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SOME PINCHING RESULTS FOR STATISTICAL SUBMANIFOLDS IN COSYMPLECTIC STATISTICAL MANIFOLDS

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Abstract. In this article, we discuss the curvature properties of statistical submanifolds in cosymplectic statistical manifolds with constant curvature. We also establish some pinching results for such submanifolds and hypersurfaces in cosymplectic statistical manifolds having constant curvature. As an application of the main result we also obtain an obstruction condition for such immersion.

1. Introduction

In 1993, B. Y. Chen [6] established the simple relationships between the invariants namely, the main extrinsic invariants and the main intrinsic invariants of submanifolds. The development of such relations is one of the most interesting fields of research in differential geometry. B. Y. Chen established bound for squared mean curvature $\|H\|^2$ in terms of the intrinsic invariant δ_M for submanifold M of a real space form $\widetilde{M}(c)$. This inequality also holds good in case of anti-invariant submanifold of complex space form [10]. Motivated by this result, a similar inequality is also obtained for C-totally real submanifolds by taking Sasakian space form as an ambient space [11]. The starting work of Chen revolves around the development of inequalities among squared mean curvature, sectional curvature and scalar curvature of a submanifold in a real space form. He also obtained the inequalities between the squared mean curvature, the shape operator and k-Ricci curvature for the submanifolds in the real space forms [9]. After that many geometers obtained similar inequalities for different submanifolds and ambient spaces [2, 3, 7, 8, 14, 17, 18].

Aydin et al. [5] derived a Chen-Ricci inequality for statistical submanifolds of a statistical manifold of constant curvature. Mihai and Mihai [19] established a Chen-Ricci inequality with respect to a sectional curvature of the ambient Hessian manifold.

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Recently, M. Aquib [1] obtained the inequality for statistical submanifolds of quaternion Kaehler-like statistical space forms.

Here, our aim is to derive Chen-Ricci inequality for statistical submanifolds in cosymplectic manifolds having constant curvature. We also obtain a non-existence result as an application of the obtained result.

2. Statistical manifolds and statistical submanifolds

In 1987, the notation of statistical manifolds was introduced by Lauritzen [15].

Definition 2.1. A statistical manifold is a triple $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ formed of a Riemannian manifold $(\tilde{M}, \tilde{g})$ and a torsion free connection subject to the following identity $(\tilde{\nabla}_E \tilde{g})(F, G) = (\tilde{\nabla}_F \tilde{g})(E, G)$, for $E, F, G \in \Gamma(T\tilde{M})$.

Given a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$, the g-dual connection of $\tilde{\nabla}$, namely $\tilde{\nabla}^*$ is defined by the following identity $\tilde{g}(\tilde{\nabla}^*_E F, G) = E\tilde{g}(F, G) - \tilde{g}(F, \tilde{\nabla}^*_E G)$, $\forall E, F, G \in \Gamma(TM)$. It is easy to check that $\tilde{\nabla}^*$ is torsion free, too, and that $(\tilde{M}, \tilde{g}, \tilde{\nabla}^*)$ is a statistical structure. The Levi Civita connection of $(\tilde{M}, \tilde{g})$, namely $\tilde{\nabla}^\circ$ is linked with $(\tilde{\nabla}, \tilde{\nabla}^*)$ in a following manner: $\tilde{\nabla}^\circ = \frac{1}{2}\big(\tilde{\nabla} + \tilde{\nabla}^*\big)$.

Let $\tilde{R}$ and $\tilde{R}^*$ be Riemannian curvature tensor fields with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively. Then we have [20]:

$$\begin{aligned}\tilde{g}(\tilde{R}(E,F)G,W) =& \tilde{g}(R(E,F)G,W)\\ &+ \tilde{g}(h(E,G),h^*(F,W)) - \tilde{g}(h^*(E,W),h(F,G)),\end{aligned} \tag{1}$$

and
$$\begin{aligned}\tilde{g}(\tilde{R}^*(E,F)G,W) =& \tilde{g}(R^*(E,F)G,W)\\ &+ \tilde{g}(h^*(E,G),h(F,W)) - \tilde{g}(h(E,W),h^*(F,G)),\end{aligned} \tag{2}$$

where h and h^* are second fundamental forms for $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

The relation between second fundamental forms h and the shape operators A are given as

$$\begin{cases}\tilde{g}(A_N E, F) = \tilde{g}(h(E,F), N),\\ \tilde{g}(A^*_N E, F) = \tilde{g}(h^*(E,F), N),\end{cases} \tag{3}$$

for any $N \in \Gamma(T\tilde{M}^\perp)$ and $E, F \in \Gamma(T\tilde{M})$.

Let K be a difference (1,2)-tensor on a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then [13],

$$K_E F = K(E,F) = \tilde{\nabla}_E F - \tilde{\nabla}^\circ_E F, \quad K_E F = K_F E \quad \text{and} \quad \tilde{g}(K_E F, G) = \tilde{g}(F, K_E G).$$

Definition 2.2 ([16]). A cosymplectic statistical manifold is a $(2m+1)$-dimensional manifold $\tilde{M}$ carrying a quintuple $(\tilde{g}, \tilde{\nabla}, \phi, \xi, \eta)$ where $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is a statistical manifold, ϕ is (1-1)-tensor on $\tilde{M}$, η is a differential 1-form, ξ is a vector field. These data are subject to the following requirements:

$$\phi^2 E = -E + \eta(E)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \tilde{g}(\phi E, F) + \tilde{g}(E, \phi F) = 0, \quad \tilde{\nabla}^\circ_E \phi = 0.$$

The structure $(\tilde{g}, \phi, \xi, \eta)$ is called a cosymplectic structure on $\tilde{M}$, and for any $E, F \in \chi(\tilde{M})$, $K_E\phi F + \phi K_E F = 0$.

For a statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ the statistical curvature tensor field $\tilde{S}$ is defined as [13]

$$\tilde{S}(E,F)G = \frac{1}{2}\{\tilde{R}(E,F)G + \tilde{R}^*(E,F)G\}. \tag{4}$$

Definition 2.3 ([16]). A cosymplectic statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, \phi, \xi)$ is said to be of constant ϕ-sectional curvature c if

$$\begin{aligned}\tilde{S}(E,F)G =& \frac{c}{4}\{\tilde{g}(F,G)E - \tilde{g}(E,G)F + \tilde{g}(E,\phi G)\phi F - \tilde{g}(F,\phi G)\phi E + 2\tilde{g}(E,\phi F)\phi G \\ &+ \eta(E)\eta(G)F - \eta(F)\eta(G)E + \tilde{g}(E,G)\eta(F)\xi - \tilde{g}(F,G)\eta(E)\xi\},\end{aligned} \tag{5}$$

holds for any $E, F, G \in \chi(\tilde{M})$.

Suppose that $\{e_1, \ldots, e_{n+1}{=}\xi\}$ is an orthonormal basis of T_xM and $\{e_{n+2}, \ldots, e_{2m+1}\}$ is an orthonormal basis of $T_x^{\perp}M$. Then, the mean curvature vector fields $\vec{H}(x)$, $\vec{H}^*(x)$, $\vec{H}^\circ(x)$ are given by

$$\begin{cases}\vec{H}(x) = \frac{1}{n+1}\sum_{\alpha=1}^{n+1} h(e_\alpha, e_\alpha),\\ \vec{H}^*(x) = \frac{1}{n+1}\sum_{\alpha=1}^{n+1} h^*(e_\alpha, e_\alpha),\\ \vec{H}^\circ(x) = \frac{1}{n+1}\sum_{\alpha=1}^{n+1} h^\circ(e_\alpha, e_\alpha).\end{cases} \tag{6}$$

We also set

$$\begin{cases}\|h\|^2 = \sum_{\alpha,\beta=1}^{n+1} \tilde{g}(h(e_\alpha, e_\beta), h(e_\alpha, e_\beta)),\\ \|h^*\|^2 = \sum_{\alpha,\beta=1}^{n+1} \tilde{g}(h^*(e_\alpha, e_\beta), h^*(e_\alpha, e_\beta)),\\ \|h^\circ\|^2 = \sum_{\alpha,\beta=1}^{n+1} \tilde{g}(h^\circ(e_\alpha, e_\beta), h^\circ(e_\alpha, e_\beta)).\end{cases} \tag{7}$$

Here, it is important to remark that a submanifold is a minimal submanifold if $\vec{H}^\circ(x) = 0$ (resp. $\vec{H}(x) = 0$, or $\vec{H}^*(x) = 0$).

If we consider a plane section $\pi \subset T_pM$ at a point p on a Riemannian manifold M and if $K(\pi)$ denotes the sectional curvature of M, then the scalar curvature τ at p is defined by $\tau(p) = \sum_{1\leq\alpha<\beta\leq n+1} K(e_\alpha \wedge e_\beta)$, for $\{e_1, \ldots, e_{n+1}\}$ as the orthonormal basis of T_pM and $\{e_{n+2}, \ldots, e_{2m+1}\}$ as the orthonormal basis of $T_p^{\perp}M$.

The normalized scalar curvature ρ is defined as $\rho = \frac{2\tau}{n(n+1)}$. We also put $h_{\alpha\beta}^{\gamma} = \tilde{g}(h(e_\alpha, e_\beta), e_\gamma)$, $h_{\alpha\beta}^{*\gamma} = \tilde{g}(h^*(e_\alpha, e_\beta), e_\gamma)$, $\alpha, \beta \in 1, \ldots, n+2, \gamma \in \{n+2, \ldots, 2m+1\}$.

3. Statistical hypersurfaces

Let us consider any two statistical manifolds (M, g, ∇) and $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then, an immersion $f : M \to \tilde{M}$ is called a statistical immersion if the statistical structure (∇, g) is the induced statistical structure by f from $(\tilde{g}, \tilde{\nabla})$ and it satisfies [12] $g = f^*\tilde{g}$, $g(\nabla_E F, G) = \tilde{g}(\tilde{\nabla}_E f_* F, f_* G)$. Further, if we consider such immersion of codimension one and $\varsigma \in \Gamma(f^*T\tilde{M})$ is the unit normal vector field of f, then from [12] we have the following Gauss and Weingarten formulas:

$$\begin{cases} \tilde{\nabla}_E f_* F = f_* \nabla_E F + h(E,F)\varsigma, \\ \tilde{\nabla}^*_E f_* F = f_* \nabla_E F + h^*(E,F)\varsigma, \end{cases}$$

$$\begin{cases} \tilde{\nabla}_E \varsigma = -f_* A^* E + \tau^*(E)\varsigma, \\ \tilde{\nabla}^*_E \varsigma = -f_* A E + \tau(E)\varsigma, \end{cases}$$

where $h, h^* \in \Gamma(TM^{(0,2)})$, $A, A^* \in \Gamma(TM^{(1,1)})$ and $\tau, \tau^* \in \Gamma(TM^*)$ satisfy

$$\begin{cases} h(E,F) = g(AE,F) \\ h^*(E,F) = g(A^*E,F), \\ \tau(E) + \tau^*(E) = 0, \end{cases}$$

for any $E, F \in \Gamma(TM)$.

Denote by $\tilde{R}, \tilde{R}^*$, R and R^* the curvature tensor fields of the connections $\tilde{\nabla}, \tilde{\nabla}^*$, ∇ and ∇^*, respectively.

We recall the following results for later use.

Proposition 3.1 ([4]). *Consider a statistical submanifold (M, g, ∇) of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Let $\tilde{\nabla}$ be the g-dual of ∇, and let ∇ be the $\tilde{g}$-dual of $\tilde{\nabla}$. Then, the Gauss, Codazzi and Ricci equations are given by*

$$\begin{aligned} \tilde{R}(E,F)G =& R(E,F)G - h(F,G)A^*E + h(E,G)A^*F + (\nabla_E h)(F,G)\varsigma \\ &- (\nabla_F h)(E,G)\varsigma + \tau^*(E)h(F,G)\varsigma - \tau^*(F)h(E,G)\varsigma, \end{aligned}$$

$$(\tilde{R}(E,F)G)^\perp = (\nabla_E h)(F,G)\varsigma - \nabla_F h)(E,G)\varsigma + \tau^*(E)h(F,G)\varsigma - \tau^*(F)h(E,G)\varsigma,$$

and

$$\begin{aligned} \tilde{R}(E,F)\varsigma =& -(\nabla_E A^*)F + (\nabla_F A^*)E - \tau^*(F)A^*E \\ &+ \tau^*(E)A^*F - h(E,A^*F)\varsigma + h(A^*E,F)\varsigma + d\tau^*(E,F)\varsigma, \end{aligned}$$

respectively.

Proposition 3.2 ([4]). *Consider a statistical submanifold (M, g, ∇^*) of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla}^*)$. Let $\tilde{\nabla}^*$ be the g-dual of ∇^*, and let ∇^* be the $\tilde{g}$-dual of $\tilde{\nabla}^*$.*

Then, the Gauss, Codazzi and Ricci equations are given by

$$\begin{aligned} \tilde{R}^*(E,F)G =& R^*(E,F)G - h^*(F,G)AE + h^*(E,G)AF + (\nabla^*_E h^*)(F,G)\varsigma \\ &- (\nabla^*_F h^*)(E,G)\varsigma + \tau(E)h^*(F,G)\varsigma - \tau(F)h^*(E,G)\varsigma, \end{aligned}$$

$$\begin{aligned} \big(\tilde{R}^*(E,F)G\big)^\perp =& (\nabla^*_E h^*)(F,G)\varsigma - (\nabla^*_F h^*)(E,G)\varsigma \\ &+ \tau(E)h^*(F,G)\varsigma - \tau(F)h^*(E,G)\varsigma \end{aligned}$$

and

$$\begin{aligned} \tilde{R}^*(E,F)\varsigma =& -(\nabla^*_E A)F + (\nabla^*_F A)E - \tau(F)AE + \tau(E)AF - h^*(E,AF)\varsigma \\ &+ h^*(AE,F)\varsigma + d\tau(E,F)\varsigma, \end{aligned}$$

respectively.

4. General inequality for cosymplectic statistical submanifolds

Consider a $(n+1)$-dimensional statistical submanifold M of a $(2m+1)$-dimensional cosymplectic statistical manifold $\tilde{M}(c)$. Then for curvature tensor fields R and R^* of ∇ and ∇^*, respectively, we use the notation

$$\begin{cases} R(E,F,G,W) = g(R(E,F)G,W), \\ R^*(E,F,G,W) = g(R^*(E,F)G,W). \end{cases}$$

Also, the mean curvature vector fields are defined as

$$\begin{aligned} H =& \frac{1}{n+1}\sum_{\alpha=1}^{n+1} h\big(e_\alpha, e_\alpha\big) = \frac{1}{n+1}\sum_{t=n+2}^{2m+1}\bigg(\sum_{\alpha=1}^{n+1} h_{\alpha\alpha}^t\bigg)e_t, \\ h_{\alpha\beta}^t =& \tilde{g}\big(h(e_\alpha, e_\beta), e_t\big), \end{aligned}$$

and

$$\begin{aligned} H^* =& \frac{1}{n+1}\sum_{\alpha=1}^{n+1} h^*\big(e_\alpha, e_\alpha\big) = \frac{1}{n+1}\sum_{t=n+2}^{2m+1}\bigg(\sum_{\alpha=1}^{n+1} h_{\alpha\alpha}^{*t}\bigg)e_t, \\ h_{\alpha\beta}^{*t} =& \tilde{g}\big(h^*(e_\alpha, e_\beta), e_t\big), \end{aligned}$$

for $\{e_1, \dots, e_{n+1}\}$ and $\{e_{n+2}, \dots, e_{2m+1}\}$ as orthonormal tangent and normal frames, respectively, on M.

Now, we prove the following result.

Proposition 4.1. *Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional cosymplectic statistical manifold of constant curvature $c \in \mathbb{R}$ and M an $(n+1)$-dimensional statistical submanifold of $\tilde{M}(c)$ such that ξ is tangent to M. Then the Ricci tensor Q satisfies*

$$\begin{aligned} Q(E,F) =& \frac{c}{4}\big[(n-1)g(E,F)+3g(\phi E, \phi F)-(n-1)\eta(E)\eta(F)\big] \\ &-\frac{1}{2}\sum_{t=n+2}^{2m+1}\big[g(A_{e_t}F, A_{e_t}^*E)+g(A_{e_t}^*F, A_{e_t}E)-g(A_{e_t}E,F)trA_{e_t}^*-g(A_{e_t}^*E,F)trA_{e_t}\big], \quad (8) \end{aligned}$$

where A_t and A_t^* are linear transformations defined by (3).

Proof. We know that

$$Q(E,F) = \sum_{\beta=1}^{n+1} g(S(e_\beta, E)F, e_\beta). \quad (9)$$

Combining (9) with (1), (2), and (4) yields

$$\begin{aligned} Q(E,F) =& \sum_{\beta=1}^{n+1} g(\tilde{S}(e_\beta, E)F, e_\beta) \\ &-\frac{1}{2}\sum_{\beta=1}^{n+1}\big[\tilde{g}(h(e_\beta, F), h^*(E, e_\beta))-\tilde{g}(h^*(e_\beta, e_\beta), h(E,F)) \\ &+\tilde{g}(h^*(e_\beta, F), h(E, e_\beta))-\tilde{g}(h(e_\beta, e_\beta), h^*(E,F))\big]. \quad (10) \end{aligned}$$

Using (5) in (10), we get

$$\begin{aligned}
Q(E,F) =& \frac{c}{4}\{g(E,F)g(e_\beta,e_\beta) - g(e_\beta,F)g(E,e_\beta) + g(e_\beta,\phi F)g(\phi E,e_\beta) \\
& - g(E,\phi F)g(\phi e_\beta,e_\beta) + 2g(e_\beta,\phi E)g(\phi F,e_\beta) + \eta(e_\beta)\eta(F)g(E,e_\beta) \\
& - \eta(E)\eta(F)g(e_\beta,e_\beta) + g(e_\beta,F)\eta(E)g(\xi,e_\beta) - g(E,F)\eta(e_\beta)g(\xi,e_\beta)\} \\
& - \frac{1}{2}\sum_{\beta=1}^{n+1}\big[\tilde{g}(h(e_\beta,F),h^*(E,e_\beta)) - \tilde{g}(h^*(e_\beta,e_\beta),h(E,F)) \\
& + \tilde{g}(h^*(e_\beta,F),h(E,e_\beta)) - \tilde{g}(h(e_\beta,e_\beta),h^*(E,F))\big] \\
=& \frac{c}{4}\big[(n-1)g(E,F) + 3g(\phi E,\phi F) - (n-1)\eta(E)\eta(F)\big] \\
& - \frac{1}{2}\sum_{\beta=1}^{n+1}\big[\tilde{g}(h(e_\beta,F),h^*(E,e_\beta)) - \tilde{g}(h^*(e_\beta,e_\beta),h(E,F)) \\
& + \tilde{g}(h^*(e_\beta,F),h(E,e_\beta)) - \tilde{g}(h(e_\beta,e_\beta),h^*(E,F))\big]. \qquad (11)
\end{aligned}$$

On the other hand we have

$$\tilde{g}(h^*(e_\beta,e_\beta),h(E,F)) = \sum_{t=n+2}^{2m+1} g(A_{e_t}E,F)g(A^*_{e_t}e_\beta,e_\beta) \qquad (12)$$

and

$$\tilde{g}(h(e_\beta,F),h^*(E,e_\beta)) = \sum_{t=n+2}^{2m+1} g(A^*_{e_t}E,e_\beta)g(A_{e_t}F,e_\beta). \qquad (13)$$

Applying (12) and (13) into (11), we get

$$\begin{aligned}
Q(E,F) =& \frac{c}{4}\big[(n-1)g(E,F) + 3g(\phi E,\phi F) - (n-1)\eta(E)\eta(F)\big] \\
& - \frac{1}{2}\sum_{\beta=1}^{n+1}\sum_{t=n+2}^{2m+1}\big[g(A^*_{e_t}E,e_\beta)g(A_{e_t}F,e_\beta)g(A_{e_t}E,F) - (A^*_{e_t}e_\beta,e_\beta) \\
& + g(A^*_{e_t}F,e_\beta)g(A_{e_t}E,e_\beta) - g(A_{e_t}e_\beta,e_\beta)g(A^*_{e_t}E,F)\big] \\
=& \frac{c}{4}\big[(n-1)g(E,F) + 3g(\phi E,\phi F) - (n-1)\eta(E)\eta(F)\big] \\
& - \frac{1}{2}\sum_{t=n+2}^{2m+1}\big[g(A_{e_t}F,A^*_{e_t}E) + g(A^*_{e_t}F,A_{e_t}E) \\
& - g(A_{e_t}E,F)trA^*_{e_t} - g(A^*_{e_t}E,F)trA_{e_t}\big],
\end{aligned}$$

which is the required equality (8). □

5. Inequalities for cosymplectic hypersurfaces

Proposition 5.1. *Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional cosymplectic statistical manifold. Let M be a hypersurface of $\tilde{M}(c)$ of a constant curvature $c \in R$. Then*

$$2\tau \geq \frac{c}{4}\Big[4m^2 - 6m + 2 + 3\|P\|^2\Big] + 4m^2\tilde{g}(H, H^*) - \|h^\circ\|^2 \tag{14}$$

and the equality in the inequality holds if

$$h^{2m+1}_{\alpha\beta} = h^{*2m+1}_{\alpha\beta}, \forall \alpha, \beta = 1, \ldots, 2m. \tag{15}$$

Proof. From (1), (2), (4) and (5) we have

$$\begin{aligned}
g(S(E,F)G,W) = \frac{c}{4}\Big\{&g(F,G)g(E,W) - g(E,G)g(F,W) + g(E,\phi G)g(\phi F, W)\\
&- g(F,\phi G)g(\phi E, W) + 2g(E,\phi F)g(\phi G, W) + \eta(E)\eta(G)g(F,W)\\
&- \eta(F)\eta(G)g(E,W) + g(E,G)\eta(F)g(\xi,W) - g(F,G)\eta(E)g(\xi,W)\Big\}\\
&- \frac{1}{2}\Big[\tilde{g}(h(E,G), h^*(F,W)) - \tilde{g}(h^*(E,W), h(F,G))\\
&+ \tilde{g}(h^*(E,G), h(F,W)) - \tilde{g}(h(E,W), h^*(F,G))\Big],
\end{aligned} \tag{16}$$

where E, F, G and $W \in (TM)$. Substituting $E = W = e_\alpha$, $F = G = e_\beta$, it is easy to see that

$$\begin{aligned}
g(S(e_\alpha,e_\beta)e_\beta,e_\alpha) = \frac{c}{4}\Big\{&g(e_\beta,e_\beta)g(e_\alpha,e_\alpha) - g(e_\alpha,e_\beta)g(e_\beta,e_\alpha) + g(e_\alpha,\phi e_\beta)g(\phi e_\beta,e_\alpha)\\
&- g(e_\beta,\phi e_\beta)g(\phi e_\alpha,e_\alpha) + 2g(e_\alpha,\phi e_\beta)g(\phi e_\beta,e_\alpha) + \eta(e_\alpha)\eta(e_\beta)g(e_\beta,e_\alpha)\\
&- \eta(e_\beta)\eta(e_\beta)g(e_\alpha,e_\alpha) + g(e_\alpha,e_\beta)\eta(e_\beta)g(\xi,e_\alpha) - g(e_\beta,e_\beta)\eta(e_\alpha)g(\xi,e_\alpha)\Big\}\\
&- \frac{1}{2}\Big[\tilde{g}(h(e_\alpha,e_\beta), h^*(e_\beta,e_\alpha)) - \tilde{g}(h^*(e_\alpha,e_\alpha), h(e_\beta,e_\beta))\\
&+ \tilde{g}(h^*(e_\alpha,e_\beta), h(e_\beta,e_\alpha)) - \tilde{g}(h(e_\alpha,e_\alpha), h^*(e_\beta,e_\beta))\Big].
\end{aligned} \tag{17}$$

If we consider $\{e_1, \ldots, e_{2m} = \xi\}$ as an orthonormal frame of M and e_{2m+1} as a unit normal vector to M, then by applying summation over $1 \leq \alpha, \beta \leq 2m$, (17) reduces to

$$\begin{aligned}
2\tau =& \frac{1}{4}\Big[4m^2 - 6m + 2 + 3\|P\|^2\Big] - \tilde{g}(h(e_\alpha,e_\beta), h^*(e_\beta,e_\alpha))\\
&+ \frac{1}{2}\tilde{g}(h^*(e_\alpha,e_\alpha), h(e_\beta,e_\beta)) + \frac{1}{2}\tilde{g}(h(e_\alpha,e_\alpha), h^*(e_\beta,e_\beta)).
\end{aligned} \tag{18}$$

On the other hand, in this case the mean curvature vector fields H and H^* can be given as

$$H = \frac{1}{2m}\left(\sum_{\alpha=1}^{2m} h_{\alpha\alpha}\right)e_{2m+1}, h_{\alpha\beta} = \tilde{g}\big(h(e_\alpha,e_\beta), e_{2m+1}\big) \tag{19}$$

and

$$H^* = \frac{1}{2m}\left(\sum_{\alpha=1}^{2m} h^*_{\alpha\alpha}\right)e_{2m+1}, h^*_{\alpha\beta} = \tilde{g}\big(h^*(e_\alpha,e_\beta), e_{2m+1}\big). \tag{20}$$

Combining (18), (19) and (20), a straightforward computation gives

$$
\begin{aligned}
2\tau =&\frac{c}{4}\big[4m^2-6m+2+3\|P\|^2\big]+4m^2g(H,H^*)-\|h^\circ\|^2+\frac{1}{4}\sum_{\alpha,\beta=1}^{2m}(h^{2m+1}_{\alpha\beta}-h^{*2m+1}_{\alpha\beta})^2\\
\geq&\frac{c}{4}\big[4m^2-6m+2+3\|P\|^2\big]+4m^2g(H,H^*)-\|h^\circ\|^2,
\end{aligned}
$$

which gives the required inequality (14) and the equality in the inequality holds if it satisfies (15). ☐

THEOREM 5.2. *Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional cosymplectic statistical manifold. Let M be a hypersurface of $\tilde{M}(c)$. Then, for each $E\in T_p(M)$, we have*

$$
Ric(E)=\frac{c}{4}\Big[(2(m-1)+3\|P\|^2\Big]+m[\tilde{g}(h^*(E,E),H)+\tilde{g}\big(h(E,E),H^*]-\sum_{\alpha=1}^{2m}h_{\alpha1}h^*_{\alpha1}.
$$

Proof. Let us consider the orthonormal frame $\{e_1,\dots,e_{2m}=\xi\}$ such that $E=G=e_1$ and $F=W=e_\alpha$, $\alpha=2,\dots,2m$. With the help of (16), we have

$$
\begin{aligned}
g(S(E,e_\alpha)e_\alpha,E)=&\frac{c}{4}\{g(e_\alpha,e_\alpha)g(E,E)-g(E,e_\alpha)g(e_\alpha,E)+g(E,\phi e_\alpha)g(\phi e_\alpha,E)\\
&-g(e_\alpha,\phi e_\alpha)g(\phi E,E)+2g(E,\phi e_\alpha)g(\phi e_\alpha,E)+\eta(E)\eta(e_\alpha)g(e_\alpha,E)\\
&-\eta(e_\alpha)\eta(e_\alpha)g(E,E)+g(E,e_\alpha)\eta(e_\alpha)g(\xi,E)-g(e_\alpha,e_\alpha)\eta(E)g(\xi,E)\}\\
&-\frac{1}{2}[\tilde{g}(h(E,e_\alpha),h^*(e_\alpha,E))-\tilde{g}(h^*(E,E),h(e_\alpha,e_\alpha))\\
&+\tilde{g}(h^*(E,e_\alpha),h(e_\alpha,E))-\tilde{g}(h(E,E),h^*(e_\alpha,e_\alpha))].
\end{aligned}
$$

Applying summation over $2\leq\alpha\leq 2m$ in the above equation, we compute

$$
\begin{aligned}
Ric(E)=&\frac{c}{4}\Big[(2m-1)+3\sum_{\alpha=2}^{2m}g^2\big(\phi E,e_\alpha\big)-\sum_{\alpha=2}^{2m}\eta^2(e_\alpha)\Big]-\sum_{\alpha=2}^{2m}\tilde{g}(h(E,e_\alpha),h^*(e_\alpha,E))\\
&+\frac{1}{2}\sum_{\alpha=2}^{2m}\tilde{g}(h^*(E,E),h(e_\alpha,e_\alpha))+\frac{1}{2}\sum_{\alpha=2}^{2m}\tilde{g}(h(E,E),h^*(e_\alpha,e_\alpha))\\
=&\frac{c}{4}\Big[2(m-1)+3\|P\|^2\Big]-\sum_{\alpha=2}^{2m}g(h(E,e_\alpha),h^*(e_\alpha,E))\\
&+\frac{1}{2}\Big[2mg(h^*(E,E),H)-2\tilde{g}(h^*(E,E),h(e_1,e_1))+2mg(h(E,E),H^*)\Big]\\
=&\frac{c}{4}\Big[2(m-1)+3\|P\|^2\Big]-\sum_{\alpha=1}^{2m}h_{\alpha1}h^*_{\alpha1}+m\Big[g(h^*(E,E),H)+g(h(E,E),H^*)\Big].
\end{aligned}
$$

This completes the proof. ☐

6. Chen-Ricci inequality for cosymplectic statistical manifolds

In this section we mainly prove the following result.

THEOREM 6.1. *Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional cosymplectic statistical manifold. Let M be an $(n+1)$-dimensional statistical submanifold of $\tilde{M}(c)$. Then for each unit $E \in T_pM$, we have*

$$\begin{aligned} Ric(E) \geq & 2Ric^{\circ}(E) - \frac{(n+1)^2}{8}\tilde{g}(H,H) - \frac{(n+1)^2}{8}\tilde{g}(H^*,H^*) \\ & + \frac{c}{8}\Big[2(n-1) + 3(\|P\|^2 - \|P_1\|^2)\Big] - 2(n)\max \tilde{K}^{\circ}(E \wedge .), \end{aligned}$$

where $\|P_1\|^2 = \sum_{2\leq\alpha\neq\beta\leq n+1} \tilde{g}^2(\phi e_\beta, e_\alpha)$. The equality holds if $2\sum_{t=n+2}^{2m+1} h^{*t}_{11} = nH^*$.

Proof. Let $\{e_1, \ldots, e_{n+1} = \xi\}$ be an orthonormal frame of M and $\{e_{n+2}, \ldots, e_{2m+1}\}$ a normal frame to M. Then by summing over $1 \leq \alpha, \beta \leq n+1$, it follows from (17) that:

$$2\tau = \frac{c}{4}\Big[n(n-1)+3\|P\|^2\Big]+(n+1)^2\tilde{g}(H,H^*)+\sum_{\alpha,\beta=1}^{n+1}\tilde{g}\big(h(e_\alpha,e_\beta),h^*(e_\beta,e_\alpha)\big). \tag{21}$$

Using the fact $2H^{\circ} = H + H^*$ in (21), we derive

$$\begin{aligned} 2\tau = & \frac{c}{4}\Big[n(n-1) + 3\|P\|^2\Big] + 2(n+1)^2\tilde{g}(H^{\circ},H^{\circ}) \\ & - \frac{(n+1)^2}{2}\tilde{g}(H,H) - \frac{(n+1)^2}{2}\tilde{g}(H^*,H^*) - 2\|h^{\circ}\|^2 + \frac{1}{2}\Big(\|h\|^2 + \|h^*\|^2\Big). \end{aligned} \tag{22}$$

On the other hand, one has

$$\begin{aligned} \|h\|^2 = & \sum_{t=n+1}^{2m+1}\Big[(h^t_{11})^2 + (h^t_{22} + \ldots + h^t_{nn})^2 + 2\sum_{1\leq\alpha<\beta\leq n}(h^t_{\alpha\beta})^2\Big] \\ & - \sum_{t=n+1}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n} h^t_{\alpha\alpha}h^t_{\beta\beta} \\ = & \frac{1}{2}\sum_{t=n+1}^{2m+1}\Big[\big(h^t_{11} + h^t_{22} + \ldots + h^t_{nn}\big)^2 + \big(h^t_{11} - h^t_{22} - \ldots - h^t_{nn}\big)^2\Big] \\ & + 2\sum_{t=n+1}^{2m+1}\sum_{1\leq\alpha<\beta\leq n}(h^t_{\alpha\beta})^2 - \sum_{t=n+1}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n} h^t_{\alpha\alpha}h^t_{\beta\beta} \\ \geq & \frac{n^2}{2}\|H\|^2 - \sum_{t=n+1}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n}\big[h^t_{\alpha\alpha}h^t_{\beta\beta} - (h^t_{\alpha\beta})^2\big]. \end{aligned} \tag{23}$$

Similarly, we have

$$\|h^*\|^2 \geq \frac{n^2}{2}\|H^*\|^2 - \sum_{t=n+1}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n}\big[h^{*t}_{\alpha\alpha}h^{*t}_{\beta\beta} - (h^{*t}_{\alpha\beta})^2\big]. \tag{24}$$

Combining (23) and (24) with (22), we find

$$\begin{aligned}2\tau \geq &\frac{c}{4}\Big[n(n-1)+3\|P\|^2\Big]+2(n+1)^2\tilde{g}(H^\circ,H^\circ)-\frac{(n+1)^2}{2}\tilde{g}(H,H)\\&-\frac{(n+1)^2}{2}\tilde{g}(H^*,H^*)-2\|h^\circ\|^2-2\sum_{t=n+2}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n+1}h^{\circ t}_{\alpha\alpha}h^{\circ t}_{\beta\beta}\\&+\sum_{t=n+2}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n+1}h^{t}_{\alpha\alpha}h^{*t}_{\beta\beta}+\frac{1}{2}\sum_{t=n+2}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n+1}\big[(h^t_{\alpha\beta})^2+(h^{*t}_{\alpha\beta})^2\big].\end{aligned}\tag{25}$$

Further, we obtain

$$\begin{aligned}\sum_{2\leq\alpha\neq\beta\leq n+1}\tilde{g}\big(S(e_\alpha,e_\beta)e_\alpha,e_\beta\big)=&\frac{c}{4}\big(n^2-3n+2+3\|P_2\|^2\big)\\&+\sum_{t=n+2}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n+1}\big(h^t_{\alpha\alpha}h^{*t}_{\beta\beta}-h^t_{\alpha\beta}h^{*t}_{\alpha\beta}\big).\end{aligned}\tag{26}$$

From (25) and (26), we reduce to

$$\begin{aligned}2\tau \geq &\frac{c}{4}\Big[n(n-1)+3\|P\|^2\Big]+2(n+1)^2\tilde{g}(H^\circ,H^\circ)-\frac{(n+1)^2}{2}\tilde{g}(H,H)\\&-\frac{(n+1)^2}{2}\tilde{g}(H^*,H^*)-2\|h^\circ\|^2-2\sum_{t=n+2}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n+1}h^{\circ t}_{\alpha\alpha}h^{\circ t}_{\beta\beta}\\&+\sum_{2\leq\alpha\neq\beta\leq n+1}\tilde{g}\big(S(e_\alpha,e_\beta)e_\alpha,e_\beta\big)-\frac{c}{4}\Big[n^2-3n+2+3\|P_2\|^2\Big]\\&+\sum_{t=n+2}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n+1}h^{t}_{\alpha\beta}h^{*t}_{\alpha\beta}+\frac{1}{2}\sum_{t=n+2}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n+1}\big[(h^t_{\alpha\beta})^2+(h^{*t}_{\alpha\beta})^2\big].\end{aligned}$$

Then, a direct computation gives

$$\begin{aligned}Ric(E)\geq&\frac{c}{8}[2(n-1)+3(\|P\|^2-\|P_1\|^2)+(n+1)^2\tilde{g}(H^\circ,H^\circ)\\-&\frac{(n+1)^2}{8}\Big[\tilde{g}(H,H)+\tilde{g}(H^*,H^*)\Big]-\|h^\circ\|^2-\sum_{t=n+2}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n+1}\Big[h^{\circ t}_{\alpha\alpha}h^{\circ t}_{\beta\beta}-(h^{\circ t}_{\alpha\beta})^2\Big].\end{aligned}\tag{27}$$

Equation (1) for Levi-Civita connection yields

$$\sum_{1\leq\alpha\neq\beta\leq n+1}\tilde{R}^\circ\big(e_\alpha,e_\beta,e_\alpha,e_\beta\big)=2\tau^\circ-(n+1)^2\tilde{g}(H^\circ,H^\circ)+\|h^\circ\|^2,\tag{28}$$

and

$$\begin{aligned}\sum_{2\leq\alpha\neq\beta\leq n+1}\tilde{R}^\circ\big(e_\alpha,e_\beta,e_\alpha,e_\beta\big)=&\sum_{2\leq\alpha\neq\beta\leq n+1}R^\circ\big(e_\alpha,e_\beta,e_\alpha,e_\beta\big)\\&-\sum_{t=n+2}^{2m+1}\sum_{2\leq\alpha\neq\beta\leq n+1}\big(h^{\circ t}_{\alpha\alpha}h^{\circ t}_{\beta\beta}-(h^{\circ t}_{\alpha\beta})^2\big).\end{aligned}\tag{29}$$

Making use of (28) and (29) in (27), it is easy to see that

$$\begin{aligned}Ric(E) \geq &\frac{c}{8}\Big[2(n-1)+3(\|P\|^2-\|P_1\|^2)\Big]+2\tau^\circ\\ &-\sum_{1\leq\alpha\neq\beta\leq n+1}\tilde{g}(\tilde{R}^\circ(e_\alpha,e_\beta)e_\beta,e_\alpha)-\frac{(n+1)^2}{8}\Big[\tilde{g}(H,H)+\tilde{g}(H^*,H^*)\Big]\\ &-\sum_{2\leq\alpha\neq\beta\leq n+1}\tilde{g}(\tilde{R}^\circ(e_\alpha,e_\beta)e_\beta,e_\alpha)+\sum_{2\leq\alpha\neq\beta\leq n}\tilde{g}(R^\circ(e_\alpha,e_\beta)e_\beta,e_\alpha).\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}Ric(E) \geq &2Ric^\circ(E)+\frac{c}{8}\Big[2(n-1)+3(\|P\|^2-\|P_1\|^2)\Big]\\ &-\frac{(n+1)^2}{8}\Big[\tilde{g}(H,H)+\tilde{g}(H^*,H^*)\Big]-2\sum_{\alpha=2}^{n}\tilde{K}^\circ(E\wedge e_\alpha)\end{aligned}$$

where $\tilde{K}^\circ(E\wedge .)$ is the maximum of the sectional curvature function of $\tilde{M}(c)$. □

As a consequence of the above theorem we have the following obstruction result.

Corollary 6.2. *Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional cosymplectic statistical manifold. Let M be an $(n+1)$-dimensional statistical submanifold of $\tilde{M}(c)$. Then for each unit $E \in T_pM$, if*

$$\begin{aligned}Ric(E) <&2Ric^\circ(E)+\frac{c}{8}\Big[2(n-1)+3(\|P\|^2-\|P_1\|^2)\Big]\\ &+\frac{(n+1)^2}{4}\tilde{g}(H,H^*)-2(n)\max\tilde{K}^\circ(E\wedge .),\end{aligned}$$

then M cannot be minimally immersed in $\tilde{M}(c)$.

Proof. Proof of the result directly follows from the above theorem. If M is a minimal submanifold, we have $H^\circ = 0$. This implies that $H + H^* = 0$. Using this in Theorem 6.1 we have the required result. □

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