MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
75, 2 (2023), 138–146
June 2023

research paper
оригинални научни рад
DOI: 10.57016/MV-mp9po679

A NEW TYPE OF WEIGHTED ORLICZ SPACES

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Abstract. In this paper, by some group action, we introduce a new type of weighted Orlicz spaces $L^{\Phi}_{w,v}(\Omega)$, where w and v are weights on Ω and Φ is a Young function. We study conditions under which $L^{\Phi}_{w,v}(G)$ is a convolution Banach algebra, where G is a locally compact group.

1. Introduction and preliminaries

Let $p \geq 1$ and $(\Omega, \mathcal{A}, \mu)$ be a measure space and $w : \Omega \to (0, \infty)$ be a measurable function. It is well-known that a complex-valued measurable function f on Ω belongs to the weighted Lebesgue space $L^p_w(\Omega)$ whenever $\int_\Omega |wf|^p \, d\mu < \infty$. If $\mathcal{L}^p_w(\Omega)$ denotes the set of all measurable functions $f : \Omega \to \mathbb{C}$ with

$$\int_\Omega |f|^p \, w \, d\mu < \infty, \tag{1}$$

then easily, $L^p_w(\Omega) = \mathcal{L}^p_{w^p}(\Omega)$. This fact trivially follows from the equality $\Phi_p(xy) = \Phi_p(x)\,\Phi_p(y)$, where $\Phi_p(\cdot) := |\cdot|^p$. In some papers, researchers prefer to consider the relation (1) for weighted Lebesgue spaces; for example see [8]. On the other hand, the situation is different for Orlicz spaces. As a natural way in [5] a measurable function $f : \Omega \to \mathbb{C}$ belongs to the weighted Orlicz space $L^{\Phi}_w(\Omega)$ if and only if $wf \in L^{\Phi}(\Omega)$, where Φ is a Young function. By considering a version of the relation (1) for the case of Orlicz spaces, we give a new version of weighted Orlicz spaces which are more complicated from the previous known one. Although these two classes are the same in the case of Lebesgue spaces, they are different for general Orlicz ones.

Definition 1.1. Let G be a locally compact group, Ω be a locally compact Hausdorff space, and μ be a Borel nonnegative measure on Ω. A continuous function $G \times \Omega \longrightarrow \Omega$, $(s, x) \mapsto sx$, $(s \in G, x \in \Omega)$ is called an *action* of G on the measure space (Ω, μ) if
(i) for each $x \in \Omega$, $ex = x$, where e is the identity element of G;

2020 Mathematics Subject Classification: 46E30, 47B37, 43A15
Keywords and phrases: Locally compact group; weighted Orlicz algebra; Young function; convolution; spaceablity; inclusion.

(ii) for each $s, t \in G$ and $x \in \Omega$, $s(tx) = (st)x$;

(iii) for each $s \in G$, the self-mapping $x \mapsto sx$ on Ω is Borel measurable and measure-preserving, that is, for each $s \in G$ and Borel subset $E \subseteq \Omega$, $sE := \{sx : x \in E\}$ is also a Borel subset of Ω and $\mu(sE) = \mu(E)$.
In this case, we simply say that G acts on (Ω, μ), and write $G \curvearrowright (\Omega, \mu)$.

Throughout, we assume that G is a locally compact group and (Ω, μ) is a Borel measure space with finite subset property. Also, we assume that $G \curvearrowright (\Omega, \mu)$. The set of all Borel measurable complex-valued functions on Ω is denoted by $\mathcal{M}_0$. Also, we denote the set of all nonnegative functions in $\mathcal{M}_0$ by $\mathcal{M}_0^+$.

Let G act on a locally compact Hausdorff space Ω, and $f : \Omega \to \mathbb{C}$. Then, for each $x \in \Omega$ and $y \in G$ we define $\mathrm{L}_y f(x) := f(yx)$.

We note that if $\Omega = G$ is a locally compact group, then G acts naturally on (G, λ) where λ is the left Haar measure of G.

Example 1.2. Let G be a locally compact group, and H be a closed subgroup of G such that $\Delta_G|_H = \Delta_H$, where Δ is the related modular function. Then G naturally acts on the quotient space G/H, and there is a Borel measure μ on G/H such that for each $x \in G$ and each Borel set $E \subseteq G/H$, we have $\mu(xE) = \mu(E)$. Indeed, $G \curvearrowright (G/H, \mu)$.

2. Weighted Orlicz type spaces

A convex mapping $\Phi : [0, \infty) \to [0, \infty)$ is called a *Young function* if $\Phi(0) = \lim_{x \to 0} \Phi(x) = 0$ and $\lim_{x \to \infty} \Phi(x) = \infty$. The *complementary* of a Young function Φ is defined by $\Psi(x) := \sup\{xy - \Phi(y) : y \geq 0\}$, $(x \geq 0)$. In this paper Φ is a Young function and Ψ is its complimentary. Also, we assume that $\Phi(x) = 0$ implies that $x = 0$. For every $x, y \geq 0$ we have the *Young's inequality*:

$$xy \leq \Phi(x) + \Psi(y). \tag{2}$$

In this paper, v and w are two continuous positive functions on Ω (called *weight* functions).

We write $w^{-1} := \frac{1}{w}$. In the sequel we assume that w, v satisfy the condition

$$M := \sup\left\{\int_\Omega \chi_F \, \mathrm{L}_y(wv) \, d\mu : y \in G \text{ and } F \subseteq \Omega \text{ with } \mu(F) < \infty\right\} < \infty. \tag{3}$$

Put $\quad \mathcal{S}_\Psi^{w,v} := \left\{h \in \mathcal{M}_0 : \int_\Omega \mathrm{L}_y(wv)\Psi(|h|) \, d\mu \leq 1 \quad \text{for all} \quad y \in G\right\}.$

Remark 2.1. (i) The relation (3) implies that the set $\mathcal{S}_\Psi^{w,v}$ is non-empty.

(ii) If $wv \in L^1(\mu)$, then easily (3) holds.

Because of the way we have defined $\mathcal{S}_\Psi^{w,v}$ by left translations, we have the following property:

$$h \in \mathcal{S}_{\Psi}^{w,v} \quad \text{if and only if} \quad \mathrm{L}_y h \in \mathcal{S}_{\Psi}^{w,v} \quad \text{for all} \quad y \in G. \tag{4}$$

This fact plays a key role in Section 3 while we discuss on convolution Banach algebras.

Definition 2.2. The space of all complex valued measurable functions f on Ω with $\|f\|_{\Phi,(w,v)}<\infty$, is denoted by $\mathcal{L}_{w,v}^{\Phi}(\Omega)$, where $\|f\|_{\Phi,(w,v)}:=\sup\left\{\int_\Omega |fh|w\,d\mu : h \in \mathcal{S}_{\Psi}^{w,v}\right\}$.

Remark 2.3. The above structure covers the classical weighted Orlicz spaces [5] because $\mathcal{L}_{w,\frac{1}{w}}^{\Phi}(\Omega) = L_w^{\Phi}(\Omega)$.

Theorem 2.4. *$\left(\mathcal{L}_{w,v}^{\Phi}(\Omega), \|\cdot\|_{\Phi,(w,v)}\right)$ is a Banach space.*

Proof. For each $f \in \mathcal{M}_0^+$ denote $\rho(f) := \sup\left\{\int_\Omega f|h|w\,d\mu : h \in \mathcal{S}_{\Psi}^{w,v}\right\}$. Clearly, $\rho(0) = 0$. Conversely, let $0 \neq f \in \mathcal{M}_0^+$ and $\rho(f) = 0$. Set $E := \{x \in \Omega : f(x) > 0\}$. Then, since μ has the finite subset property, there exists a subset F of E such that $0 < \mu(F) < \infty$. Now, since w, v satisfy (3), by properties of Φ, setting $\alpha := \max\{1, M\,\Psi(1)\}$ we have $h_0 := \frac{\chi_F}{\alpha} \in \mathcal{S}_{\Psi}^{w,v}$. This implies that $\int_\Omega f h_0 w\,d\mu = \frac{1}{\alpha}\int_F fw\,d\mu = 0$ and so $\mu(F) = 0$. This contradiction shows that $f = 0$ a.e. Moreover, easily ρ is subadditive and for each $a \geq 0$, $\rho(af) = a\rho(f)$. Also, for each increasing sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{M}_0^+$, if $f_n \uparrow f$, by Monoton Convegence Theorem we have $\rho(f_n) \uparrow \rho(f)$. So, by [11, Section 30, Theorem 2] $\left(\mathcal{L}_{w,v}^{\Phi}(\Omega), \|\cdot\|_{\Phi,(w,v)}\right)$ is a Banach space. □

The following definition gives a version of Luxembourg norm for $\mathcal{L}_{w,v}^{\Phi}(\Omega)$ (see [6, page 54] for more details).

Definition 2.5. For each $f \in \mathcal{M}_0$ we define

$$\mathcal{N}_{\Phi}^{w,v}(f) := \inf\left\{\lambda > 0 : \frac{f}{\lambda} \in \mathcal{S}_{\Phi}^{w,v}\right\} = \inf\left\{\lambda > 0 : \sup_{y\in G}\int_\Omega \mathrm{L}_y(wv)\Phi(\frac{|f|}{\lambda})\,d\mu \leq 1\right\}.$$

Since Φ is increasing, we have

$$\mathcal{N}_{\Phi}^{w,v}(f) \leq 1 \quad \text{if and only if} \quad f \in \mathcal{S}_{\Phi}^{w,v}. \tag{5}$$

Also,

$$\text{if} \quad \mathcal{N}_{\Phi}^{w,v}(f) \neq 0, \quad \text{then} \quad \int_\Omega \mathrm{L}_y(wv)\Phi(\frac{|f|}{\mathcal{N}_{\Phi}^{w,v}(f)})\,d\mu \leq 1 \ \text{ for all } \ y \in G. \tag{6}$$

Proposition 2.6. *For each $f, g \in \mathcal{M}_0$ we have*
(i) Hölder's inequality

$$\int_\Omega \mathrm{L}_y(wv)|fg|\,d\mu \leq 2\,\mathcal{N}_{\Phi}^{w,v}(f)\mathcal{N}_{\Psi}^{w,v}(g) \tag{7}$$

for all $y \in G$.

(ii) If $v \geq 1$, then $\|f\|_{\Phi,(w,v)} \leq 2\mathcal{N}_{\Phi}^{w,v}(f)$.

Proof. (i) If $\mathcal{N}_\Phi^{w,v}(f) = 0$ or $\mathcal{N}_\Psi^{w,v}(g) = 0$, then both sides of the inequality (7) are zero. Otherwise, by the Young's inequality (2) for all $y \in G$ we have

$$\mathrm{L}_y(wv)\frac{|f|}{\mathcal{N}_\Phi^{w,v}(f)}\frac{|g|}{\mathcal{N}_\Psi^{w,v}(g)} \le \mathrm{L}_y(wv)\ \left[\Phi(\frac{|f|}{\mathcal{N}_\Phi^{w,v}(f)}) + \Psi(\frac{|g|}{\mathcal{N}_\Psi^{w,v}(g)})\right].$$

So by (6), $\displaystyle\int_\Omega \mathrm{L}_y(wv)\frac{|f|}{\mathcal{N}_\Phi^{w,v}(f)}\frac{|g|}{\mathcal{N}_\Psi^{w,v}(g)}\,d\mu \le 2$, and the proof is complete.

(ii) By $v \ge 1$, Hölder's inequality (7) and the relation (5) we have

$$\begin{aligned}\|f\|_{\Phi,(w,v)} &= \sup\left\{\int_\Omega |fg|w\,d\mu : g \in \mathcal{S}_\Psi^{w,v}\right\}\\ &\le 2\sup\{\mathcal{N}_\Phi^{w,v}(f)\mathcal{N}_\Psi^{w,v}(g) : \mathcal{N}_\Psi^{w,v}(g) \le 1\} = 2\mathcal{N}_\Phi^{w,v}(f).\end{aligned}$$ □

Next, we show that $\mathcal{N}_\Phi^{w,v}$ is a complete norm on $\mathcal{L}_{w,v}^\Phi(\Omega)$.

THEOREM 2.7. *Let $\rho : \mathcal{M}_0^+ \to [0,\infty]$ be defined by*

$$\rho(f) = \inf\left\{\lambda > 0 : \sup_{y\in G}\int_\Omega \mathrm{L}_y(wv)\Phi(\frac{f}{\lambda})\,d\mu \le 1\right\}.$$

Then, the followings hold.
(a) $\rho(f) = 0$ if and only if $f = 0$.

(b) $\rho(f_1 + f_2) \le \rho(f_1) + \rho(f_2)$ for all $f_1, f_2 \in \mathcal{M}_0^+$.

(c) $\rho(\alpha f) = \alpha\rho(f)$ for all $\alpha > 0$ and $f \in \mathcal{M}_0^+$.

(d) If $f_1, f_2 \in \mathcal{M}_0^+$ and $f_1 \le f_2$ μ-a.e., then $\rho(f_1) \le \rho(f_2)$.
If Φ is continuous, then
(e) If $\{f_n\}_{n=1}^\infty \subseteq \mathcal{M}_0^+$, $f \in \mathcal{M}_0^+$ and $f_n \longrightarrow f$ μ-a.e., then

$$\rho(f) \le \liminf_n \rho(f_n). \tag{8}$$

(f) If $\{f_n\}_{n=1}^\infty \subseteq \mathcal{M}_0^+$ and $f_n \uparrow f \in \mathcal{M}_0^+$ μ-a.e., then $\rho(f_n) \uparrow \rho(f)$.

Proof. (a) Clearly $\rho(0) = 0$. Conversely, let $f \in \mathcal{M}_0^+$ and $\rho(f) = 0$. Put $E := \{x \in \Omega : |f(x)| > 0\}$. In contrast, let $\mu(E) > 0$. Then, since μ has finite subset property, there exists a Borel set $F \subseteq E$ such that $0 < \mu(F) < \infty$. By the assumption $\rho(f) = 0$, we have

$$\begin{aligned}\int_\Omega \mathrm{L}_y(wv)\Phi(f\chi_F)\,d\mu &= \int_F \mathrm{L}_y(wv)\Phi(f)\,d\mu \le \lambda\int_F \mathrm{L}_y(wv)\Phi(\frac{f}{\lambda})\,d\mu\\ &\le \lambda\int_\Omega \mathrm{L}_y(wv)\Phi(\frac{f}{\lambda})\,d\mu \le \lambda\end{aligned}$$

for all $0 < \lambda < 1$ and $y \in G$. Therefore, $\|wv\Phi(f)\chi_F\|_1 = 0$ and so $\chi_F = 0$ μ a.e. or equivalently $\mu(F) = 0$, a contradiction. Hence, $f = 0$.

(b) Let $f_1, f_2 \in \mathcal{M}_0^+$. For each $\lambda_1, \lambda_2 > 0$ satisfying

$$\sup_{y\in G}\int_\Omega \mathrm{L}_y(wv)\Phi\left(\frac{f_i}{\lambda_i}\right) \le 1, \qquad (i = 1, 2) \tag{9}$$

since Φ is convex we have

$$\int_\Omega \mathrm{L}_y(wv)\Phi\Big(\frac{f_1+f_2}{\lambda_1+\lambda_2}\Big)\,d\mu \le \int_\Omega \mathrm{L}_y(wv)\left[\frac{\lambda_1}{\lambda_1+\lambda_2}\Phi\left(\frac{f_1}{\lambda_1}\right)+\frac{\lambda_2}{\lambda_1+\lambda_2}\Phi\left(\frac{f_2}{\lambda_2}\right)\right]d\mu$$
$$\le \frac{\lambda_1}{\lambda_1+\lambda_2}+\frac{\lambda_2}{\lambda_1+\lambda_2}=1,$$

for all $y\in G$. This implies that $\rho(f_1+f_2)\le\lambda_1+\lambda_2$. Taking infimum on λ_1 and λ_2 satisfying (9), we have $\rho(f_1+f_2)\le\rho(f_1)+\rho(f_2)$.

(c) and (d) are clear.

(e) Put $\eta:=\liminf_n \rho(f_n)$. If $f=0$ or $\eta=\infty$ then (8) holds. So let $f\neq 0$ and $\eta<\infty$. Without loss generality, we assume that $\rho(f_n)>0$ for all $n\in\mathbb{N}$.

By (a), there is a $\lambda_0>0$ such that for each $\lambda\le\lambda_0$ we have

$$\sup_{y\in G}\int_G \mathrm{L}_y(wv)\Phi(\frac{f}{\lambda})>1,\quad \text{and so}\quad 1<\liminf_n\left(\sup_{y\in G}\int_\Omega \mathrm{L}_y(wv)\Phi(\frac{f_n}{\lambda})\,d\mu\right),$$

thanks to Fatou's Lemma. Hence for each $n_0\in\mathbb{N}$ there exists some $n\ge n_0$ such that $1<\sup_{y\in G}\int_G \mathrm{L}_y(wv)\Phi(\frac{f_n}{\lambda})$, and so $\rho(f_n)\ge\lambda_0>0$.

In contrast, let $\eta=0$. Then, there is a subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ such that $\lim_{k\to\infty}\rho(f_{n_k})=0$. So there is $k_0\in\mathbb{N}$ such that $\rho(f_{n_k})\le 1$ for all $k\ge k_0$. So,

$$\frac{1}{\rho(f_{n_k})}\int_\Omega \mathrm{L}_y(wv)\Phi(f_{n_k})\,d\mu\le\int_\Omega \mathrm{L}_y(wv)\Phi(\frac{f_{n_k}}{\mathcal{N}_\Phi^{w,v}(f_{n_k})})\,d\mu\le 1 \tag{10}$$

for all $k\ge k_0$ and $y\in G$ thanks to convexity of Φ and (6). Using the Fatou's Lemma we have,

$$0\le\int_\Omega \mathrm{L}_y(wv)\Phi(f)\,d\mu\le\liminf_{k\to\infty}\int_\Omega \mathrm{L}_y(wv)\Phi(f_{n_k})\,d\mu\le\liminf_{k\to\infty}\rho(f_{n_k})=0$$

for all $y\in G$. Therefore, $\int_\Omega wv\Phi(f)\,d\mu=0$ and so $\Phi(f)=0$ μ-a.e. or equivalently $f=0$ μ-a.e., a contradiction. So, $0<\eta<\infty$. In this situation for each $\gamma>\eta$, $\rho(f_n)<\eta$ for sufficiently large n. Therefore for each $y\in G$,

$$\int_\Omega \mathrm{L}_y(wv)\Phi(\frac{f_n}{\gamma})\,d\mu\le\int_\Omega \mathrm{L}_y(wv)\Phi(\frac{f_n}{\rho(f_n)})\,d\mu\le 1. \tag{11}$$

From here, using the Fatou's Lemma and (11), we get

$$\sup_{y\in G}\int_\Omega \mathrm{L}_y(wv)\Phi(\frac{f}{\gamma})\,d\mu\le\sup_{y\in G}\left(\liminf_{n\to\infty}\int_\Omega \mathrm{L}_y(wv)\Phi(\frac{f_n}{\gamma})\,d\mu\right)\le 1.$$

So, by the definition of ρ, we have $\rho(f)\le\gamma$. This implies that $\rho(f)\le\eta=\liminf_{n\to\infty}\rho(f_n)$.

(f) Let $\{f_n\}_{n=1}^\infty\subseteq\mathcal{M}_0^+$ and $f_n\uparrow f\in\mathcal{M}_0^+$ μ-a.e. Then, by (d) and (e) we have $\rho(f)\le\liminf_{n\to\infty}\rho(f_n)\le\limsup_{n\to\infty}\rho(f_n)\le\rho(f)$. This completes the proof. □

Proposition 2.8. *Let Φ be a continuous Young function. Then, $\big(L_{w,v}^\Phi(\Omega),\mathcal{N}_\Phi^{w,v}\big)$ is a Banach space.*

Proof. Thanks to [11, Section 30, Theorem 2], the proof is directly obtained by the parts (a), (b), (c) and (f) of Theorem 2.7. □

Let $p > 0$. We define $\Phi_p(x) := x^p$ for each $x > 0$. Then, we denote $\mathcal{L}^p_{w,v} := \mathcal{L}^{\Phi_p}_{w,v}$. Easily, for each $f \in \mathcal{M}_0$ we have $\mathcal{N}^{w,v}_p(f) := \mathcal{N}^{w,v}_{\Phi_p}(f) = \sup_{y\in G}\left(\int_\Omega L_y(vw)\,|f|^p\,d\mu\right)^{\frac{1}{p}}$ for all $f \in \mathcal{L}^p_{(w,v)}$.

Next, we give some equivalent conditions for inclusion of these weighted Lebesgue type spaces. Note that the first condition is independent of the choice of p, q.

Theorem 2.9. *Let $G \curvearrowright (\Omega,\mu)$. Then, the following statements are equivalent:*
(i) $\inf\left\{\sup_{y\in G}\int_E L_y(vw)\,d\mu : E \in \mathcal{A},\, \mu(E) > 0\right\} > 0$.

(ii) For each $p > 0$, $\mathcal{L}^p_{w,v} \subseteq L^\infty(\Omega,\mu)$.

(iii) For each $p, q > 0$ with $p < q$ we have $\mathcal{L}^p_{w,v} \subseteq \mathcal{L}^q_{w,v}$.

Proof. (i) $\Longrightarrow$ (ii): Let the condition (i) hold. Let $p > 0$ and $f \in \mathcal{L}^p_{(w,v)}$. For each $n \in \mathbb{N}$ we put $E_n := \{x \in \Omega : |f(x)| > n\}$. Then, $n\,\chi_F E_n \le |f|$, and so

$$n \sup_{y\in G}\left(\int_{E_n} L_y(vw)\,d\mu\right)^{\frac{1}{p}} \le \mathcal{N}^{w,v}_p(f).$$

This implies that $\lim_{n\to\infty}\sup_{y\in G}\left(\int_{E_n} L_y(vw)\,d\mu\right) = 0$. Therefore, by (i), for some $n \in \mathbb{N}$ we have $\mu(E_n) = 0$ i.e. $f \in L^\infty(\Omega)$.

(ii) $\Longrightarrow$ (iii): Let (ii) hold. Let $0 < p < q$ and $f \in \mathcal{L}^p_{w,v}$. Then, by (ii), there is some $k > 0$ such that $|f| \le k$ a.e. Therefore,

$$\sup_{y\in G}\left(\int_\Omega L_y(vw)\,\frac{|f|^q}{k^q}\,d\mu\right) \le \sup_{y\in G}\left(\int_\Omega L_y(vw)\,\frac{|f|^p}{k^p}\,d\mu\right) < \infty,$$

because $0 \le \frac{|f|}{k} < 1$. This shows that $f \in \mathcal{L}^q_{w,v}$.

(iii) $\Longrightarrow$ (i) Let (iii) hold. Then, there exists a constant $C > 0$ such that for each $f \in \mathcal{L}^p_{w,v}$,

$$\mathcal{N}^{w,v}_q(f) \le C\,\mathcal{N}^{w,v}_p(f). \tag{12}$$

Assume that $E \in \mathcal{A}$, $\mu(E) > 0$ and $\sup_{y\in G} L_y(vw)\,d\mu < \infty$. So, $\chi_E \in \mathcal{L}^p_{w,v}$, and by the relation (12), $0 < C^{\frac{pq}{p-q}} \le \sup_{y\in G}\left(\int_E L_y(vw)\,d\mu\right)$, and this completes the proof. □

The next result is a weighted version of [10, Theorem 1] which can be concluded directly from Theorem 2.9 by taking $G := \{e\}$ and $v \equiv 1$.

Corollary 2.10. *The following conditions are equivalent.*
(i) $\mathcal{L}^p_w(\mu) \subseteq \mathcal{L}^q_w(\mu)$ for some p, q with $0 < p < q < \infty$.

(ii) $\inf\{\int_E w\,d\mu : E \in \mathcal{A}$ *with* $\mu(E) > 0\} > 0$.

(iii) $\mathcal{L}^p_w(\mu) \subseteq \mathcal{L}^q_w(\mu)$ for all p, q with $0 < p < q < \infty$.

3. Weighted Orlicz type convolution algebras

Let G be a locally compact group with a left Haar measure λ. In this section, we set $\Omega := G$ and consider the natural act of G on itself. The space $\mathcal{L}^{\Phi}_{w,v}(G)$ is called a *convolution Banach algebra* if there exists a constant $c > 0$ such that $f * g \in \mathcal{L}^{\Phi}_{w,v}(G)$ and $\|f * g\|_{\Phi,(w,v)} \le c\,\|f\|_{\Phi,(w,v)}\,\|g\|_{\Phi,(w,v)}$, for all $f, g \in \mathcal{L}^{\Phi}_{w,v}(G)$.

Next, we give some sufficient condition for a weighted Orlicz space $\big(\mathcal{L}^{\Phi}_{w,v}(G), \|\cdot\|_{\Phi,(w,v)}\big)$ to be a convolution Banach algebra.

Theorem 3.1. *Let (Φ, Ψ) be a complementary pair of Young functions and v, w be weight functions with w satisfying (3). If w is submultiplicative then $\mathcal{L}^{\Phi}_{w,v}(G)$ is an L^1_w-module.*

Proof. Let $f, g \in L^{\Phi}_{w,v}(G)$. Then since w is submultiplicative,

$$\begin{aligned}
\|f * g\|_{\Phi,(w,v)} &= \sup\left\{\int_G |f * g||h|w\, d\lambda(x) :\, h \in \mathcal{S}^{w,v}_{\Psi}\right\}\\
&\le \sup\left\{\int_G \int_G |f(y)||g(y^{-1}x)||h(x)|w(x)\, d\lambda(x) :\, h \in \mathcal{S}^{w,v}_{\Psi}\right\}\\
&= \sup\left\{\int_G |f(y)| \int_G |g(x)||\mathrm{L}_{y-1}h(x)|w(yx)\, d\lambda(x) :\, h \in \mathcal{S}^{w,v}_{\Psi}\right\}\\
&\le \sup\left\{\int_G |f(y)|w(y)(\int_G |g(x)||\mathrm{L}_{y-1}h(x)|w(x))\, d\lambda(x) :\, h \in \mathcal{S}^{w,v}_{\Psi}\right\}\\
&\le \|f\|_{1,w}\|g\|_{\Phi,(w,v)},
\end{aligned}$$

following from (4). □

Corollary 3.2. *Let (Φ, Ψ) be a complementary pair of Young functions and v, w be weight functions with w satisfying (3). If w is submultiplicative and $\mathcal{L}^{\Phi}_{w,v}(G) \subseteq L^1_w(G)$, then $\big(\mathcal{L}^{\Phi}_{w,v}(G), \|\cdot\|_{\Phi,(w,v)}\big)$ is a convolution Banach algebra.*

Proof. By Theorem 3.1, for all $f, g \in \mathcal{L}^{\Phi}_{w,v}(G)$ we have $\|f*g\|_{\Phi,(w,v)} \le \|f\|_{1,w}\|g\|_{\Phi,(w,v)}$. Now by the assumption $\mathcal{L}^{\Phi}_{w,v}(G) \subseteq L^1_w(G)$, there exists a $c > 0$ such that $\|f\|_{1,w} \le c\|f\|_{\Phi,(w,v)}$. So for all $f, g \in \mathcal{L}^{\Phi}_{w,v}(G)$ we have, $\|f * g\|_{\Phi,(w,v)} \le c\|f\|_{\Phi,(w,v)}\|g\|_{\Phi,(w,v)}$. Hence $\big(\mathcal{L}^{\Phi}_{w,v}(G), \|\cdot\|_{\Phi,(w,v)}\big)$ is a convolution Banach algebra. □

Definition 3.3. Let $\mathcal{E}$ be a topological vector space. We say that a relation $\sim$ on $\mathcal{E}$ has property (D) if the following conditions hold:
(i) If (x_n) is a sequence in $\mathcal{E}$ such that $x_n \sim x_m$ for all distinct index m, n, then for each finite subsets A, B of $\mathbb{N}$ we have $\sum_{n\in A} \alpha_n x_n \sim \sum_{m\in B} \beta_m x_m$, where α_n and β_m's are arbitrary scalars.

(ii) If a sequence (x_n) converges to x in $\mathcal{E}$ and for some $y \in \mathcal{E}$, $x_n \sim y$ for all $n \in \mathbb{N}$, then $x \sim y$.

We recall the following result from [1]. Note that a subset S of a Banach space X is called *spaceable* if the set $S \cup \{0\}$ contains a closed infinite-dimensional subspace of X.

Theorem 3.4. *Let $(\mathcal{E}, \|\cdot\|)$ be a Banach space, $\sim$ be a relation on $\mathcal{E}$ with property (D), and B be a nonempty subset of $\mathcal{E}$. Assume that:*
(i) there is a constant $k>0$ such that $\|x+y\| \geq k\,\|x\|$ for all $x, y \in \mathcal{E}$ with $x \sim y$;

(ii) B is a cone;

(iii) if $x, y \in \mathcal{E}$ such that $x+y \in B$ and $x \sim y$ then $x, y \in B$;

(iv) there is an infinite sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{E}-B$ such that for each distinct $m, n \in \mathbb{N}$, $x_m \sim x_n$.
Then, $\mathcal{E}-B$ is spaceable in $\mathcal{E}$.

Theorem 3.5. *Let G be a compactly generated non-compact abelian group. Let Φ be a Young function such that for two nonnegative sequences (α_n) and (β_n) we have*

$$\sum_{n=1}^{\infty} \Phi(\alpha_n) < \infty, \quad \sum_{n=1}^{\infty} \Phi(\beta_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty. \tag{13}$$

Then, the set $\{(f,g) : |f| * |g| \notin \mathcal{L}^{\Phi}_{w,v}\}$ is spaceable in $\mathcal{L}^{\Phi}_{w,v} \times \mathcal{L}^{\Phi}_{w,v}$.

Proof. We consider the cone $B := \{(f,g) \in \mathcal{L}^{\Phi}_{w,v} \times \mathcal{L}^{\Phi}_{w,v} : |f| * |g| \in \mathcal{L}^{\Phi}_{w,v}\}$. For each $f_i, g_i \in \mathcal{L}^{\Phi}_{w,v}$ $(i=1,2)$ we say that $(f_1,g_1) \sim (f_2,g_2)$ whenever $\lambda(\sigma(f_1) \cap \sigma(f_2)) = \lambda(\sigma(g_1) \cap \sigma(g_2)) = 0$, where $\sigma(h) := \{x \in \Omega : h(x) \neq 0\}$ for each function $h : G \to \mathbb{C}$. One can easily see that the relation $\sim$ satisfies Definition 3.3 and by solidity, the conditions (1) and (3) in Theorem 3.4 hold. Let U be a compact symmetric neighborhood of e in G and V be a compact symmetric neighborhood of e with $VV \subseteq U$. By [3, 9.26(b)] and [2, Lemma 1.1] there are an element $a \in G$ and some $k \in \mathbb{N}$ such that for each $n \geq k$, $U \cap Ua^n = \varnothing$. By (13), there is an infinite partition $\mathcal{P}$ of $\mathbb{N}$ whose elements are infinite subsets of $\mathbb{N}$ such that for each $N \in \mathcal{P}$, $\sum_{n \in N} \Phi(\alpha_n) < \frac{1}{M\,\lambda(V)}$ and $\sum_{n\in N} \Phi(\beta_n) < \frac{1}{M\,\lambda(VV)}$, where M is defined as (3). Define $f_N(x) := \sum_{n \in N} \alpha_n \chi_{Va^{-nk}}(x)$ and $g(x) := \sum_{n\in N} \beta_n \chi_{VVa^{nk}}(x)$, for all $x \in G$. So, for each $y \in G$,

$$\begin{aligned}
&\int_G \mathrm{L}_y(wv)(x)\Phi(f_N(x))\,d\lambda(x) = \int_{\bigcup_{n\in N} Va^{-nk}} \mathrm{L}_y(wv)(x)\,\Phi(f_N(x))\,d\lambda(x) \\
&= \sum_{n \in N} \int_{Va^{-nk}} \mathrm{L}_y(wv)(x)\,\Phi(f_N(x))\,d\lambda(x) = \sum_{n\in N} \int_{Va^{-nk}} \mathrm{L}_y(wv)(x)\,\Phi(\alpha_n)\,d\lambda(x) \\
&= M\,\lambda(V) \sum_{n \in N} \Phi(\alpha_n) < 1.
\end{aligned}$$

Similarly, $\sup_{y\in G} \int_G \mathrm{L}_y(wv)(x)\,\Phi(g_N(x))\,d\lambda(x) = M\,\lambda(VV) \sum_{n\in N} \Phi(\beta_n) < 1$. So, $f, g \in \mathcal{L}^{\Phi}_{w,v}(G)$. But, for each $x \in V$,

$$(f_N * g_N)(x) = \sum_{n\in N} \alpha_n \int_{Va^{-nk}} \sum_{m \in N} \beta_m \chi_{VVa^{mk}}(y^{-1}x)\,d\lambda(y)$$

$$= \sum_{n\in N} \alpha_n \int_{Va^{-nk}} \beta_n \, d\lambda(y) = \lambda(V) \sum_{n\in N} \alpha_n\beta_n = \infty.$$

This implies that $\{(f_N, g_N)\}_{N\in\mathcal{P}}$ is an infinite sequence in $(\mathcal{L}^{\Phi}_{w,v} \times \mathcal{L}^{\Phi}_{w,v}) - B$. Trivially, for each distinct $N, N' \in \mathcal{P}$ we have $(f_N, g_N) \sim (f_{N'}, g_{N'})$. Therefore, by Theorem 3.4 the proof is complete. □

Remark 3.6. Following the same proof, one can see that the above result holds if we replace the norm $\|\cdot\|_{\Phi,(w,v)}$ by $\mathcal{N}^{w,v}_{\Phi}$.

Since the Young function Φ_p for each $p \geq 1$ satisfies the sequence condition (13) in Theorem 3.5, we deduce the next result.

Corollary 3.7. *Let G be a compactly generated non-compact abelian group and $p \geq 1$. Then, for each $p > 1$, the set $\{(f,g) : |f| * |g| \notin \mathcal{L}^p_{w,v}\}$ is spaceable in $\mathcal{L}^p_{w,v} \times \mathcal{L}^p_{w,v}$.*

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(received 03.11.2021; in revised form 19.01.2022; available online 12.09.2022)

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