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A GENERALIZATION OF NONSINGULAR REGULAR MAGIC SQUARES

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Abstract. A generalization of regular magic squares with magic sum μ is an sq-corner (or square corner) magic square. It is a magic square satisfying the condition that the sum of 4 entries, square symmetrically placed with respect to the center, equals $\frac{4\mu}{n}$. Using the sq-corner magic squares of order n, a construction of sq-corner magic squares of order $n+2$ is derived. Moreover, this construction provides some nonsingular classical sq-corner magic squares of all orders. In particular, a nonsingular regular magic square of any odd order can be constructed under this new method, as well.

1. Introduction

An $n \times n$ matrix M over $\mathbb{C}$ whose sum of n entries in any row and any column equals a constant μ is called a *semi-magic square*, and if n entries on each of its cross diagonals also sum to μ, then M is called a *magic square* with a *magic sum* μ. One of the special types of magic squares widely studied is a regular magic square, an $n \times n$ complex magic square $M = [m_{i,j}]$ such that

$$m_{i,j} + m_{n+1-i,n+1-j} = \frac{2\mu}{n}.$$

Mattingly showed in [6] that a regular magic square of any even order is singular. However, this is not the case for an odd-order regular magic square, which leads to many attempts to construct a nonsingular regular magic square of odd order. Lee and et. al. introduced in [3] a construction of nonsingular regular magic squares whose orders are odd primes and powers of odd primes by using a centroskew S-circulant matrix A with the first row of A defined as $a_j = j - 1$ for $j = 1, 2, \dots, \left(\frac{n+1}{2}\right)$. Their construction also lead to further study of singularity of regular magic squares, e.g., see [2, 4]. In our work, we are more interested in studying the singularity of its generalization. Rungratgasame and et. al. introduced in [7] a generalization of regular magic squares, called corner magic squares, which will be defined by Definition 1.1.

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To be precise, we shall call a corner magic square a *sq-corner magic square*. This work will give a construction of a nonsingular sq-corner magic square of any order $n \geq 3$.

Definition 1.1. An $n \times n$ complex magic square $M = [m_{i,j}]$ with a magic sum μ is said to be a *sq-corner* (*square corner*) *magic square* if

$$m_{i,i} + m_{(n+1-i),(n+1-i)} + m_{i,(n+1-i)} + m_{(n+1-i),i} = \frac{4\mu}{n}, \quad \text{for all } i = 1, 2, 3, \ldots, n.$$

Both regular and sq-corner magic squares can be symbolically illustrated in the following table where the same symbols represent associated entries added to be a constant.

Magic squares	$n = 4$	$n = 5$	$n = 6$
regular	$\begin{bmatrix} \spadesuit & \triangledown & \triangle & \clubsuit \\ \triangleleft & \heartsuit & \diamondsuit & \triangleright \\ \triangleright & \diamondsuit & \heartsuit & \triangleleft \\ \clubsuit & \triangle & \triangledown & \spadesuit \end{bmatrix}$	$\begin{bmatrix} \spadesuit & \triangledown & \circ & \triangle & \clubsuit \\ \triangleleft & \heartsuit & \diamondsuit & \bullet & \triangleright \\ \odot & \oplus & \bigstar & \oplus & \odot \\ \triangleright & \bullet & \diamondsuit & \heartsuit & \triangleleft \\ \clubsuit & \triangle & \circ & \triangledown & \spadesuit \end{bmatrix}$	$\begin{bmatrix} \spadesuit & \triangledown & \circ & \bullet & \triangle & \clubsuit \\ \triangleleft & \square & \heartsuit & \diamondsuit & \blacksquare & \triangleright \\ \odot & \oslash & \boxplus & \bigstar & \ominus & \otimes \\ \otimes & \ominus & \bigstar & \boxplus & \oslash & \odot \\ \triangleright & \blacksquare & \diamondsuit & \heartsuit & \square & \triangleleft \\ \clubsuit & \triangle & \bullet & \circ & \triangledown & \spadesuit \end{bmatrix}$
sq-corner	$\begin{bmatrix} \spadesuit & & & \spadesuit \\ & \boxtimes & \boxtimes & \\ & \boxtimes & \boxtimes & \\ \spadesuit & & & \spadesuit \end{bmatrix}$	$\begin{bmatrix} \spadesuit & & & & \spadesuit \\ & \heartsuit & & \heartsuit & \\ & & \clubsuit & & \\ & \heartsuit & & \heartsuit & \\ \spadesuit & & & & \spadesuit \end{bmatrix}$	$\begin{bmatrix} \spadesuit & & & & & \spadesuit \\ & \circledast & & & \circledast & \\ & & \bigstar & \bigstar & & \\ & & \bigstar & \bigstar & & \\ & \circledast & & & \circledast & \\ \spadesuit & & & & & \spadesuit \end{bmatrix}$

It is obvious that a regular magic square is sq-corner. However, a sq-corner magic square need not be regular, e.g. the magic square with Frénicle index 175:

$$F_{175} = \begin{bmatrix} 1 & 12 & 8 & 13 \\ 14 & 7 & 11 & 2 \\ 15 & 6 & 10 & 3 \\ 4 & 9 & 5 & 16 \end{bmatrix}$$

is regular and sq-corner whereas the magic squares in [5] with Frénicle indices 181 and 268 in Dudeney Group XI and VII, respectively

$$F_{181} = \begin{bmatrix} 1 & 12 & 13 & 8 \\ 16 & 9 & 4 & 5 \\ 2 & 7 & 14 & 11 \\ 15 & 6 & 3 & 10 \end{bmatrix} \quad \text{and} \quad F_{268} = \begin{bmatrix} 2 & 5 & 16 & 11 \\ 8 & 12 & 1 & 13 \\ 9 & 7 & 14 & 4 \\ 15 & 10 & 3 & 6 \end{bmatrix},$$

are sq-corner but not regular.

The matrices F_{181} and F_{268} are examples of nonsingular sq-corner magic squares. In particular, these show that a sq-corner magic square of even order need not be singular. To study the singularity of sq-corner magic squares that we construct in this paper, we will find a method to determine their determinants.

2. A construction of nonsingular sq-corner magic squares

Recall that a square matrix is nonsingular if all of its eigenvalues are nonzero. In [1], Amir-Moéz and Fredricks showed the connection between eigenvalues of a magic square and its related magic square as follows.

THEOREM 2.1. *If M is an $n \times n$ magic square and ρ is a complex number, then $M + \rho E$ has the same eigenvalues as M except that μ is replaced by $\mu + \rho n$.*

For any $n \times n$ magic square M, the *corresponding zero magic square* of order n is defined to be $Z_M = M - \frac{\mu}{n}E$. From Theorem 2.1, Z_M has the same eigenvalues as M except that μ is replaced by 0. It implies that Z_M has no repeated zero eigenvalue if M is nonsingular. Next, we will construct an extended zero magic square of order $(n+2) \times (n+2)$ when a magic square of order n is given. For any $n \in \mathbb{N}$, we say that $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ is *zero-sum* if $\sum_{i=1}^{n} a_i = 0$.

DEFINITION 2.2. Let Z be a zero magic square of order n. For a zero-sum $\vec{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$, we define an extended $(n+2) \times (n+2)$ matrix with respect to Z, denoted by $\mathcal{S}_Z\vec{a}$, as follows:

$$\mathcal{S}_Z\vec{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n+1} & 0 \\ a_2 & & & & -a_2 \\ \vdots & & Z & & \vdots \\ a_{n+1} & & & & -a_{n+1} \\ 0 & -a_2 & \cdots & -a_{n+1} & -a_1 \end{bmatrix}.$$

Then $\mathcal{S}_Z\vec{a}$ is a zero magic square of order $n+2$. In particular, if Z is a zero sq-corner magic square, then so is $\mathcal{S}_Z\vec{a}$.

EXAMPLE 2.3. Let us consider the following regular zero magic square of order 5 produced by an S-circulant matrix (see [2]),

$$C = \begin{bmatrix} 0 & 1 & 2 & -2 & -1 \\ -1 & 0 & 1 & 2 & -2 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & -2 & -1 & 0 & 1 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix}.$$

We choose $\vec{a} = (-3, 1, 0, 1, 0, 1)$. Then

$$\mathcal{S}_C\vec{a} = \begin{bmatrix} -3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 & -2 & -1 & -1 \\ 0 & -1 & 0 & 1 & 2 & -2 & 0 \\ 1 & -2 & -1 & 0 & 1 & 2 & -1 \\ 0 & 2 & -2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & -2 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 & 3 \end{bmatrix}$$

is a zero sq-corner magic square of order 7 which has $1575x + 230x^3 - 10x^5 - x^7$ as its characteristic polynomial, i.e., $\mathcal{S}_C\vec{a}$ has no repeated zero eigenvalue.

We directly obtain the following proposition from Definition 2.2 to construct a regular zero magic square of any odd order. Let J denote the permutation matrix obtained by writing 1 in each of the cross diagonal entries and 0 elsewhere, that is,

$$J = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}.$$

Proposition 2.4. *For a regular zero magic square Z of odd order n, the sq-corner magic square with respect to Z, $\mathcal{S}_Z\vec{a}$, where $\vec{a} = (x, \vec{v}, y, \vec{v}J)$, $\vec{v}$ is an $1 \times \left(\frac{n-1}{2}\right)$ matrix and J is of order $\frac{n-1}{2}$, viewed as*

$$\begin{bmatrix} x & \vec{v} & y & \vec{v}J & 0 \\ \vec{v}^T & & & & -\vec{v}^T \\ y & & Z & & -y \\ J\vec{v}^T & & & & -J\vec{v}^T \\ 0 & -\vec{v} & -y & -\vec{v}J & -x \end{bmatrix},$$

is also a regular zero magic square of order $n+2$.

The main result of this work is to derive a nonsingular magic square of order $n+2$ once a nonsingular sq-corner magic square of order n is given. Here we shall begin with a definition of a submatrix and some of its properties in order to find a determinant of a matrix later on.

Definition 2.5. Let A be an $n \times n$ matrix. For index sets $\mathcal{X}$, $\mathcal{Y} \subseteq \{1, \dots, n\}$, let $A[\mathcal{X}, \mathcal{Y}]$ be a submatrix of A obtained by keeping entries positioned on the rows and columns with indices in $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $\mathcal{X} = \mathcal{Y}$, then $A[\mathcal{X}, \mathcal{X}]$ is a *principal submatrix* of A, denoted by $A[\mathcal{X}]$. Let $\mathcal{X}^c = \{1, \dots, n\} \setminus \mathcal{X}$ denote the index set complementary to $\mathcal{X}$. Then $A[\mathcal{X}^c] = A[\{1, \dots, n\} \setminus \mathcal{X}]$.

Let $\mathbf{e}$ denote a column vector containing all 1's and $\mathbf{e}_i$ a column vector whose i^{th} row entry is 1 and 0 elsewhere. For an $n \times n$ matrix M and a vector $\vec{a} \in \mathbb{R}^n$, we define $[M]^R_{(\vec{a},k)}$ and $[M]^C_{(\vec{a},k)}$ as matrices formed by replacing the k^{th} row and the k^{th} column of M by the vector $\vec{a}$, respectively.

The next lemma shows that the determinant of a matrix can be written in terms of a determinant of a principal submatrix.

Lemma 2.6. *Let Z be an $n \times n$ zero magic square. Then* $\det(Z + \lambda E) = n^2 \lambda \det Z[\{t\}^c]$ *for $t \in \{1, \dots, n\}$.*

Proof. The result immediately holds for $\lambda = 0$. Since the magic sum of Z is zero, we can apply elementary column operations to have that $\det(Z + \lambda E) = \det[Z + \lambda E]^C_{(n\lambda\mathbf{e},t)}$. Furthermore, $\det([Z + \lambda E]^C_{(n\lambda\mathbf{e},t)}) = \det \begin{bmatrix} Z[\{t\}^c] & n\lambda\mathbf{e} \\ \mathbf{0} & n^2\lambda \end{bmatrix}$. Hence, $\det(Z + \lambda E) = n^2 \lambda \det Z[\{t\}^c]$. □

Lemma 2.7. *Let M be an $(n+1) \times n$ matrix of the form $[\vec{a}, Z]^T$ where $\vec{a} \in \mathbb{R}^n$ and Z is a zero magic square of order n. Then for $i \in \{3, \dots, n+1\}$,* $\det M[\{i\}^c, \{1, \dots, n\}] = (-1)^i \det M[\{2\}^c, \{1, \dots, n\}]$.

Proof. Since Z is a zero magic square, $-\mathbf{e}_i^T Z = \mathbf{e}_1^T Z + \cdots + \mathbf{e}_{i-1}^T Z + \mathbf{e}_{i+1}^T Z + \cdots + \mathbf{e}_n^T Z$, and hence the desired results can be obtained by using elementary row operations. □

THEOREM 2.8. *Let Z be an $n\times n$ zero magic square. Let $\vec{a} = (a_1, a_2, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$ be zero-sum. For $\lambda \in \mathbb{R}$, $\det(\mathcal{S}_Z\vec{a} + \lambda E) = -a_1^2(n+2)^2\lambda \det Z[\{1\}^c]$.*

Proof. By using elementary row and column operations and $a_1 = -a_2 - a_3 - \cdots - a_{n+1}$, the determinant of $\mathcal{S}_Z\vec{a} + \lambda E$ is

$$\begin{aligned}
&\det\begin{bmatrix} a_1+\lambda & a_2+\lambda & a_3+\lambda & \cdots & a_{n+1}+\lambda & \lambda \\ a_2+\lambda & & & & & -a_2+\lambda \\ a_3+\lambda & & Z+\lambda E & & & -a_3+\lambda \\ \vdots & & & & & \vdots \\ a_{n+1}+\lambda & & & & & -a_{n+1}+\lambda \\ \lambda & -a_2+\lambda & -a_3+\lambda & \cdots & -a_{n+1}+\lambda & -a_1+\lambda \end{bmatrix} \\
=&\det\begin{bmatrix} a_1+\lambda & a_2+\lambda & a_3+\lambda & \cdots & a_{n+1}+\lambda & a_1+2\lambda \\ a_2+\lambda & & & & & 2\lambda \\ a_3+\lambda & & Z+\lambda E & & & 2\lambda \\ \vdots & & & & & \vdots \\ a_{n+1}+\lambda & & & & & 2\lambda \\ a_1+2\lambda & 2\lambda & 2\lambda & \cdots & 2\lambda & 4\lambda \end{bmatrix} \\
=&\det\begin{bmatrix} a_1+\lambda & a_2+\lambda & a_3+\lambda & \cdots & a_{n+1}+\lambda & a_1+2\lambda \\ a_2+\lambda & & & & & 2\lambda \\ a_3+\lambda & & Z+\lambda E & & & 2\lambda \\ \vdots & & & & & \vdots \\ a_{n+1}+\lambda & & & & & 2\lambda \\ (n+2)\lambda & (n+2)\lambda & (n+2)\lambda & \cdots & (n+2)\lambda & 2(n+2)\lambda \end{bmatrix} \\
=&\det\begin{bmatrix} a_1+\lambda & a_2+\lambda & a_3+\lambda & \cdots & a_{n+1}+\lambda & (n+2)\lambda \\ a_2+\lambda & & & & & (n+2)\lambda \\ a_3+\lambda & & Z+\lambda E & & & (n+2)\lambda \\ \vdots & & & & & \vdots \\ a_{n+1}+\lambda & & & & & (n+2)\lambda \\ (n+2)\lambda & (n+2)\lambda & (n+2)\lambda & \cdots & (n+2)\lambda & (n+2)^2\lambda \end{bmatrix} \\
=&(n+2)^2\det\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n+1} & 0 \\ a_2 & & & & & 0 \\ a_3 & & Z & & & 0 \\ \vdots & & & & & \vdots \\ a_{n+1} & & & & & 0 \\ \lambda & \lambda & \lambda & \cdots & \lambda & \lambda \end{bmatrix}.
\end{aligned}$$

We write

$$\det(\mathcal{S}_Z\vec{a}+\lambda E) = (n+2)^2\lambda \det Q, \quad \text{where} \quad Q = \begin{bmatrix} a_1 & a_2 & \cdots & a_n & a_{n+1} \\ a_2 & & & & \\ \vdots & & Z & & \\ a_n & & & & \\ a_{n+1} & & & & \end{bmatrix}.$$

To determine the formula of the determinant of Q, we pick the first column of Q and find all of its cofactors. Let $\vec{b} = (a_2, a_3, \ldots, a_{n+1})$. For $k \geq 3$, $Q[\{k\}^c, \{1\}^c]$ can be viewed as the submatrix of $[\vec{b}, Z]^T$ by deleting the $(k-1)^{\text{th}}$ row of Z. By applying Lemma 2.7, $\det Q[\{k\}^c, \{1\}^c] = (-1)^k \det Q[\{2\}^c, \{1\}^c]$. From the assumption of $\sum_{j\geq 2} a_j = -a_1$, we have

$$\begin{aligned} \det Q &= \sum_{i=1}^{n+1} a_i(-1)^{i+1} \det Q[\{i\}^c, \{1\}^c] \\ &= -a_2 \det Q[\{2\}^c, \{1\}^c] + \sum_{i=3}^{n+1} a_i(-1)^{i+1}(-1)^i \det Q[\{2\}^c, \{1\}^c] \\ &= -(a_2 + a_3 + \cdots + a_{n+1}) \cdot \det Q[\{2\}^c, \{1\}^c] = a_1 \cdot \det([Z]^R_{(\vec{b},1)}). \end{aligned}$$

By applying elementary column operations, we get

$$\det([Z]^R_{(\vec{b},1)}) = \det \begin{bmatrix} -a_1 & a_3 \cdots & a_n \\ 0 & & \\ \vdots & Z[\{1\}^c] & \\ 0 & & \end{bmatrix}.$$

Then $\det Q = a_1(-a_1) \cdot \det Z[\{1\}^c]$. □

From Theorem 2.8 and Lemma 2.6, the following corollary is obtained immediately.

Corollary 2.9. *For a zero-sum* $\vec{a} = (a_1, \ldots, a_{n+1})$,

$$\det(\mathcal{S}_Z\vec{a}+\lambda E) = -a_1^2\left(\frac{n+2}{n}\right)^2 \det(Z+\lambda E).$$

Example 2.10. Let us consider the following nonsingular regular magic square of order 9,

$$M = \begin{bmatrix} 0 & 5 & 4 & 5 & 4 & 5 & 4 & 5 & 4 \\ 5 & 1 & 5 & 4 & 5 & 4 & 5 & 4 & 3 \\ 4 & 5 & 4 & 5 & 6 & 2 & 3 & 3 & 4 \\ 5 & 4 & 3 & 4 & 5 & 6 & 2 & 4 & 3 \\ 4 & 5 & 2 & 3 & 4 & 5 & 6 & 3 & 4 \\ 5 & 4 & 6 & 2 & 3 & 4 & 5 & 4 & 3 \\ 4 & 5 & 5 & 6 & 2 & 3 & 4 & 3 & 4 \\ 5 & 4 & 3 & 4 & 3 & 4 & 3 & 7 & 3 \\ 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 8 \end{bmatrix}$$

with the magic sum 36. This regular magic square is originally an extended matrix from the S-circulant matrix C of order 5 given in Example 2.3. To be precise, $M = \mathcal{S}_{\mathcal{S}_C \vec{a}} \vec{b} + 4E$, where $\vec{a} = (-3, 1, 0, 1, 0, 1)$ and $\vec{b} = (-4, 1, 0, 1, 0, 1, 0, 1)$. Using Corollary 2.9,

$$\det M = -(-4)^2 \left(\frac{9}{7}\right)^2 \left[-(-3)^2 \left(\frac{7}{5}\right)^2 \det(C + 4E) \right] = 1166400,$$

showing that M is nonsingular.

Theorem 2.8 and Corollary 2.9 provide us the following result.

Theorem 2.11. *Let M be a nonsingular magic sq-corner square of order n. For a zero-sum $\vec{a} = (a_1, a_2, a_3, \ldots, a_{n+1})$ and $\lambda \in \mathbb{R}$, the sq-corner magic square $\mathcal{S}_{Z_M} \vec{a} + \lambda E$ is nonsingular if and only if $a_1, \lambda \neq 0$.*

Proof. By Theorem 2.1, $\det(Z_M + \lambda E) \neq 0$ if and only if $\lambda \neq 0$. By Corollary 2.9, $\det(\mathcal{S}_{Z_M} \vec{a} + \lambda E) \neq 0$ if and only if $a_1 \neq 0$ and $\lambda \neq 0$. □

Our construction here shows that starting from any nonsingular sq-corner magic square of order 3, by repeatedly choosing appropriate λ and $\vec{a}$, we can construct nonsingular sq-corner magic square of any odd order, and similarly for the even order cases.

From Proposition 2.4, a noteworthy special case of Theorem 2.11 is the next corollary.

Corollary 2.12. *For a regular zero magic square Z of odd order n with no repeated zero eigenvalue, the regular zero magic squares $\mathcal{S}_Z(x, \vec{v}, y, \vec{v}J)$ of order $n+2$ have also no repeated zero eigenvalue when x is nonzero.*

In conclusion, we construct a nonsingular sq-corner magic square $\mathcal{S}_{Z_M} \vec{a} + \lambda E$ of order $n+2$ when we know a nonsingular sq-corner magic square M of order n by choosing $\lambda \neq 0$ and $\vec{a} = (a_1, a_2, \ldots, a_{n+1})$ such that $a_1 \neq 0$. If we begin with

$$\begin{bmatrix} -13 & -4 & 8 & 9 \\ -4 & 7 & -9 & 6 \\ 8 & -9 & 11 & -10 \\ 9 & 6 & -10 & -5 \end{bmatrix},$$

by our construction, we can derive a sequence of nonsingular sq-corner magic squares of even orders: $4, 6, 8, 10, \ldots$. Also, if M is a nonsingular regular magic square of order 3, then this construction will provide a sequence of nonsingular regular magic squares of odd orders: $3, 5, 7, 9, \ldots$. Our construction here depends on the choices of λ and $\vec{a}$ which gives another form of a nonsingular regular magic square different from those given in [2, 3].

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