], Poyraz introduced the notion of golden GCR-lightlike submanifold of a golden semi-Riemannian manifold and found some equivalent conditions for integrability and totally geodesic foliation of distributions. In [12], Poyraz introduced the notion of screen semi-invariant lightlike submanifolds of a golden semi-Riemannian manifold and found equivalant conditions for integrability of distributions. He proved some results for totally umbilical screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds.

The purpose of this paper is to study slant lightlike submanifolds of a golden semi-Riemannian manifold. The paper is arranged as follows. In Section 2, some definitions and basic results about lightlike submanifolds and golden semi-Riemannian manifolds are given. In Section 3, we study slant lightlike submanifolds of a golden semi-Riemannian manifold, with examples and investigate the integrability of distributions. In Section 4, we study curvature invariant and irrotational lightlike submanifolds of a golden semi-Riemannian manifold.

2. Preliminaries

Let $\overline{M}$ be a differentiable manifold. If a (1, 1) type tensor field $\tilde{P}$ on $\overline{M}$ satisfies the following equation $\tilde{P}^2 = \tilde{P} + I$, then $\tilde{P}$ is called a golden structure on $\overline{M}$, where I is the identity transformation. Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\overline{M}$. If $\tilde{P}$ satisfies the equation

$$\overline{g}(\tilde{P}U, W) = \overline{g}(U, \tilde{P}W), \tag{1}$$

for all $U, W \in \Gamma(T\overline{M})$, then $(\overline{M}, \overline{g}, \tilde{P})$ is called a golden semi-Riemannian manifold [13]. Also, if $\tilde{P}$ is integrable then we have [2]

$$\overline{\nabla}_U \tilde{P}W = \tilde{P}\overline{\nabla}_U W. \tag{2}$$

Now, from (1), we get

$$\overline{g}(\tilde{P}U, \tilde{P}W) = \overline{g}(\tilde{P}U, W) + \overline{g}(U, W), \tag{3}$$

for all $U, W \in \Gamma(T\overline{M})$.

We denote real space forms with constant sectional curvatures c_p and c_q by M_p and M_q, respectively. Then similar calculations of semi-Riemannian product real space form [17], the Riemannian curvature tensor $\overline{R}$ of a locally golden product space form $(\overline{M} = M_p(c_p) \times M_q(c_q), \overline{g}, \tilde{P})$ is given as follows

$$\overline{R}(X,Y)Z =(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}})\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y$$

$$+\overline{g}(\tilde{P}Y,Z)\tilde{P}X-\overline{g}(\tilde{P}X,Z)\tilde{P}Y\} \tag{4}$$
$$+(-\frac{(1-\psi)c_p+\psi c_q}{4})\{\overline{g}(\tilde{P}Y,Z)X-\overline{g}(\tilde{P}X,Z)Y$$
$$+\overline{g}(Y,Z)\tilde{P}X-\overline{g}(X,Z)\tilde{P}Y\},$$

where $\psi=\frac{1+\sqrt{5}}{2}\approx 1.618\ldots)$ is Golden proportion and $X,Y,Z\in\Gamma(T\overline{M})$.

A submanifold (M^m,g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n},\overline{g})$ is called a lightlike submanifold [3] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1\leq r\leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is $TM=RadTM\oplus_{orth}S(TM)$.

Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i,N_j)=\delta_{ij}$ and $\overline{g}(N_i,N_j)=0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM)=ltr(TM)\oplus_{orth}S(TM^{\perp}),\quad T\overline{M}|_M=TM\oplus tr(TM),$$
$$T\overline{M}|_M=S(TM)\oplus_{orth}[RadTM\oplus ltr(TM)]\oplus_{orth}S(TM^{\perp}).$$

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_XY=\nabla_XY+h(X,Y),\quad \overline{\nabla}_XV=-A_VX+\nabla^t_XV, \tag{5}$$

for all $X,Y\in\Gamma(TM)$ and $V\in\Gamma(tr(TM))$, where ∇_XY,A_VX belong to $\Gamma(TM)$ and $h(X,Y),\nabla^t_XV$ belong to $\Gamma(tr(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (5), for any $X,Y\in\Gamma(TM)$, $N\in\Gamma(ltr(TM))$ and $W\in\Gamma(S(TM^{\perp}))$, we have

$$\overline{\nabla}_XY=\nabla_XY+h^l(X,Y)+h^s(X,Y), \tag{6}$$
$$\overline{\nabla}_XN=-A_NX+\nabla^l_XN+D^s(X,N), \tag{7}$$
$$\overline{\nabla}_XW=-A_WX+\nabla^s_XW+D^l(X,W), \tag{8}$$

where $h^l(X,Y)=L(h(X,Y))$, $h^s(X,Y)=S(h(X,Y))$, $D^l(X,W)=L(\nabla^t_XW)$, $D^s(X,N)=S(\nabla^t_XN)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M, respectively.

Also, by using (5), (6)-(8) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W)+\overline{g}(Y,D^l(X,W))=g(A_WX,Y),$$
$$\overline{g}(D^s(X,N),W)=\overline{g}(N,A_WX).$$

Now, denote the projection of TM on $S(TM)$ by $\tilde{S}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X,Y\in\Gamma(TM)$ and $\xi\in$

$\Gamma(RadTM)$, we have

$$\nabla_X \tilde{S}Y = \nabla^*_X \tilde{S}Y + h^*(X, \tilde{S}Y), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi.$$

By using above equations, we obtain $\overline{g}(h^l(X, \tilde{S}Y), \xi) = g(A^*_\xi X, \tilde{S}Y)$, $\overline{g}(h^*(X, \tilde{S}Y), N) = g(A_N X, \tilde{S}Y)$, $\overline{g}(h^l(X, \xi), \xi) = 0$, $A^*_\xi \xi = 0$.

It is important to note that in general ∇ is not a metric connection on M. Since $\overline{\nabla}$ is a metric connection on $\overline{M}$, by using (6), we get $(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y)$, for all $X, Y, Z \in \Gamma(T\overline{M})$.

Definition 2.1 ([8]). A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be irrotational if $\overline{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.

Gauss equation for lightlike submanifold of semi-Riemannian manifold is given in [3]:

$$\begin{aligned}\overline{R}(X, Y)Z =& R(X, Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ &+ (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) \\ &+ (\nabla_X h^s)(Y, Z) - (\nabla_Y h^s)(X, Z) + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)),\end{aligned} \tag{9}$$

for any $X, Y, Z \in \Gamma(TM)$.

3. Slant lightlike submanifolds

In this section, we study slant lightlike submanifolds of golden semi-Riemannian manifolds. First, we give the following lemmas which will be used to define slant notion on the screen distribution.

Lemma 3.1. *Let (M, g) be a q-lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$ of index $2q$. Suppose that $\tilde{P}RadTM$ is a distribution on M such that $RadTM \cap \tilde{P}RadTM = \{0\}$. Then $\tilde{P}ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\tilde{P}RadTM \cap \tilde{P}ltr(TM) = \{0\}$.*

Proof. Since, by hypothesis, $\tilde{P}RadTM$ is a distribution on M such that $\tilde{P}RadTM \cap RadTM = 0$, we have $\tilde{P}RadTM \subset S(TM)$. Now we claim that $ltr(TM)$ is not invariant with respect to $\tilde{P}$. Let us suppose the contrary. Choose $\xi \in \Gamma(RadTM)$ and $N \in \Gamma ltr(TM)$ such that $\overline{g}(N, \xi) = 1$. Then from (3), we have $1 = \overline{g}(\xi, N) = \overline{g}(\tilde{P}\xi, \tilde{P}N) - \overline{g}(\tilde{P}\xi, N) = 0$, due to $\tilde{P}\xi \in \Gamma S(TM)$ and $\tilde{P}N \in \Gamma ltr(TM)$. This is a contradiction, so $ltr(TM)$ is not invariant with respect to $\tilde{P}$. Also $\tilde{P}N$ does not belong to $S(TM^\perp)$, since $S(TM^\perp)$ is orthogonal to $S(TM)$, $\overline{g}(\tilde{P}N, \tilde{P}\xi)$ must be zero, but from (3) we have $\overline{g}(\tilde{P}N, \tilde{P}\xi) = \overline{g}(\tilde{P}\xi, N) + \overline{g}(N, \xi) \neq 0$, for some $\xi \in \Gamma RadTM$, this is again a contradiction. Thus we conclude that $\tilde{P}ltr(TM)$ is a distribution on M. Moreover, $\tilde{P}N$ does not belong to $Rad(TM)$. Indeed, if $\tilde{P}N \in \Gamma Rad(TM)$, we would have $\tilde{P}^2 N = \tilde{P}N + N \in \Gamma(\tilde{P}RadTM)$, but this is impossible. Finally, let $\tilde{P}N \in \Gamma(\tilde{P}RadTM)$, we obtain $\tilde{P}^2 N = \tilde{P}N + N \in \Gamma(\tilde{P}RadTM + RadTM)$, this is not possible. Hence $\tilde{P}N$ does not belong to $\tilde{P}RadTM$. Thus we conclude that $\tilde{P}ltr(TM) \subset S(TM)$ and $\tilde{P}RadTM \cap \tilde{P}ltr(TM) = \{0\}$. □

Lemma 3.2. *Let (M, g) be a q-lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$ of index $2q$. Suppose $\tilde{P}RadTM$ is a distribution on M such that $RadTM \cap \tilde{P}RadTM = \{0\}$. Then any complementary distribution to $\tilde{P}RadTM \oplus \tilde{P}ltr(TM)$ in $S(TM)$ is Riemannian.*

Proof. Let M be an m-dimensional q-lightlike submanifold of an $(m+n)$-dimensional golden semi-Riemannian manifold $\overline{M}$ of index $2q$. From Lemma 3.1, we have $\tilde{P}RadTM$ $\cap\tilde{P}ltr(TM) = \{0\}$ and $\tilde{P}RadTM \oplus \tilde{P}ltr(TM) \subset S(TM)$. We denote the complementary distribution to $\tilde{P}RadTM \oplus \tilde{P}ltr(TM)$ in $S(TM)$ by D. Then we have a local orthonormal frame of fields on $\overline{M}$ along M $\{\xi_i, N_i, \tilde{P}\xi_i, \tilde{P}N_i, X_\alpha, W_a\}$, $i \in \{1, 2, \ldots, q\}$, $\alpha \in \{3q+1, \ldots, m\}, a \in \{q+1, \ldots, n\}$, where $\{\xi_i\}$ and $\{N_i\}$ are lightlike bases of $RadTM$ and $ltrTM$, respectively and $\{X_\alpha\}$ and $\{W_a\}$ are orthonormal bases of D and $S(TM^\perp)$, respectively.

Now, from the bases $\{\xi_1, \ldots, \xi_q, N_1, \ldots, N_q, \tilde{P}\xi_1, \ldots, \tilde{P}\xi_q, \tilde{P}N_1, \ldots, \tilde{P}N_q\}$ of $RadTM \oplus ltrTM \oplus \tilde{P}RadTM \oplus \tilde{P}ltr(TM)$, we can construct an orthonormal bases $\{U_1, \ldots, U_{2q}, V_1, \ldots, V_{2q}\}$ as follows:

$$\begin{aligned}
U_1 =& \frac{1}{\sqrt{2}}(\xi_1 + N_1),\ U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1)\\
U_3 =& \frac{1}{\sqrt{2}}(\xi_2 + N_2),\ U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2)\\
&\ldots,\ \ldots\\
&\ldots,\ \ldots\\
U_{2q-1} =& \frac{1}{\sqrt{2}}(\xi_q + N_q),\ U_{2q} = \frac{1}{\sqrt{2}}(\xi_q - N_q)\\
V_1 =& \frac{1}{\sqrt{2}}(\tilde{P}\xi_1 + \tilde{P}N_1),\ V_2 = \frac{1}{\sqrt{2}}(\tilde{P}\xi_1 - \tilde{P}N_1)\\
V_3 =& \frac{1}{\sqrt{2}}(\tilde{P}\xi_2 + \tilde{P}N_2),\ V_4 = \frac{1}{\sqrt{2}}(\tilde{P}\xi_2 - \tilde{P}N_2)\\
&\ldots,\ \ldots\\
&\ldots,\ \ldots\\
V_{2q-1} =& \frac{1}{\sqrt{2}}(\tilde{P}\xi_q + \tilde{P}N_q),\ V_{2q} = \frac{1}{\sqrt{2}}(\tilde{P}\xi_q - \tilde{P}N_q).
\end{aligned}$$

Hence, span$\{\xi_i, N_i, \tilde{P}\xi_i, \tilde{P}N_i\}$ is a non-degenerate space of constant index $2q$. Thus we conclude that $RadTM \oplus ltr(TM) \oplus \tilde{P}RadTM \oplus \tilde{P}ltr(TM)$ is non-degenerate and of constant index $2q$ on $\overline{M}$. Since index$(T\overline{M})$ = index$(RadTM \oplus ltr(TM) \oplus \tilde{P}RadTM \oplus \tilde{P}ltr(TM))$ + index $(D \oplus_{orth} S(TM^\perp))$, we have $2q = 2q$ + index $(D + S(TM^\perp))$. Thus, $D \oplus_{orth} S(TM^\perp)$ is Riemannian, i.e., index $(D \oplus_{orth} S(TM^\perp)) = 0$. Hence D is Riemannian. □

Definition 3.3. Let (M, g) be a q-lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$ of index $2q$ such that $2q < \dim(M)$. Then we say that M is a slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:
(i) $\tilde{P}RadTM$ is a distribution on M such that $RadTM \cap \tilde{P}RadTM = \{0\}$,

(ii) there exists a non-degenerate orthogonal complementary distribution D on M such that $S(TM) = (\tilde{P}RadTM \oplus \tilde{P}ltr(TM)) \oplus_{orth} D$,

(iii) the distribution D is slant with angle $\theta(\neq 0)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D)_x$, the angle θ between $\tilde{P}X$ and the vector subspace $(D)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D)_x$.

This constant angle θ is called the slant angle of distribution D. A slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TM = RadTM \oplus_{orth} (\tilde{P}RadTM \oplus \tilde{P}ltr(TM)) \oplus_{orth} D. \tag{10}$$

Now, for any vector field X tangent to M, we put

$$\tilde{P}X = PX + FX, \tag{11}$$

where PX and FX are tangential and transversal parts of $\tilde{P}X$ respectively. We denote the projections on $RadTM$, $\tilde{P}RadTM$, $\tilde{P}ltr(TM)$ and D in TM by P_1, P_2, P_3 and P_4 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2 respectively. Thus, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + P_4X. \tag{12}$$

Now applying $\tilde{P}$ to (12), we have

$$\tilde{P}X = \tilde{P}P_1X + \tilde{P}P_2X + \tilde{P}P_3X + \tilde{P}P_4X, \tag{13}$$

which gives

$$\tilde{P}X = \tilde{P}P_1X + \tilde{P}P_2X + \tilde{P}P_3X + PP_4X + FP_4X, \tag{14}$$

where $\tilde{P}P_2X = K_1\tilde{P}P_2X + K_2\tilde{P}P_2X$, $\tilde{P}P_3X = L_1\tilde{P}P_3X + L_2\tilde{P}P_3X$ and PP_4X (resp. FP_4X) denotes the tangential (resp. transversal) component of $\tilde{P}P_4X$. Thus we get $\tilde{P}P_1X \in \Gamma(\tilde{P}RadTM), K_1\tilde{P}P_2X \in \Gamma(RadTM), K_2\tilde{P}P_2X \in \Gamma(\tilde{P}RadTM), L_1\tilde{P}P_3X \in \Gamma(ltr(TM)), L_2\tilde{P}P_3X \in \Gamma(\tilde{P}ltr(TM)), PP_4X \in \Gamma(D)$ and $FP_4X \in \Gamma(S(TM^\perp))$. Also, for any $W \in \Gamma(tr(TM))$, we have $W = Q_1W + Q_2W$. Applying $\tilde{P}$ to it, we obtain $\tilde{P}W = \tilde{P}Q_1W + \tilde{P}Q_2W$, which gives

$$\tilde{P}W = \tilde{P}Q_1W + BQ_2W + CQ_2W, \tag{15}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of $\tilde{P}Q_2W$. Thus we get $\tilde{P}Q_1W \in \Gamma(\tilde{P}ltr(TM))$, $BQ_2W \in \Gamma(D)$ and $CQ_2W \in \Gamma(S(TM^\perp))$.

Now, by using (2), (6)-(8), (12)-(14), (15) and identifying the components on $RadTM$, $\tilde{P}RadTM$, $\tilde{P}ltr(TM)$, D, $ltr(TM)$ and $S(TM^\perp)$, we obtain

$$\begin{aligned} &P_1(\nabla_X\tilde{P}P_1Y) + P_1(\nabla_X\tilde{P}P_2Y) + P_1(\nabla_XL_2\tilde{P}P_3Y) + P_1(\nabla_XPP_4Y) \\ &= P_1(A_{FP_4Y}X) + P_1(A_{L_1\tilde{P}P_3Y}X) + K_1\tilde{P}P_2\nabla_XY, \end{aligned} \tag{16}$$

$$\begin{aligned}&P_2(\nabla_X\tilde{P}P_1Y)+P_2(\nabla_X\tilde{P}P_2Y)+P_2(\nabla_XL_2\tilde{P}P_3Y)+P_2(\nabla_XPP_4Y)\\&=P_2(A_{FP_4Y}X)+P_2(A_{L_1\tilde{P}P_3Y}X)+K_2\tilde{P}P_2\nabla_XY+\tilde{P}P_1\nabla_XY,\end{aligned}\tag{17}$$

$$\begin{aligned}&P_3(\nabla_X\tilde{P}P_1Y)+P_3(\nabla_X\tilde{P}P_2Y)+P_3(\nabla_XL_2\tilde{P}P_3Y)+P_3(\nabla_XPP_4Y)\\&=P_3(A_{FP_4Y}X)+P_3(A_{L_1\tilde{P}P_3Y}X)+L_2\tilde{P}P_3\nabla_XY+\tilde{P}h^l(X,Y),\end{aligned}\tag{18}$$

$$\begin{aligned}&P_4(\nabla_X\tilde{P}P_1Y)+P_4(\nabla_X\tilde{P}P_2Y)+P_4(\nabla_XL_2\tilde{P}P_3Y)+P_4(\nabla_XPP_4Y)\\&=P_4(A_{FP_4Y}X)+P_4(A_{L_1\tilde{P}P_3Y}X)+PP_4\nabla_XY+BQ_2h^s(X,Y),\end{aligned}\tag{19}$$

$$\begin{aligned}&h^l(X,\tilde{P}P_1Y)+h^l(X,K_1\tilde{P}P_2Y)+h^l(X,K_2\tilde{P}P_2Y)+h^l(X,PP_4Y)\\&+h^l(X,L_2\tilde{P}P_3Y)=L_1\tilde{P}P_3\nabla_XY-\nabla^l_sL_1\tilde{P}P_3Y-D^l(X,FP_4Y),\end{aligned}\tag{20}$$

$$\begin{aligned}&h^s(X,\tilde{P}P_1Y)+h^s(X,K_1\tilde{P}P_2Y)+h^s(X,K_2\tilde{P}P_2Y)+h^s(X,PP_4Y)+h^s(X,\\&L_2\tilde{P}P_3Y)=FP_4\nabla_XY+CQ_2h^s(X,Y)-\nabla^s_XFP_4Y-D^s(X,L_1\tilde{P}P_3Y).\end{aligned}\tag{21}$$

Example 3.4. Let $(\mathbb{R}^8_2,\overline{g},\tilde{P})$ be a golden semi-Riemannian manifold, where metric $\overline{g}$ is of signature $(-,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1,\partial x^2,\partial x^3,\partial x^4,\partial x^5,\partial x^6,\partial x^7,\partial x^8\}$ with $(x^1,x^2,x^3,x^4,x^5,x^6,x^7,x^8)$ being standard coordinates of $\mathbb{R}^8_2$.

Take $\tilde{P}(\partial x^1,\ldots,\partial x^8)=((1-\psi)\partial x^1,\psi\partial x^2,\psi\partial x^3,(1-\psi)\partial x^4,\psi\partial x^5,\psi\partial x^6,(1-\psi)\partial x^7,(1-\psi)\partial x^8)$, where $\psi=\frac{1+\sqrt{5}}{2}$ *and* $(1-\psi)=\frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2-x-1=0$. Thus, $\tilde{P^2}=\tilde{P}+I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^8_2$. Suppose M is a submanifold of $\mathbb{R}^8_2$ given by $x^1=\psi u^1+u^2-u^3$, $x^2=u^1-\psi u^2+\psi u^3$, $x^3=u^1+\psi u^2+\psi u^3$, $x^4=\psi u^1-u^2-u^3$, $x^5=\psi u^4$, $x^6=\psi u^5$, $x^7=(1-\psi)u^4$, $x^8=(1-\psi)u^5$. The local frame of TM is given by $\{Z_1,Z_2,Z_3,Z_4,Z_5\}$, where $Z_1=\psi\partial x^1+\partial x^2+\partial x^3+\psi\partial x^4$, $Z_2=\partial x^1-\psi\partial x^2+\psi\partial x^3-\partial x^4$, $Z_3=-\partial x^1+\psi\partial x^2+\psi\partial x^3-\partial x^4$, $Z_4=\psi\partial x^5+(1-\psi)\partial x^7$ and $Z_5=\psi\partial x^6+(1-\psi)\partial x^8$.

Hence, $RadTM=\operatorname{span}\{Z_1\}$ and $S(TM)=\operatorname{span}\{Z_2,Z_3,Z_4,Z_5\}$. Now $ltr(TM)$ is spanned by $N_1=\frac{1}{2(1+\psi^2)}(-\psi\partial x^1-\partial x^2+\partial x^3+\psi\partial x^4)$ and $S(TM^\perp)$ is spanned by $W_1=(1-\psi)\partial x^5-\psi\partial x^7$, $W_2=(1-\psi)\partial x^6-\psi\partial x^8$. It follows that $\tilde{P}Z_1=Z_3$ and $\tilde{P}N_1=Z_2$ and distribution $D=\operatorname{span}\{Z_4,Z_5\}$ is a slant distribution with slant angle $\theta=\arccos(\frac{4}{\sqrt{21}})$. Hence M is a slant 1-lightlike submanifold of $\mathbb{R}^8_2$.

Example 3.5. Let $(\mathbb{R}^8_2,\overline{g},\tilde{P})$ be a golden semi-Riemannian manifold, where metric $\overline{g}$ is of signature $(+,-,+,-,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1,\partial x^2,\partial x^3,\partial x^4,\partial x^5,\partial x^6,\partial x^7,\partial x^8\}$ and $(x^1,x^2,x^3,x^4,x^5,x^6,x^7,x^8)$ representing standard coordinates of $\mathbb{R}^8_2$.

Take $\tilde{P}(\partial x^1,\ldots,\partial x^8)=(\psi\partial x^1,\psi\partial x^2,(1-\psi)\partial x^3,(1-\psi)\partial x^4,(1-\psi)\partial x^5,\psi\partial x^6,(1-\psi)\partial x^7,\psi\partial x^8)$, where $\psi=\frac{1+\sqrt{5}}{2}$ *and* $(1-\psi)=\frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2-x-1=0$. Thus $\tilde{P^2}=\tilde{P}+I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^8_2$. Suppose M is a submanifold of $\mathbb{R}^8_2$ given by $x^1=u^1+\psi u^2-\psi u^3$, $x^2=u^1+\psi u^2+\psi u^3$, $x^3=\psi u^1-u^2+u^3$, $x^4=\psi u^1-u^2-u^3$, $x^5=\psi u^4$, $x^6=(1-\psi)u^4$, $x^7=\psi u^5$, $x^8=(1-\psi)u^5$. The local frame of TM is given by $\{Z_1,Z_2,Z_3,Z_4,Z_5\}$, where $Z_1=\partial x^1+\partial x^2+\psi\partial x^3+\psi\partial x^4$, $Z_2=\psi\partial x^1+\psi\partial x^2-\partial x^3-\partial x^4$, $Z_3=-\psi\partial x^1+\psi\partial x^2+\partial x^3-\partial x^4$, $Z_4=\psi\partial x^5+(1-\psi)\partial x^6$

and $Z_5 = \psi\partial x^7 + (1-\psi)\partial x^8$.

Hence, $RadTM = \text{span}\{Z_1\}$ and $S(TM) = \text{span}\{Z_2, Z_3, Z_4, Z_5\}$. Now, $ltr(TM)$ is spanned by $N_1 = -\frac{1}{2(1+\psi^2)}(-\partial x^1 + \partial x^2 - \psi\partial x^3 + \psi\partial x^4)$ and $S(TM^\perp)$ is spanned by $W_1 = (1-\psi)\partial x^5 - \psi\partial x^6$, $W_2 = (1-\psi)\partial x^7 - \psi\partial x^8$. It follows that $\tilde{P}Z_1 = Z_2$ and $\tilde{P}N_1 = Z_3$ and distribution $D = \text{span}\{Z_4, Z_5\}$ is a slant distribution with slant angle $\theta = \arccos(1/\sqrt{6})$. Hence M is a slant 1-lightlike submanifold of $\mathbb{R}^8_2$.

Theorem 3.6. *Let (M, g) be a q-lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$ of index $2q$. Then M is a slant lightlike submanifold of $\overline{M}$ if and only if*
(i) $\tilde{P}RadTM$ is a distribution on M such that $RadTM \cap \tilde{P}RadTM = 0$,
(ii) the screen distribution $S(TM)$ split as $S(TM)=(\tilde{P}RadTM \oplus \tilde{P}ltr(TM)) \oplus_{orth} D$,
(iii) there exists a constant $\lambda \in [0,1)$ such that $P^2X = \lambda(PX+X)$, for all $X \in \Gamma(D)$. Moreover, in this case $\lambda = \cos^2\theta$ and θ is the slant angle of D.

Proof. Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the distribution $\tilde{P}RadTM$ is a distribution on M such that $RadTM \cap \tilde{P}RadTM = 0$ and $S(TM) = (\tilde{P}RadTM \oplus \tilde{P}ltr(TM)) \oplus_{orth} D$.

Now for any $X \in \Gamma(D)$ we have $|PX| = |\tilde{P}X|\cos\theta$, which implies

$$\cos\theta = \frac{|PX|}{|\tilde{P}X|}. \tag{22}$$

In view of (22), we get $\cos^2\theta = \frac{|PX|^2}{|\tilde{P}X|^2} = \frac{g(PX,PX)}{g(\tilde{P}X,\tilde{P}X)} = \frac{g(X,P^2X)}{g(X,\tilde{P}^2X)}$, which gives

$$g(X, P^2X) = \cos^2\theta\, g(X, \tilde{P^2}X). \tag{23}$$

Since M is a slant lightlike submanifold, $\cos^2\theta = \lambda(constant) \in [0,1)$ and therefore from (23), we get $g(X, P^2X) = \lambda g(X, \tilde{P^2}X) = g(X, \lambda\tilde{P^2}X) = g(X, \lambda(\tilde{P}X + X))$, which implies

$$g(X, P^2X - \lambda(PX + X)) = 0. \tag{24}$$

Since $P^2X - \lambda(PX+X) \in \Gamma(D)$ and the induced metric $g = g|_{D\times D}$ is non-degenerate (positive definite), from (24), we have $(P^2X - \lambda(PX+X)) = 0$, which implies

$$P^2X = \lambda(PX + X), \tag{25}$$

for all $X \in \Gamma(D)$. This proves (iii).

Conversely, suppose that conditions (i), (ii) and (iii) are satisfied. From (iii), we have $P^2X = \lambda(PX+X)$, for all $X \in \Gamma(D)$, where $\lambda(\text{constant}) \in [0,1)$. Now,

$$\begin{aligned}\cos\theta =& \frac{g(\tilde{P}X, PX)}{|\tilde{P}X||PX|} = \frac{g(X, \tilde{P}PX)}{|\tilde{P}X||PX|} = \frac{g(X, P^2X)}{|\tilde{P}X||PX|} = \frac{g(X, \lambda(PX+X))}{|\tilde{P}X||PX|}\\ =& \frac{g(X, \lambda(\tilde{P}X + X))}{|\tilde{P}X||PX|} = \lambda\frac{g(X, \tilde{P^2}X)}{|\tilde{P}X||PX|} = \lambda\frac{g(\tilde{P}X, \tilde{P}X)}{|\tilde{P}X||PX|}.\end{aligned}$$

From the above equation, we get $\cos\theta = \lambda\frac{|\tilde{P}X|}{|PX|}$. Therefore, this with (22) gives $\cos^2\theta = \lambda(\text{constant})$. Hence (M, g) is a slant lightlike submanifold. □

Corollary 3.7. *Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$ with slant angle θ; then for any $X, Y \in \Gamma(D)$, we have*
(i) $g(PX, PY) = \cos^2\theta\ (g(X, Y) + g(X, PY))$,

(ii) $g(FX, FY) = \sin^2\theta\ (g(X, Y) + g(PX, Y))$.

Proof. From (1), (11) and (25), we obtain (i). Moreover, we get (ii) from (1), (11) and (i). Hence, the proof is complete. □

Theorem 3.8. *Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then $RadTM$ is integrable if and only if*
(i) $P_1(\nabla_X \tilde{P}P_1Y) - P_1(\nabla_Y \tilde{P}P_1X) + P_2(\nabla_X \tilde{P}P_1Y) - P_2(\nabla_Y \tilde{P}P_1X) = \tilde{P}P_1[X, Y]$,

(ii) $P_3(\nabla_X \tilde{P}P_1Y) - P_3(\nabla_Y \tilde{P}P_1X) = h^l(Y, \tilde{P}P_1X) - h^l(X, \tilde{P}P_1Y)$,

(iii) $P_4(\nabla_X \tilde{P}P_1Y) = P_4(\nabla_Y \tilde{P}P_1X)$ *and* $h^s(X, \tilde{P}P_1Y) = h^s(Y, \tilde{P}P_1X)$, *for all* $X, Y \in \Gamma(RadTM)$.

Proof. Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$. Let $X, Y \in \Gamma(RadTM)$. From (16) and (17), we have $P_1(\nabla_X \tilde{P}P_1Y) + P_2(\nabla_X \tilde{P}P_1Y) - \tilde{P}P_1\nabla_X Y = \tilde{P}P_2\nabla_X Y$, which gives $P_1(\nabla_X \tilde{P}P_1Y) - P_1(\nabla_Y \tilde{P}P_1X) + P_2(\nabla_X \tilde{P}P_1Y) - P_2(\nabla_Y \tilde{P}P_1X) - \tilde{P}P_1[X, Y] = \tilde{P}P_2[X, Y]$. From (18) and (20), we get $P_3(\nabla_X \tilde{P}P_1Y) + h^l(X, \tilde{P}P_1Y) - \tilde{P}h^l(X, Y) = \tilde{P}P_3\nabla_X Y$, which implies $P_3(\nabla_X \tilde{P}P_1Y) - P_3(\nabla_Y \tilde{P}P_1X) + h^l(X, \tilde{P}P_1Y) - h^l(Y, \tilde{P}P_1X) = \tilde{P}P_3[X, Y]$. From (19), we have $P_4(\nabla_X \tilde{P}P_1Y) = PP_4\nabla_X Y + BQ_2h^s(X, Y)$, which gives $P_4(\nabla_X \tilde{P}P_1Y) - P_4(\nabla_Y \tilde{P}P_1X) = PP_4[X, Y]$. In view of (21), we have $h^s(X, \tilde{P}P_1Y) = CQ_2h^s(X, Y) + FP_4\nabla_X Y$, which gives $h^s(X, \tilde{P}P_1Y) - h^s(Y, \tilde{P}P_1X) = FP_4[X, Y]$. This concludes the proof. □

Theorem 3.9. *Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then $\tilde{P}RadTM$ is integrable if and only if*
(i) $P_2(\nabla_X \tilde{P}P_2Y) - P_2(\nabla_Y \tilde{P}P_2X) = K_2\tilde{P}P_2\nabla_X Y - K_2\tilde{P}P_2\nabla_Y X$,

(ii) $P_3(\nabla_X \tilde{P}P_2Y) - P_3(\nabla_Y \tilde{P}P_2X) + h^l(X, K_1\tilde{P}P_2Y) - h^l(Y, K_1\tilde{P}P_2X) = - h^l(X, K_2\tilde{P}P_2Y) + h^l(Y, K_2\tilde{P}P_2X)$,

(iii) $P_4(\nabla_X \tilde{P}P_2Y) = P_4(\nabla_Y \tilde{P}P_2X)$ *and* $h^s(X, K_1\tilde{P}P_2Y) - h^s(Y, K_1\tilde{P}P_2X) = h^s(Y, K_2\tilde{P}P_2X) - h^s(X, K_2\tilde{P}P_2Y)$, *for all* $X, Y \in \Gamma(\tilde{P}RadTM)$.

Proof. Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$. Let $X, Y \in \Gamma(\tilde{P}RadTM)$. From (18) and (20), we have $P_3(\nabla_X \tilde{P}P_2Y) - \tilde{P}h^l(X, Y) + h^l(X, K_1\tilde{P}P_2Y) + h^l(X, K_2\tilde{P}P_2Y) = \tilde{P}P_3\nabla_X Y$, which gives $P_3(\nabla_X \tilde{P}P_2Y) - P_3(\nabla_Y \tilde{P}P_2X) + h^l(X, K_1\tilde{P}P_2Y) - h^l(Y, K_1\tilde{P}P_2X) + h^l(X, K_2\tilde{P}P_2Y) - h^l(Y, K_2\tilde{P}P_2X) = \tilde{P}P_3[X, Y]$. From (17), we get $P_2(\nabla_X \tilde{P}P_2Y) - K_2\tilde{P}P_2\nabla_X Y = \tilde{P}P_1\nabla_X Y$, which implies $P_2(\nabla_X \tilde{P}P_2Y) - P_2(\nabla_Y \tilde{P}P_2X) - K_2\tilde{P}P_2\nabla_X Y + K_2\tilde{P}P_2\nabla_Y X = \tilde{P}P_1[X, Y]$. From (19), we have $P_4(\nabla_X \tilde{P}P_2Y) = PP_4\nabla_X Y + BQ_2h^s(X, Y)$, which gives $P_4(\nabla_X \tilde{P}P_2Y) - P_4(\nabla_Y \tilde{P}P_2X) = PP_4[X, Y]$. In view of (21), we have $h^s(X, K_1\tilde{P}P_1Y) + h^s(X, K_2\tilde{P}P_2Y) = CQ_2h^s(X, Y) + FP_4\nabla_X Y$, which gives $h^s(X, K_1\tilde{P}P_2Y) - h^s(Y, K_1\tilde{P}P_2X) + h^s(X, K_2\tilde{P}P_2Y) - h^s(Y, K_2\tilde{P}P_2Y) = FP_4[X, Y]$. This concludes the proof. □

Theorem 3.10. *Let* (M, g) *be a slant lightlike submanifold of a golden semi-Riemannian manifold* $(\overline{M}, \overline{g})$. *Then* $\tilde{P}ltr(TM)$ *is integrable if and only if*
(i) $P_2(\nabla_X L_2\tilde{P}P_3Y) - P_2(\nabla_Y L_2\tilde{P}P_3X) - P_2(A_{L_1\tilde{P}P_3Y}X) + P_2(A_{L_1\tilde{P}P_3X}Y) = K_2\tilde{P}P_2\nabla_XY - K_2\tilde{P}P_2\nabla_YX$,

(ii) $P_1(\nabla_X L_2\tilde{P}P_3Y) - P_1(\nabla_Y L_2\tilde{P}P_3X) + P_2(\nabla_X L_2\tilde{P}P_3Y) - P_2(\nabla_Y L_2\tilde{P}P_3X) + P_1(A_{L_1\tilde{P}P_3X}Y) - P_1(A_{L_1\tilde{P}P_3Y}X) + P_2(A_{L_1\tilde{P}P_3X}Y) - P_2(A_{L_1\tilde{P}P_3Y}X) = \tilde{P}P_1[X, Y]$,

(iii) $P_4(\nabla_X L_2\tilde{P}P_3Y) - P_4(\nabla_Y L_2\tilde{P}P_3X) = P_4(A_{L_1\tilde{P}P_3Y}X) - P_4(A_{L_1\tilde{P}P_3X}Y)$ *and* $D^s(X, L_1\tilde{P}P_3Y) - D^s(Y, L_1\tilde{P}P_3X) = h^s(Y, L_2\tilde{P}P_3X) - h^s(X, L_2\tilde{P}P_3Y)$, *for all* $X, Y \in \Gamma(\tilde{P}ltr(TM))$.

Proof. Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$. Let $X, Y \in \Gamma(\tilde{P}ltr(TM))$. From (16) and (17), we have $P_1(\nabla_X L_2\tilde{P}P_3Y) + P_2(\nabla_X L_2\tilde{P}P_3Y) - P_1(A_{L_1\tilde{P}P_3Y}X) - P_2(A_{L_1\tilde{P}P_3Y}X) - \tilde{P}P_1\nabla_XY = \tilde{P}P_2\nabla_XY$, which gives $P_1(\nabla_X L_2\tilde{P}P_3Y) - P_1(\nabla_Y L_2\tilde{P}P_3X) + P_2(\nabla_X L_2\tilde{P}P_3Y) - P_2(\nabla_Y L_2\tilde{P}P_3X) - P_1(A_{L_1\tilde{P}P_3Y}X) + P_1(A_{L_1\tilde{P}P_3X}Y) - P_2(A_{L_1\tilde{P}P_3Y}X) + P_2(A_{L_1\tilde{P}P_3X}Y) - \tilde{P}P_1[X, Y] = \tilde{P}P_2[X, Y]$. From (17), we get $P_2(\nabla_X L_2\tilde{P}P_3Y) - P_2(A_{L_1\tilde{P}P_3Y}X) - K_2\tilde{P}P_2\nabla_XY = \tilde{P}P_1\nabla_XY$, which implies $P_2(\nabla_X L_2\tilde{P}P_3Y) - P_2(\nabla_Y L_2\tilde{P}P_3X) - P_2(A_{L_1\tilde{P}P_3Y}X) + P_2(A_{L_1\tilde{P}P_3X}Y) - K_2\tilde{P}P_2\nabla_XY + K_2\tilde{P}P_2\nabla_YX = \tilde{P}P_1[X, Y]$. From (19), we have $P_4(\nabla_X L_2\tilde{P}P_3Y) - P_4(A_{L_1\tilde{P}P_3Y}X) = PP_4\nabla_XY + BQ_2h^s(X, Y)$, which gives $P_4(\nabla_X L_2\tilde{P}P_3Y) - P_4(\nabla_Y L_2\tilde{P}P_3X) - P_4(A_{L_1\tilde{P}P_3Y}X) + P_4(A_{L_1\tilde{P}P_3X}Y) = PP_4[X, Y]$. In view of (21), we have $D^s(X, L_1\tilde{P}P_3Y) + h^s(X, L_2\tilde{P}P_3Y) = CQ_2h^s(X, Y) + FP_4\nabla_XY$, which gives $D^s(X, L_1\tilde{P}P_3Y) - D^s(Y, L_1\tilde{P}P_3X) + h^s(X, L_2\tilde{P}P_3Y) - h^s(Y, L_2\tilde{P}P_3X) = FP_4[X, Y]$. This concludes the theorem. □

Theorem 3.11. *Let* (M, g) *be a slant lightlike submanifold of a golden semi-Riemannian manifold* $(\overline{M}, \overline{g})$. *Then* D *is integrable if and only if*
(i) $P_2(\nabla_X PP_4Y) - P_2(\nabla_Y PP_4X) - P_2(A_{FP_4Y}X) + P_2(A_{FP_4X}Y) = K_2\tilde{P}P_2\nabla_XY - K_2\tilde{P}P_2\nabla_YX$,

(ii) $P_1(\nabla_X PP_4Y) - P_1(\nabla_Y PP_4X) - P_1(A_{FP_4Y}X) + P_1(A_{FP_4X}Y) + P_2(\nabla_X PP_4Y) - P_2(\nabla_Y PP_4X) - P_2(A_{FP_4Y}X) + P_2(A_{FP_4X}Y) = \tilde{P}P_1[X, Y]$,

(iii) $P_3(\nabla_X PP_4Y) - P_3(\nabla_Y PP_4X) - P_3(A_{FP_4Y}X) + P_3(A_{FP_4X}Y) + h^l(X, PP_4Y) - h^l(Y, PP_4X) = -D^l(X, FP_4Y) + D^l(Y, FP_4X)$, *for all* $X, Y \in \Gamma(D)$.

Proof. Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$. Let $X, Y \in \Gamma(D)$. From (16) and (17), we have $P_1(\nabla_X PP_4Y) - P_1(A_{FP_4Y}X) + P_2(\nabla_X PP_4Y) - P_2(A_{FP_4Y}X) - \tilde{P}P_1\nabla_XY = \tilde{P}P_2\nabla_XY$, which gives $P_1(\nabla_X PP_4Y) - P_1(\nabla_Y PP_4X) - P_1(A_{FP_4Y}X) + P_1(A_{FP_4X}Y) + P_2(\nabla_X PP_4Y) - P_2(\nabla_Y PP_4X) - P_2(A_{FP_4Y}X) + P_2(A_{FP_4X}Y) - \tilde{P}P_1[X, Y] = \tilde{P}P_2[X, Y]$. From (17), we get $P_2(\nabla_X PP_4Y) - P_2(A_{FP_4Y}X) - K_2\tilde{P}P_2\nabla_XY = \tilde{P}P_1\nabla_XY$, which implies $P_2(\nabla_X PP_4Y) - P_2(\nabla_Y PP_4X) - P_2(A_{FP_4Y}X) + P_2(A_{FP_4X}Y) - K_2\tilde{P}P_2\nabla_XY + K_2\tilde{P}P_2\nabla_YX = \tilde{P}P_1[X, Y]$. From (18) and (20), we have $P_3(\nabla_X PP_4Y) - P_3(A_{FP_4Y}X) - \tilde{P}h^l(X, Y) + h^l(X, PP_4Y) + D^l(X, FP_4Y) = \tilde{P}P_3\nabla_XY$, which implies $P_3(\nabla_X PP_4Y) -$

$P_3(\nabla_Y PP_4X) - P_3(A_{FP_4Y}X) + P_3(A_{FP_4X}Y) + h^l(X, PP_4Y) - h^l(Y, PP_4X) + D^l(X, FP_4Y) - D^l(Y, FP_4X) = \tilde{P}P_3[X, Y]$. This concludes the proof. □

THEOREM 3.12. *Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the induced connection ∇ is a metric connection if and only if*
(i) $\tilde{P}P_1\nabla_X\tilde{P}Y + K_2\tilde{P}P_2\nabla_X\tilde{P}Y = P_2\nabla_X\tilde{P}Y$,

(ii) $L_2\tilde{P}P_3\nabla_X\tilde{P}Y + \tilde{P}h^l(X, \tilde{P}Y) = P_3\nabla_X\tilde{P}Y$,

(iii) $PP_4\nabla_X\tilde{P}Y + BQ_2h^s(X, \tilde{P}Y) = P_4\nabla_X\tilde{P}Y$, *for all* $X \in \Gamma(TM)$ *and* $Y \in \Gamma(RadTM)$.

Proof. Let (M, g) be a slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the induced connection ∇ is a metric connection if and only if $RadTM$ is parallel distribution with respect to ∇ [3]. For all $X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$, we have $\overline{\nabla}_XY = \tilde{P}\overline{\nabla}_X\tilde{P}Y - \overline{\nabla}_X\tilde{P}Y$. From (6), (12), (14) and (15), we obtain $\overline{\nabla}_XY = \tilde{P}P_1\nabla_X\tilde{P}Y + K_1\tilde{P}P_2\nabla_X\tilde{P}Y + K_2\tilde{P}P_2\nabla_X\tilde{P}Y + L_1\tilde{P}P_3\nabla_X\tilde{P}Y + L_2\tilde{P}P_3\nabla_X\tilde{P}Y + PP_4\nabla_X\tilde{P}Y + FP_4\nabla_X\tilde{P}Y + \tilde{P}h^l(X, \tilde{P}Y) + BQ_2h^s(X, \tilde{P}Y) + CQ_2h^s(X, \tilde{P}Y) - P_1\nabla_X\tilde{P}Y - P_2\nabla_X\tilde{P}Y - P_3\nabla_X\tilde{P}Y - P_4\nabla_X\tilde{P}Y - h^l(X, \tilde{P}Y) - h^s(X, \tilde{P}Y)$. On comparing tangential components of both sides of above equation, we obtain $\nabla_XY = \tilde{P}P_1\nabla_X\tilde{P}Y + K_1\tilde{P}P_2\nabla_X\tilde{P}Y + K_2\tilde{P}P_2\nabla_X\tilde{P}Y + L_2\tilde{P}P_3\nabla_X\tilde{P}Y + PP_4\nabla_X\tilde{P}Y + \tilde{P}h^l(X, \tilde{P}Y) + BQ_2h^s(X, \tilde{P}Y) - P_1\nabla_X\tilde{P}Y - P_2\nabla_X\tilde{P}Y - P_3\nabla_X\tilde{P}Y - P_4\nabla_X\tilde{P}Y$, which completes the proof. □

4. Curvature properties of slant lightlike submanifolds

In this section, we study curvature properties of slant lightlike submanifolds of golden semi-Riemannian manifolds.

From (4) and (11), we get the Riemannian curvature of locally golden product space form $(\overline{M} = M_p(c_p) \times M_q(c_q), \overline{g}, \tilde{P})$ as

$$\begin{aligned}\overline{R}(X, Y)Z =& (-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}})\{\overline{g}(Y, Z)X - \overline{g}(X, Z)Y + \overline{g}(\tilde{P}Y, Z)PX \\ &+ \overline{g}(\tilde{P}Y, Z)FX - \overline{g}(\tilde{P}X, Z)PY - \overline{g}(\tilde{P}X, Z)FY\} \\ &+ (-\frac{(1-\psi)c_p + \psi c_q}{4})\{\overline{g}(\tilde{P}Y, Z)X + \overline{g}(Y, Z)PX \\ &+ \overline{g}(Y, Z)FX - \overline{g}(\tilde{P}X, Z)Y - \overline{g}(X, Z)PY - \overline{g}(X, Z)FY\},\end{aligned} \tag{26}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$.

Also, from (9) and (26), we obtain the equations of Gauss and Codazzi for the submanifold M, respectively as

$$\begin{aligned}R(X, Y)Z =& (-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}})\{\overline{g}(Y, Z)X - \overline{g}(X, Z)Y + \overline{g}(\tilde{P}Y, Z)PX - \overline{g}(\tilde{P}X, Z)PY\} \\ &+ (-\frac{(1-\psi)c_p + \psi c_q}{4})\{\overline{g}(\tilde{P}Y, Z)X - \overline{g}(\tilde{P}X, Z)Y \\ &+ \overline{g}(Y, Z)PX - \overline{g}(X, Z)PY + A_{h(Y,Z)}X - A_{h(X,Z)}Y\},\end{aligned}$$

and

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = (-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}})\{\overline{g}(\tilde{P}Y,Z)FX - \overline{g}(\tilde{P}X,Z)FY\} + (-\frac{(1-\psi)c_p + \psi c_q}{4})\{\overline{g}(Y,Z)FX - \overline{g}(X,Z)FY\}, \tag{27}$$

for any $X, Y, Z \in \Gamma(TM)$.

DEFINITION 4.1. A lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is called curvature-invariant lightlike submanifold if

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0, \tag{28}$$

for all $X, Y, Z \in \Gamma(TM)$.

THEOREM 4.2. *There is no curvature invariant slant lightlike submanifold in any semi-Riemannian locally golden product space form* ($\overline{M} = M_p(c_p) \times M_q(c_q)$) *with* $c_p, c_q \neq 0$.

Proof. Suppose that (M, g) is a curvature invariant slant lightlike submanifold of a semi-Riemannian golden product space form ($\overline{M} = M_p(c_p) \times M_q(c_q), \overline{g}, \tilde{P}$) with $c_p, c_q \neq 0$. Since M is curvature invariant, then from (27) and (28), we have

$$(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}})\{\overline{g}(\tilde{P}Y,Z)FX - \overline{g}(\tilde{P}X,Z)FY\} + (-\frac{(1-\psi)c_p + \psi c_q}{4})\{\overline{g}(Y,Z)FX - \overline{g}(X,Z)FY\} = 0, \tag{29}$$

for any $X, Y, Z \in \Gamma(TM)$.

Let $X \in \Gamma(D)$, $Y \in \Gamma RadTM$ and $Z \in \Gamma\tilde{P}ltr(TM)$ then from (29), we get

$$(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}})FX = 0. \tag{30}$$

Also, let $X \in \Gamma(D)$, $Y \in \Gamma\tilde{P}RadTM$ and $Z \in \Gamma\tilde{P}ltr(TM)$; then from (29), we obtain

$$(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}})FX + (-\frac{(1-\psi)c_p + \psi c_q}{4})FX = 0. \tag{31}$$

From (30) and (31), we get $c_p, c_q = 0$. This completes the proof. □

PROPOSITION 4.3 ([7]). *Let* (M, g) *be an irrotational q-lightlike submanifold of a semi-Riemannian manifold* $(\overline{M}, \overline{g})$. *Then we have the following equation*

$$\overline{g}(\overline{R}(X,Y)Z, \xi) = 0, \tag{32}$$

for all $X, Y, Z \in \Gamma(TM)$ *and* $\xi \in \Gamma RadTM$.

THEOREM 4.4. *Let* (M, g) *be an irrotational slant lightlike submanifold of a locally golden product space form* ($\overline{M} = M_p(c_p) \times M_q(c_q), \overline{g}, \tilde{P}$). *Then* $c_p, c_q = 0$.

Proof. Suppose that (M, g) is an irrotational slant lightlike submanifold of a locally golden product space form $(\overline{M}, \overline{g})$. Taking scalar product with ξ of (4) and using (1), we get

$$\overline{g}(\overline{R}(X,Y)Z,\xi) =(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}})\{\overline{g}(\tilde{P}Y,Z)\overline{g}(X,\tilde{P}\xi)-\overline{g}(\tilde{P}X,Z)\overline{g}(Y,\tilde{P}\xi)\} + (-\frac{(1-\psi)c_p+\psi c_q}{4})\{\overline{g}(Y,Z)\overline{g}(X,\tilde{P}\xi)-\overline{g}(X,Z)\overline{g}(Y,\tilde{P}\xi)\}. \tag{33}$$

From (32) and (33), we obtain

$$(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}})\{\overline{g}(\tilde{P}Y,Z)\overline{g}(X,\tilde{P}\xi)-\overline{g}(\tilde{P}X,Z)\overline{g}(Y,\tilde{P}\xi)\} + (-\frac{(1-\psi)c_p+\psi c_q}{4})\{\overline{g}(Y,Z)\overline{g}(X,\tilde{P}\xi)-\overline{g}(X,Z)\overline{g}(Y,\tilde{P}\xi)\}=0. \tag{34}$$

Putting $X\in\Gamma\tilde{P}ltr(TM)$, $Y\in\Gamma RadTM$ and $Z\in\Gamma\tilde{P}ltr(TM)$ in (34), we obtain

$$(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}})=0. \tag{35}$$

Putting $X\in\Gamma\tilde{P}ltr(TM)$, $Y\in\Gamma\tilde{P}RadTM$ and $Z\in\Gamma\tilde{P}ltr(TM)$ in (4.10), we obtain

$$(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}})+(-\frac{(1-\psi)c_p+\psi c_q}{4})=0. \tag{36}$$

From (35) and (36), we get $c_p, c_q=0$, which completes the proof. □

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