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EXISTENCE OF ONE WEAK SOLUTION FOR ELLIPTIC EQUATIONS INVOLVING A GENERAL OPERATOR IN DIVERGENCE FORM

S. Amirkhanlou, Mohsen Khaleghi Moghadam and Yasser Khalili

Abstract. In this article, we establish the existence of at least one non-trivial classical solution for a class of elliptic equations involving a general operator in divergence form, subject to Dirichlet boundary conditions in a smooth bounded domain in $\mathbb{R}^N$. A critical point result for differentiable functionals is discussed. Our technical approach is based on variational methods. In addition, an example to illustrate our results is given.

1. Introduction

The purpose of this paper is to establish the existence of at least one weak solution for the following elliptic Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = \lambda k(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $p > N$, $a : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a suitable continuous map of gradient type, and λ is a positive real parameter. Further, $f : \mathbb{R} \to \mathbb{R}$ and $k : \bar{\Omega} \to \mathbb{R}^+$ are two continuous functions.

The operator $-\operatorname{div}(a(x,\nabla u))$ arises, for example, from the expression of the p-Laplacian in curvilinear coordinates. We refer to the overview papers [2, 3, 8, 13] for the investigation on Dirichlet problems involving a general operator in divergence form. For example, De Nápoli and Mariani in [2] studied the existence of solutions to equations of p-Laplacian type. They proved the existence of at least one solution, and under further assumptions, the existence of infinitely many solutions. In order to apply mountain pass results, they introduced a notion of uniformly convex functional that generalizes the notion of uniformly convex norm. Duc and Vu in [3] studied the

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non-uniform case. The authors established the existence and multiplicity of weak solutions of a problem involving a uniformly convex elliptic operator in divergence form. They discussed the existence of one nontrivial solution by the mountain pass lemma, when the nonlinearity has a $(p-1)$-superlinear growth at infinity, and two nontrivial solutions by minimization and mountain pass when the nonlinear term has a $(p-1)$-sublinear growth at infinity. Molica Bisci and Repovš in [8], exploiting variational methods, investigated the existence of three weak solutions for the problem (1). They analyzed several special cases and presented a concrete example of an application by finding the existence of three nontrivial weak solutions for an uniformly elliptic second-order problem on a bounded Euclidean domain.

In [1], Colasuonno, Pucci and Varga studied different and very general classes of elliptic operators in divergence form looking at the existence of multiple weak solutions. Their contributions represent a nice improvement, in several directions, of the results obtained by Kristály et al. in [4] in which a uniform Dirichlet problem with parameter is investigated.

Our goal is to establish some new criteria for system (1) to have at least one non-trivial classical solution by applying the following critical points theorem due to Ricceri [11, Theorem 2.1].

Theorem 1.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and for every $r > \inf_X \Phi$, let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

The above result is related to the celebrated *three critical points theorem* of Pucci and Serrin [9, 10]. We refer the interested reader to the papers in which Theorem 1.1 has been successfully employed to the existence of at least one nontrivial solution for boundary-value problems.

In [6], Molica Bisci and Rădulescu, applying mountain pass results studied the existence of solutions to nonlocal equations involving the p-Laplacian. More precisely, they proved the existence of at least one nontrivial weak solution, and under additional assumptions, the existence of infinitely many weak solutions. In [5], they also established, by using an abstract linking theorem for smooth functionals, a multiplicity result on the existence of weak solutions for a nonlocal Neumann problem driven by a nonhomogeneous elliptic differential operator.

Inspired by the above results, in the present paper, we are interested to discuss the existence of at least one weak solution for the problem (1). Precisely, in Theorem 3.1 we establish the existence of at least one weak solution for the problem (1) under an asymptotical behaviour of the nonlinear datum at zero. We present Example 3.7 in which the hypotheses of Theorem 3.1 are fulfilled. We also list some consequences and the main results.

We refer to the recent monograph by Molica Bisci, Rădulescu and Servadei [7] for related problems concerning the variational analysis of solutions of some classes of nonlocal problems.

2. Preliminaries

In this section, we first introduce some notations and some necessary definitions. Assume that Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$. Further, denote by X the space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \Big(\int_\Omega |\nabla u(x)|^p \, dx\Big)^{1/p}.$$

The functional $I_\lambda : X \to \mathbb{R}$ associated with (1) is introduced as $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$, for every $u \in X$, where

$$\Phi(u) := \int_\Omega A(x, \nabla u(x)) \, dx \qquad \text{and} \qquad \Psi(u) := \int_\Omega k(x)F(u(x)) \, dx,$$

for every $u \in X$, where $k : \bar{\Omega} \to \mathbb{R}^+$ is a positive and continuous function, and $F(s) = \int_0^s f(t)dt$, for every $s \in \mathbb{R}$. By standard arguments, Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) := \int_\Omega a(x, \nabla u(x))\nabla v(x) \, dx,$$

for every $v \in X$. Moreover, Ψ is a Gâteaux differentiable sequentially weakly upper continuous functional whose Gâteaux derivative is given by

$$\Psi'(u)(v) := \int_\Omega k(x)f(u(x))v(x) \, dx,$$

for every $v \in X$. Fixing the real parameter λ, a function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of (1) if $u \in X$ and

$$\int_\Omega a(x, \nabla u(x))\nabla v(x) \, dx - \lambda \int_\Omega k(x)f(u(x))v(x) \, dx = 0,$$

for every $v \in X$. Therefore, the critical points of I_λ are exactly the weak solutions of (1). If $p > N$, let

$$\sup\left\{\frac{\max_{x\in\overline{\Omega}} |u(x)|}{\|u\|} : u \in W_0^{1,p}(\Omega), u \neq 0\right\} < +\infty.$$

It is well-known [12, formula (6b)] that by putting

$$m := \frac{N^{-\frac{1}{p}}}{\sqrt{\pi}} \Big[\Gamma\Big(1 + \frac{N}{2}\Big)\Big]^{\frac{1}{N}} \Big(\frac{p-1}{p-N}\Big)^{1-\frac{1}{p}} (\text{meas}(\Omega))^{\frac{1}{N}-\frac{1}{p}},$$

one has

$$\|u\|_\infty = \max_{x\in\bar\Omega}|u(x)| \leq m\|u\|, \tag{2}$$

for every $u \in X$. Here Γ is the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz \qquad (\forall t > 0)\,,$$

and "meas(Ω)" denotes the usual Lebesgue measure of Ω.

3. Main results

In this section, we formulate our main results on the existence of at least one weak solution for the problem (1).

Let $p \geq 1$ and let $\Omega \subseteq \mathbb{R}^N$ be a bounded Euclidean domain, where $N \geq 2$. Further, let $A : \bar\Omega \times \mathbb{R}^N \to \mathbb{R}$ and let $A = A(x,\xi)$ be a continuous function in $\bar\Omega \times \mathbb{R}^N$, with continuous gradient $a(x,\xi) := \nabla_\xi A(x,\xi) : \bar\Omega \times \mathbb{R}^N \to \mathbb{R}^N$, and assume that the following conditions hold:

(α_1) $A(x,0) = 0$, for all $x \in \Omega$.

(α_2) A satisfies $\Lambda_1|\xi|^p \leq A(x,\xi) \leq \Lambda_2|\xi|^p$ for all $x \in \bar\Omega$, $\xi \in \mathbb{R}^N$, where Λ_1 and Λ_2 are positive constants.

(α_3) a satisfies the growth condition $|a(x,\xi)| \leq c(1+|\xi|^{p-1})$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$, $c > 0$.

(α_4) A is p-uniformly convex, that is $A(x, \frac{\xi+\eta}{2}) \leq \frac{1}{2}A(x,\xi) + \frac{1}{2}A(x,\eta) - k|\xi-\eta|^p$, for every $x \in \bar\Omega$, $\xi, \eta \in \mathbb{R}^N$ and some $k > 0$.

Our main result is the following theorem.

THEOREM 3.1. *Assume that*

$$\sup_{\gamma>0}\frac{\gamma^p}{\Big\{\Big(\max\limits_{|\xi|\leq\gamma}F(\xi)\Big)\|k\|_{L^1}\Big\}} > \frac{m^p}{\Lambda_1}, \tag{3}$$

and

$$\limsup_{\xi\to0^+}\frac{F(\xi)}{|\xi|^p} = +\infty, \tag{4}$$

$$\liminf_{\xi\to0^+}\frac{F(\xi)}{|\xi|^p} > -\infty. \tag{5}$$

There exists a positive number γ such that problem (1) *has a non-zero weak solution u for every λ belonging to interval Λ defined by*

$$\Lambda := \Big]0, \frac{\Lambda_1\gamma^p}{m^p\Big\{\Big(\max\limits_{|\xi|\leq\gamma}F(\xi)\Big)\|k\|_{L^1}\Big\}}\Big[,$$

and problem (1) *admits at least one nontrivial and nonnegative weak solution in X.*

Moreover, we have $\lim_{\lambda\to 0^+}\|u_\lambda\|_X = 0$ *and the real function*

$$\lambda \to \int_\Omega a(x,\nabla u(x))\nabla v(x)\,dx - \lambda\int_\Omega k(x)f(u(x))v(x)\,dx$$

is negative and strictly decreasing in the open interval Λ.

Proof. Fix $\lambda\in\Lambda$. Our aim is to apply Theorem 1.1 where Φ and Ψ are the functionals introduced in Section 2. Clearly, Φ is coercive since, by condition (α_2), it follows that $\Phi(u)\ge \Lambda_1\|u\|^p\to+\infty$, when $\|u\|\to\infty$. As seen before, the functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 1.1. Note that the critical points of the functional I are the solutions of the problem (1). Now, we look on the existence of critical points of the functional $I_\lambda := \Phi-\lambda\Psi$ in X. The condition (3) ensures that there exists $\bar\gamma>0$ such that

$$\frac{\bar\gamma^p}{\Big(\max\limits_{|\xi|\le\bar\gamma}F(\xi)\Big)\|k\|_{L^1(\Omega)}} > \frac{m^p}{\Lambda_1}.$$

To this end, set $r:=\frac{\Lambda_1\bar\gamma^p}{m^p}$. Let $u\in X$ be such that $\Phi(u)<r$, i.e. $\int_\Omega A(x,\nabla u(x))\,dx<r$. Hence the above relation together with condition (α_2) implies that $\|u\|<\left(\frac{r}{\Lambda_1}\right)^{\frac{1}{p}}$. Owing to (2) we have $\|u\|_\infty\le\bar\gamma$. By simple calculations and from the definition of $\varphi(r)$, one has

$$\begin{aligned}\varphi(r) &= \inf_{\Phi(u)<r}\frac{\Big(\sup_{\Phi(v)<r}\Psi(v)\Big)-\Psi(u)}{r-\Phi(u)}\\ &\le \frac{\sup_{\Phi(v)<r}\Psi(v)}{r}\le m^p\,\frac{\Big(\max\limits_{|\xi|\le\bar\gamma}F(\xi)\Big)\|k\|_{L^1}}{\Lambda_1\bar\gamma^p}.\end{aligned}$$

since $0\in\Phi^{-1}(-\infty,r)$ and $\Phi(0_X)=\Psi(0_X)=0$. Hence, putting

$$\lambda^* = \frac{\Lambda_1\bar\gamma^p}{m^p\Big\{\Big(\max\limits_{|\xi|\le\bar\gamma}F(\xi)\Big)\|k\|_{L^1}\Big\}}.$$

Theorem 1.1 ensures that for every $\lambda\in(0,\lambda^*)\subseteq(0,\frac{1}{\varphi(r)})$, the functional I_λ admits at least one critical point (local minimum) $u_\lambda\in\Phi^{-1}(-\infty,r)$. Now for every fixed $\lambda\in(0,\lambda^*)$ we show that $u_\lambda\ne0$ and the map $(0,\lambda^*)\ni\lambda\mapsto I_\lambda(u_\lambda)$, is negative. To this end, let us verify that

$$\limsup_{\|u\|\to0^+}\frac{\Psi(u)}{\Phi(u)}=+\infty. \tag{6}$$

Owing to our assumptions (4) and (5), we can fix a sequence $\{\xi_n\}\subset\mathbb{R}^+$ converging to zero and two constants σ, κ (with $\sigma>0$) such that $\lim_{n\to+\infty}\frac{F(\xi_n)}{|\xi_n|^p}=+\infty$, $F(\xi)\ge\kappa|\xi|^p$, for every $\xi\in[0,\sigma]$. Now, fix two sets $C,D\subset\Omega$ of positive measures with $C\subset D$ and a function $v\in X$ such that:

(i) $v(x)\in[0,1]$ for every $x\in\Omega$, (ii) $v(x)=1$ for every $x\in C$,

(iii) $v(x)=0$ for every $x\in\Omega\setminus D$.

Hence, fix $M > 0$ and consider a real positive number η with

$$M < \frac{\eta \operatorname{meas}(C) + \kappa \int_{D\setminus C} |v(x)|^p \, dx}{\Lambda_1 \|v\|^p}.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \sigma$ and $F(\xi_n) \geq \eta|\xi_n|^p$, for every $n > n_0$. Now, for every $n > n_0$, recalling the properties of the function v (that is $0 \leq \xi_n v(t) < \sigma$ for n sufficiently large), one has

$$\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = \frac{\int_C F(\xi_n)\, dx + \int_{D\setminus C} F(\xi_n v(x))\, dx}{\Phi(\xi_n v)} > \frac{\eta \operatorname{meas}(C) + \kappa \int_{D\setminus C} |v(x)|^p \, dx}{\Lambda_1 \|v\|^p} > M.$$

Since M can be considered arbitrarily large, it follows that $\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty$, from which (6) follows. Hence, there exists a sequence $\{w_n\} \subset X$ strongly converging to zero, $w_n \in \Phi^{-1}(-\infty, r)$ and $I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0$. Since u_λ is a global minimum of the restriction of I_λ to the set $\Phi^{-1}(-\infty, r)$, we deduce that

$$I_\lambda(u_\lambda) < 0, \tag{7}$$

so that u_λ is not trivial. From (7) we easily see that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda), \tag{8}$$

is negative.

Now, we prove that $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$. Since Φ is coercive and for $\lambda \in (0, \lambda^*)$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, one has that there exists a positive constant L such that $\|u_\lambda\| \leq L$ for every $\lambda \in (0, \lambda^*)$. Therefore, there exists a positive constant N such that

$$\Big| \int_\Omega k(x) f(u_\lambda(x)) u_\lambda(x)\, dx \Big| \leq N\|u_\lambda\| \leq NL, \tag{9}$$

for every $\lambda \in (0, \lambda^*)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v) = 0$ for any $v \in X$ and every $\lambda \in (0, \lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$; that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_\Omega k(x) f(u_\lambda(x)) u_\lambda(x)\, dx, \tag{10}$$

for every $\lambda \in (0, \lambda^*)$. Then, since $0 \leq \Lambda_1\|u_\lambda\|^p \leq \Phi'(u_\lambda)(u_\lambda)$, by (10) it follows that $0 \leq \Lambda_1\|u_\lambda\|^p \leq \lambda \int_\Omega k(x) f(u_\lambda(x)) u_\lambda(x)\, dx$, for any $\lambda \in (0, \lambda^*)$. Letting $\lambda \to 0^+$, by (9) we have $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$. Finally, we show that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$. For our aim we see that for any $u \in X$,

$$I_\lambda(u) = \lambda\Big(\frac{\Phi(u)}{\lambda} - \Psi(u)\Big). \tag{11}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Also, let

$$m_{\lambda_i} = \Big(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i})\Big) = \inf_{v\in\Phi^{-1}(-\infty,r)} \Big(\frac{\Phi(v)}{\lambda_i} - \Psi(v)\Big),$$

for $i = 1, 2$. Clearly, (8) and (11), since $\lambda > 0$, imply that $m_{\lambda i} < 0$, for $i = 1, 2$. Moreover, since $0 < \lambda_1 < \lambda_2$, we have $m_{\lambda_2} \leq m_{\lambda_1}$. Hence, we obtain $I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1})$, which means the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$. Since $\lambda < \lambda^*$ is arbitrary, we observe $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$. The proof is complete. □

Remark 3.2. If f is non-negative then the solution ensured in Theorem 3.1 is non-negative. Indeed, let u_* be a non-trivial weak solution of the problem (1), then u_* is non-negative. Arguing by a contradiction, assume that the set $\mathcal{A} = \{x \in \Omega; u_*(x) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(x) = \min\{u_*(x), 0\}$. Using this fact we get that u_* is also a solution of (1), so for every $\bar{v} \in X$ we have

$$\int_\Omega a(x, \nabla u(x))\nabla v(x)\,dx - \lambda \int_\Omega k(x) f(u(x)) v(x)\,dx = 0,$$

and by choosing $\bar{v} = u_*$ and since f is non-negative, we have

$$0 \le \Lambda_1 \|u_*\|^p_{\mathcal{A}} \le \int_{\mathcal{A}} \Big(a(x, \nabla u_*(x))\nabla u_*(x)\,dx\Big)dt = \lambda \int_{\mathcal{A}} k(x) f(u_*(x)) u_*(x)\,dx \le 0,$$

hence $\|u_*\|^2_{\mathcal{A}} \le 0$ which contradicts the fact that u_* is a non-trivial solution. Hence, u_* is positive.

Remark 3.3. We observe that Theorem 3.1 is a bifurcation result in the sense that the pair $(0, 0)$ belongs to the closure of the set

$$\{(u_\lambda, \lambda) \in X \times (0, +\infty) : u_\lambda \text{ is a non-trivial weak solution of (1)}\}$$

in $X \times \mathbb{R}$. Indeed, by Theorem 3.1 we have that $\|u_\lambda\| \to 0 \quad \text{as} \quad \lambda \to 0$. Hence, there exist two sequences $\{u_j\}$ in X and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that $\lambda_j \to 0^+$ and $\|u_j\| \to 0$, as $j \to +\infty$. Moreover, we emphasize that due to the fact that the map $(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing, for every $\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, the solutions u_{λ_1} and u_{λ_2} ensured by Theorem 3.1 are different.

Remark 3.4. Here, employing Ricceri's variational principle, we are looking for the existence of critical points of the functional I_λ naturally associated to system (1). We emphasize that by direct minimization, we cannot argue, in general for finding the critical points of I_λ. Because, in general, I_λ can be unbounded from the following in X. Indeed, for example, in the case when $f(\xi) = 1 + |\xi|^{\gamma-2}\xi$ for $\xi \in \mathbb{R}$ with $\gamma > 2$, for any fixed $u \in X \backslash \{0\}$ and $\iota \in \mathbb{R}$, we obtain

$$I_\lambda(\iota u) = \Phi(\iota u) - \lambda \int_\Omega F(\iota u(x))\,dx \le \iota^p \Lambda_1 \|u\|^p - \lambda\iota\|u\|_{L^1} - \lambda\frac{\iota^\gamma}{\gamma}\|u\|^\gamma_{L^\gamma} \to -\infty$$

as $\iota \to +\infty$. Hence, we cannot use direct minimization to find critical points of the functional I_λ.

Remark 3.5. For a fixed $\bar{\gamma} > 0$ let

$$\frac{\bar{\gamma}^p}{\Big(\max\limits_{|\xi| \le \bar{\gamma}} F(\xi)\Big)\|k\|_{L^1}} > \frac{m^p}{\Lambda_1}.$$

Then the result of Theorem 3.1 holds with $\|u_\lambda\|_\infty \le \bar{\gamma}$ where u_λ is the ensured weak solution in X.

Remark 3.6. If in Theorem 3.1 the function $f(\xi) \ge 0$ for every $\xi \in \mathbb{R}$, then the condition (3) acquires the simpler form

$$\sup_{\gamma>0}\frac{\gamma^p}{F(\gamma)\|k\|_{L^1}} > \frac{m^p}{\Lambda_1}. \tag{12}$$

Moreover, if the assumption

$$\limsup_{\gamma\to+\infty}\frac{\gamma^p}{F(\gamma)\|k\|_{L^1}} > \frac{m^p}{\Lambda_1},$$

is satisfied, then condition (12) automatically holds.

Now we present an example in which the hypotheses of Theorem 3.1 are satisfied.

EXAMPLE 3.7. Let $\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 4\}$. Consider the autonomous problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = \lambda\log(1+u^4), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where $k(x)=1$. By choosing $p=4$, we have

$$\lim_{\xi\to 0^+}\frac{F(\xi)}{\xi^4} = \lim_{\xi\to 0^+}\frac{\int_0^\xi \log(1+t^4)\,dt}{\xi^4} = +\infty.$$

Hence, Theorem 3.1 implies that problem (13) admits at least one weak solution in X.

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Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

E-mail: amirkhanlou.s@gmail.com

Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, Sari, Iran

E-mail: m.khaleghi@sanru.ac.ir

Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, Sari, Iran

E-mail: y.khalili@sanru.ac.ir