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THE σ-POINT-FINITE cn-NETWORKS (ck-NETWORKS) OF PIXLEY-ROY HYPERSPACES

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Abstract. In this paper, we study the relation between a space X satisfying certain generalized metric properties and the Pixley-Roy hyperspace $\mathcal{F}[X]$ over X satisfying the same properties. We prove that if X has a σ-point-finite cn-network (resp., ck-network), then $\mathcal{F}[X]$ also has a σ-point-finite cn-network (resp., ck-network).

1. Introduction and preliminaries

The generalized metric properties on Pixley-Roy hyperspaces have been studied by many authors [1, 2, 5–10]. They considered several generalized metric properties and studied the relation between a space X satisfying such property and its Pixley-Roy hyperspaces satisfying the same property.

Throughout this paper, all spaces are assumed to be T_1 and regular, $\mathbb{N}$ denotes the set of all positive integers. Moreover, if $\mathcal{P}$ is a family of subsets of a space X and $G \subset X$, denote $(\mathcal{P})_G = \{P \in \mathcal{P} : P \cap G \neq \emptyset\}$.

The *Pixley-Roy hyperspace* $\mathcal{F}[X]$ over a space X, defined by C. Pixley and P. Roy in [8], is the set of all non-empty finite subsets of X with the topology generated by the sets of the form $[F, V] = \{G \in \mathcal{F}[X] : F \subset G \subset V\}$, where $F \in \mathcal{F}[X]$ and V is an open subset in X containing F. It is known that $\mathcal{F}[X]$ is always zero-dimensional, completely regular (see [2]).

For each $F \in \mathcal{F}[X]$ and $A \subset X$, denote $[F, A] = \{H \in \mathcal{F}[X] : F \subset H \subset A\}$.

Definition 1.1. A family $\mathcal{P}$ of subsets of a space X is said to be:
(i) a *network* [3] for X, if for any neighborhood U of a point $x \in X$, there exists a set $P \in \mathcal{P}$ such that $x \in P \subset U$.

(ii) a *cn-network* [4] for X, if for any neighborhood U of a point $x \in X$, the set $\bigcup\{P \in \mathcal{P} : x \in P \subset U\}$ is a neighborhood of x.

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(iii) a *ck-network* [4] for X, if for any neighborhood U of a point $x \in X$, there is a neighborhood U_x of x such that for each compact subset $K \subset U_x$, there exists a finite subfamily $\mathcal{F} \subset \mathcal{P}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset U$.

(iv) *point-finite* [3], if the family $\{P \in \mathcal{P} : x \in P\}$ is finite for each $x \in X$.

(v) *σ-point-finite*, if $\mathcal{P}$ can be expressed as $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ is point-finite, and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$.

Remark 1.2 ([4]). *ck*-networks $\Longrightarrow$ *cn*-networks $\Longrightarrow$ networks.

In this paper, we study the relation between a space X satisfying certain generalized metric properties and the Pixley-Roy hyperspace $\mathcal{F}[X]$ over X satisfying the same properties. We prove that if X has a σ-point-finite *cn*-network (resp., *ck*-network), then $\mathcal{F}[X]$ also has a σ-point-finite *cn*-network (resp., *ck*-network).

2. Main results

Theorem 2.1. *Let X be a space. If X has a σ-point-finite cn-network (resp., ck-network), then so does $\mathcal{F}[X]$.*

Proof. Assume that $\mathcal{P} = \bigcup\{\mathcal{P}_k : k \in \mathbb{N}\}$ is a *cn*-network (resp., *ck*-network) for X, where each $\mathcal{P}_k$ is point-finite and $\mathcal{P}_k \subset \mathcal{P}_{k+1}$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, if we put

$$\mathfrak{P}_k = \Big\{[F, \bigcup \mathcal{H}] : F \in \mathcal{F}[X] \text{ and } \mathcal{H} \text{ is a finite subfamily of } (\mathcal{P}_k)_F\Big\},$$

then $\mathfrak{P}_k \subset \mathfrak{P}_{k+1}$. Moreover, $\mathfrak{P}_k$ is point-finite for $\mathcal{F}[X]$ for each $k \in \mathbb{N}$. Indeed, for each $k \in \mathbb{N}$, let $F \in \mathcal{F}[X]$ and $\mathcal{W} \in \mathfrak{P}_k$ such that $F \in \mathcal{W}$. Then there exist $G \in \mathcal{F}[X]$ and a finite subfamily $\mathcal{H} \subset (\mathcal{P}_k)_G$ such that $\mathcal{W} = [G, \bigcup \mathcal{H}]$ and $G \subset F \subset \bigcup \mathcal{H}$.

Since $\mathcal{H} \subset (\mathcal{P}_k)_G$, we have $\mathcal{H} \subset (\mathcal{P}_k)_F$. This implies that

$$\{\mathcal{W} \in \mathfrak{P}_k : F \in \mathcal{W}\} \subset \{[G, \bigcup \mathcal{H}] : G \subset F,\ \mathcal{H} \text{ is a finite subfamily of } (\mathcal{P}_k)_F\}.$$

Since $\mathcal{P}_k$ is point-finite for X and the set F is finite, $(\mathcal{P}_k)_F$ is finite and the set $\{G : G \subset F\}$ is finite. Hence, $\{[G, \bigcup \mathcal{H}] : G \subset F,\ \mathcal{H}$ is a finite subfamily of $(\mathcal{P}_k)_F\}$ is finite. Therefore, $\{\mathcal{W} \in \mathfrak{P}_k : F \in \mathcal{W}\}$ is finite. This proves that $\mathfrak{P}_k$ is point-finite for $\mathcal{F}[X]$. Therefore, $\mathfrak{P} = \bigcup\{\mathfrak{P}_k : k \in \mathbb{N}\}$ is a σ-point-finite family for $\mathcal{F}[X]$.

Next, we prove that

$$\mathfrak{P} = \Big\{[F, \bigcup \mathcal{H}] : F \in \mathcal{F}[X] \text{ and } \mathcal{H} \text{ is a finite subfamily of } (\mathcal{P})_F\Big\}.$$

It is clear that $\mathfrak{P} \subset \{[F, \bigcup \mathcal{H}] : F \in \mathcal{F}[X]$ and $\mathcal{H}$ is a finite subfamily of $(\mathcal{P})_F\}$. Now, take any $\mathcal{W} \in \{[F, \bigcup \mathcal{H}] : F \in \mathcal{F}[X]$ and $\mathcal{H}$ is a finite subfamily of $(\mathcal{P})_F\}$. Then there exist $F \in \mathcal{F}[X]$ and a finite subfamily $\mathcal{H} = \{P_i : i \leq s\}$ of $(\mathcal{P})_F$ such that $\mathcal{W} = [F, \bigcup \mathcal{H}]$. Since $\mathcal{P} = \bigcup\{\mathcal{P}_k : k \in \mathbb{N}\}$, there exists $k_i \in \mathbb{N}$ such that $P_i \in (\mathcal{P}_{k_i})_F$ for each $i \leq s$. If we put $m = \max\{k_i : i \leq s\}$, then $P_1, \ldots, P_s \in (\mathcal{P}_m)_F$. This implies that $\mathcal{W} \in \mathfrak{P}_m \subset \mathfrak{P}$.

Finally, let $F \in \mathcal{F}[X]$ and $\mathcal{U}$ be an open neighborhood of F in $\mathcal{F}[X]$. Then there is an open set V in X such that $F \in [F, V] \subset \mathcal{U}$.

(1) Let $\mathcal{P}$ be a cn-network for X. Put $\mathcal{Q} = \{P \in (\mathcal{P})_F : P \subset V\}$. Then $\bigcup \mathcal{Q}$ is a neighborhood of F in X. This implies that there exists W open in X such that $F \subset W \subset \bigcup \mathcal{Q}$. Thus, $F \in [F, W] \subset [F, \bigcup \mathcal{Q}]$. Moreover, we have

$$[F, \bigcup \mathcal{Q}] \subset \bigcup\{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}.$$

Indeed, suppose that $H \in [F, \bigcup \mathcal{Q}]$, then $F \subset H \subset \bigcup \mathcal{Q}$. Since the set H is finite, there is a finite subfamily $\mathcal{G} \subset \mathcal{Q}$ such that $H \subset \bigcup \mathcal{G}$. On the other hand, since $\bigcup \mathcal{G} \subset \bigcup \mathcal{Q} \subset V$, it shows that $[F, \bigcup \mathcal{G}] \subset [F, V] \subset \mathcal{U}$. Furthermore, since $\mathcal{G} \subset \mathcal{Q} \subset (\mathcal{P})_F$, we have $[F, \bigcup \mathcal{G}] \in \{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}$. Since $H \in [F, \bigcup \mathcal{G}]$, this implies that $H \in \bigcup\{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}$.

Therefore, $F \in [F, W] \subset [F, \bigcup \mathcal{Q}] \subset \bigcup\{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}$.

Since the set $[F, W]$ is open in $\mathcal{F}[X]$, we conclude that $\bigcup\{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}$ is a neighborhood of F in $\mathcal{F}[X]$. This shows that $\mathfrak{P}$ is a cn-network for $\mathcal{F}[X]$.

(2) Suppose that $\mathcal{P}$ is a ck-network for X. Then for each $x \in F$, there exists a neighborhood V_x of x such that $V_x \subset V$ and for each compact subset $A_x \subset V_x$, there exists a finite subfamily $\mathcal{A}_x$ of $\mathcal{P}$ satisfying $x \in \bigcap \mathcal{A}_x$ and $A_x \subset \bigcup \mathcal{A}_x \subset V$. For each $x \in F$, since X is regular, there is an open set O_x in X such that $x \in O_x \subset \overline{O}_x \subset V_x$. Put $\mathcal{V}_F = [F, \bigcup_{x \in F} O_x]$, then for each compact subset $\mathcal{K} \subset \mathcal{V}_F$, we have $\bigcup \mathcal{K} \subset \bigcup_{x \in F} \overline{O}_x$.

Claim $\bigcup \mathcal{K}$ is compact in X.

In fact, take any open cover $\mathcal{L}$ of $\bigcup \mathcal{K}$ in X. Then for each $F \in \mathcal{K}$, we have that $F \subset \bigcup \mathcal{K} \subset \bigcup \mathcal{L}$. Hence, for each $x \in F$, there exists $U_x \in \mathcal{L}$ such that $x \in U_x$. Since $F \in [F, \bigcup_{x \in F} U_x]$ for each $F \in \mathcal{K}$, $\mathfrak{U} = \{[F, \bigcup_{x \in F} U_x] : F \in \mathcal{K}\}$ is an open cover of $\mathcal{K}$ in $\mathcal{F}[X]$. On the other hand, since $\mathcal{K}$ is compact, there exists a finite subfamily $\mathfrak{V}$ of $\mathfrak{U}$ such that $\mathcal{K} \subset \bigcup \mathfrak{V}$. This implies that there exist $F_1, \ldots, F_m \in \mathcal{K}$ such that $\mathfrak{V} = \Big\{[F_1, \bigcup_{x \in F_1} U_x], \ldots, [F_m, \bigcup_{x \in F_m} U_x]\Big\}$. Put $\mathcal{V} = \{U_x : x \in F_i, i \leq m\}$. Since each set F_i is finite, $\mathcal{V}$ is a finite subfamily of $\mathcal{L}$. Thus, we only need to prove that $\bigcup \mathcal{K} \subset \bigcup \mathcal{V}$. Indeed, let $z \in \bigcup \mathcal{K}$, then $z \in A$ for some $A \in \mathcal{K}$. This implies that there exists $i \leq m$ such that $A \in [F_i, \bigcup_{x \in F_i} U_x]$, hence $A \subset \bigcup_{x \in F_i} U_x$. Therefore, $z \in U_x \subset \bigcup \mathcal{V}$ for some $x \in F_i$.

By **Claim**, $K_x = (\bigcup \mathcal{K}) \cap \overline{O}_x$ is compact in X and $K_x \subset V_x$ for each $x \in F$. This implies that there is a finite subfamily $\mathcal{F}_x \subset \mathcal{P}$ such that $x \in \bigcap \mathcal{F}_x$ and $K_x \subset \bigcup \mathcal{F}_x \subset V$. Suppose that $\mathcal{H} = \bigcup_{x \in F} \mathcal{F}_x$ and $\mathfrak{F} = \{[F, \bigcup \mathcal{G}] \neq \emptyset : \mathcal{G} \subset \mathcal{H}\}$. Then $F \in \bigcap \mathfrak{F}$ and $\bigcup \mathfrak{F} \subset [F, V]$. On the other hand, since the set F is finite and $\mathcal{H} \subset (\mathcal{P})_F$, the family $\mathfrak{F}$ is a finite subfamily of $\mathfrak{P}$. Furthermore, we have $\mathcal{K} \subset \bigcup \mathfrak{F}$. In fact, take any $H \in \mathcal{K}$, then $H \subset \bigcup \mathcal{K}$. For each $y \in H$, since $\bigcup \mathcal{K} = \bigcup_{x \in F} K_x$, there exists $x_y \in F$ such that $y \in K_{x_y} \subset \bigcup \mathcal{F}_{x_y}$.

Put $\mathcal{G} = \bigcup_{y \in H} \mathcal{F}_{x_y}$. Then since $H \in \mathcal{K} \subset \mathcal{V}_F$, $F \subset H$. This implies that $H \in [F, \bigcup \mathcal{G}] \in \mathfrak{F}$. Hence, $H \in \bigcup \mathfrak{F}$. It shows that $\mathcal{K} \subset \bigcup \mathfrak{F}$. Thus, $\mathcal{K} \subset \bigcup \mathfrak{F} \subset [F, V] \subset \mathcal{U}$. Therefore, $\mathfrak{P}$ is a ck-network for $\mathcal{F}[X]$. $\square$

Question: Let X be a space. If $\mathcal{F}[X]$ has a σ-point-finite cn-network (resp., ck-network), then does X have a σ-point-finite cn-network (resp., ck-network)?

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