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LINEAR COMBINATIONS OF UNIVALENT HARMONIC MAPPINGS WITH COMPLEX COEFFICIENTS

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Abstract. We study the linear combinations $f(z) = \lambda f_1(z) + (1-\lambda)f_2(z)$ of two univalent harmonic mappings f_1 and f_2 in the cases when λ is some complex number. We determine the radius of close-to-convexity of f and establish some sufficient conditions for f to be locally-univalent and sense-preserving. Some known results reduce to particular cases of our general results.

1. Introduction and preliminaries

A complex-valued harmonic function f in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ can be represented as $f(z) = h(z) + \overline{g(z)}$, where both h and g are analytic in $\mathbb{D}$. The collection of all such functions is denoted by $\mathcal{H}$. In 1936, Lewy [6] proved that a harmonic function $f \in \mathcal{H}$, $f(z) = h(z)+\overline{g(z)}$, is sense-preserving and locally-univalent, if and only if its Jacobian, $J_f(z) = |h'(z)|^2 - |g'(z)|^2$, is positive or equivalently the dilatation function $w(z) = g'(z)/h'(z)$, $h'(z) \neq 0$ in $\mathbb{D}$, has the property $|w(z)| < 1$ in $\mathbb{D}$. We denote by $S_{\mathcal{H}}$ the subclass of $\mathcal{H}$ consisting of harmonic sense-preserving and univalent functions in $\mathbb{D}$ normalized by the conditions $f(0) = 0$ and $f_z(0) = 1$. Such mappings can be represented as

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad z \in \mathbb{D}.$$

Additionally, if a function f also satisfies $f_{\overline{z}}(0) = 0$, then the class of such functions is denoted by $S^0_{\mathcal{H}}$. The classical family S of normalized, univalent and analytic functions in $\mathbb{D}$ is a subclass of $S_{\mathcal{H}}$ with $g(z) \equiv 0$.

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the direction $\varphi \in [0, \pi)$ if the intersection of Ω and every line parallel to the one passing through the origin and $e^{i\varphi}$ is connected or empty. In particular, a domain is convex in the horizontal direction (CHD) if

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every line parallel to the real axis has either an empty or a connected intersection with the domain. A domain convex in every direction is a convex domain. A function is said to be convex in the direction φ if it maps the open unit disk $\mathbb{D}$ onto a domain convex in the direction φ. A domain $\Omega \subset \mathbb{C}$ is close-to-convex if the complement of Ω can be written as the union of non intersecting half lines. A normalized analytic function f in $\mathbb{D}$ is close-to-convex if there exists an analytic convex function ϕ in $\mathbb{D}$ such that $\mathfrak{Re}\left(\frac{f'(z)}{\phi'(z)}\right) > 0$, $z \in \mathbb{D}$. Clunie and Sheil-Small [2] obtained the following sufficient condition for a harmonic function to be close-to-convex.

Lemma 1.1. *If h, g are analytic in $\mathbb{D}$ with $|h'(0)| > |g'(0)|$ and $h(z) + \epsilon g(z)$ is close-to-convex for each ϵ with $|\epsilon| = 1$, then $f(z) = h(z) + \overline{g(z)}$ is harmonic close-to-convex in $\mathbb{D}$.*

As a consequence of the above lemma several other sufficient conditions for a harmonic function to be close-to-convex have been established, for example see [1,4, 11,12]. In particular, we recall the following sufficient conditions for close-to-convexity of a harmonic function.

Theorem 1.2 ([11]). *Let $f(z) = h(z) + \overline{g(z)}$, where h and g are analytic functions in $\mathbb{D}$ such that $h(0) = g(0) = 0$ and $h'(0) = 1$. Further, let ϕ be univalent, analytic and convex in $\mathbb{D}$. If f satisfies*

$$\mathfrak{Re}\left(e^{i\theta}\frac{h'(z)}{\phi'(z)}\right) > \left|\frac{g'(z)}{\phi'(z)}\right|$$

for all $z \in \mathbb{D}$ and for some real θ, then f is sense-preserving, harmonic univalent and close-to-convex in $\mathbb{D}$.

Theorem 1.3 ([1]). *Let $f(z) = h(z) + \overline{g(z)}$ be a sense-preserving harmonic mapping in $\mathbb{D}$, where $h \in S^*$ and $g(0) = 0$. If H and G are analytic functions defined by the relations $zH'(z) = h(z)$, $zG'(z) = -g(z)$, $H(0) = G(0) = 0$, then for each $|\lambda| \leq 1$, the harmonic function $F_\lambda(z) = H(z) + \overline{\lambda G(z)}$ is sense-preserving and close-to-convex in $\mathbb{D}$. In particular, $F(z) = H(z) + \overline{G(z)}$ is a close-to-convex mapping in $\mathbb{D}$.*

For two analytic functions f_1 and f_2 and a real number λ, $0 \leq \lambda \leq 1$, the function $\lambda f_1(z) + (1-\lambda) f_2(z) = f(z)(say)$ is called the linear combination of f_1 and f_2. Generally, the function f may not possess the same properties as those possessed by f_1 and f_2. For example, linear combination of two univalent functions may not be univalent. It is therefore a subject of interest to find conditions on f_1 and f_2 so that their linear combination has desired property. For more details we refer to [7,8,13–16]. Linear combination of two harmonic mappings $f_1(z) = h_1(z) + \overline{g_1(z)}$ and $f_2 = h_2(z) + \overline{g_2(z)}$, for $0 \leq \lambda \leq 1$, can be written as

$$f(z) = \lambda f_1(z) + (1-\lambda) f_2(z) = \lambda h_1(z) + (1-\lambda) h_2(z) + \overline{\lambda g_1(z) + (1-\lambda) g_2(z)}.$$

In 2012, Dorff and Rolf [3] presented sufficient conditions so that linear combination of two suitably harmonic functions is univalent and convex in the direction of imaginary axis. Several other authors also presented beautiful results in this direction, for example see [5, 17, 18, 20].

In 1971, Stump [19] studied linear combinations of two analytic functions by taking the constant λ a complex number instead of a real one. Motivated by this, in the present article we investigate the linear combination $f = \lambda f_1 + (1-\lambda) f_2$ of two univalent harmonic functions f_1 and f_2 where the constant λ is a complex number. Apart from proving several sufficient conditions for f to be sense-preserving and locally-univalent, the radius of close-to-convexity of f is obtained. Some known results reduce to particular cases of the results presented here. In order to prove our main results, we shall require the following results.

LEMMA 1.4. *A harmonic function $f = h + \overline{g}$, locally-univalent in $\mathbb{D}$, is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction θ, if and only if $h - e^{2i\theta} g$ is an analytic univalent mapping of $\mathbb{D}$ onto a domain convex in direction θ.*

LEMMA 1.5. *If $|u-a| \le d$ and $|v-a| \le d$ where a and d are real and $a > d \ge 0$, and $w = u\frac{1}{1+Ae^{i\alpha}} + v\frac{1}{1+A^{-1}e^{-i\alpha}}$, where A is real and $A > 0$ and $\alpha \in [0, \pi)$, then $\Re(w) \ge a - d\sec\alpha/2$.*

LEMMA 1.6. *Let f be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) \ne 0$ and let $K(z) = \frac{z}{(1+ze^{i\theta})(1+ze^{-i\theta})}$, $\theta \in \mathbb{R}$. If $\mathfrak{Re}\left(\frac{zf'(z)}{K(z)}\right) > 0$ $(z \in \mathbb{D})$, then f is convex in the horizontal direction (CHD).*

Lemma 1.4 is due to Clunie and Sheil-Small [2], whereas Lemmas 1.5 and 1.6 are given by Stump [19] and Pommerenke [10], respectively.

2. Main results

It is well known that every close-to-convex function in $\mathbb{D}$ is univalent in $\mathbb{D}$ and also that univalency and other geometric properties of individual mappings are not, generally, transferred to their convex combination. It is therefore of interest to find radius of the largest disc in which convex combination of two analytic/harmonic mappings is close-to-convex or exhibits other geometric properties like starlikeness, convexity etc. We refer the reader to [4, 9] for some glimpse of the radius problems of harmonic mappings.

In the first theorem below, we obtain radius of close-to-convexity of the linear combination of two univalent harmonic mappings which are shears of same convex univalent function in $\mathbb{D}$.

THEOREM 2.1. *Let ϕ be convex univalent in $\mathbb{D}$. For $j = 1, 2$, let $f_j(z) = h_j(z) + \overline{g_j(z)} \in S_{\mathcal{H}}$ such that $h_j(z) + g_j(z) = \phi(z)$. Then for a complex number λ, $F(z) = \lambda f_1(z) + (1-\lambda) f_2(z)$ is close-to-convex in $|z| < R$, where R is the least positive root of the equation $br^2 + r\left(b + \sec\frac{\beta}{2}\right) - 1 = 0$, with $b = |\lambda| + |1-\lambda|$ and $\beta = \arg\left(\frac{\lambda}{1-\lambda}\right)$, $0 \le \beta < \pi$.*

Proof. If $w_j(z) = g_j'(z)/h_j'(z)$ is a dilatation of f_j $(j = 1, 2)$, then $h_j'(z) = \frac{\phi'(z)}{1+w_j(z)}$ and $g_j'(z) = \frac{\phi' w_j(z)}{1+w_j(z)}$. Since $w_j(0) = 0$ and $|w_j(z)| \leq |z| = r < 1$, for $j = 1, 2$, it follows that

$$\left|\frac{1}{1+w_j(z)} - \frac{1}{1-r^2}\right| \leq \frac{r}{1-r^2} \tag{1}$$

and

$$\left|\frac{w_j(z)}{1+w_j(z)}\right| \leq \frac{r}{1-r}.$$

Let

$$\begin{aligned} F(z) &= \lambda f_1(z) + (1-\lambda) f_2(z) = \lambda h_1(z) + (1-\lambda) h_2(z) + \overline{(\overline{\lambda} g_1(z) + (1-\overline{\lambda}) g_2(z))} \\ &= H(z) + \overline{G(z)} \quad \text{(say)}, \end{aligned}$$

where $H(z) = \lambda h_1(z) + (1-\lambda) h_2(z)$ and $G(z) = \overline{\lambda} g_1(z) + (1-\overline{\lambda}) g_2(z)$. Now

$$\begin{aligned} \frac{H'(z)}{\phi'(z)} &= \frac{\lambda h_1'(z) + (1-\lambda) h_2'(z)}{\phi'(z)} \\ &= \frac{h_1'(z)}{\phi'(z)} \left(1 + \left(\frac{\lambda}{1-\lambda}\right)^{-1}\right)^{-1} + \frac{h_2'(z)}{\phi'(z)} \left(1 + \left(\frac{\lambda}{1-\lambda}\right)\right)^{-1}. \end{aligned}$$

In view of Lemma 1.5 and (1), we have

$$\mathfrak{Re}\left(\frac{H'(z)}{\phi'(z)}\right) \geq \frac{1}{1-r^2} - \frac{r}{1-r^2} \sec\frac{\beta}{2}. \tag{2}$$

Also,

$$\begin{aligned} \left|\frac{G'(z)}{\phi'(z)}\right| &= \left|\frac{\overline{\lambda} g_1'(z) + (1-\overline{\lambda}) g_2'(z)}{\phi'(z)}\right| \leq |\overline{\lambda}| \left|\frac{g_1'(z)}{\phi'(z)}\right| + |1-\overline{\lambda}| \left|\frac{g_2'(z)}{\phi'(z)}\right| \\ &= |\lambda| \left|\frac{w_1(z)}{1+w_1(z)}\right| + |1-\lambda| \left|\frac{w_2(z)}{1+w_2(z)}\right| \leq (|\lambda| + |1-\lambda|) \frac{r}{1-r}. \end{aligned}$$

Therefore,

$$\left|\frac{G'(z)}{\phi'(z)}\right| \leq \frac{br}{1-r}. \tag{3}$$

From equations (2) and (3) we have,

$$\mathfrak{Re}\left(\frac{H'(z)}{\phi'(z)}\right) - \left|\frac{G'(z)}{\phi'(z)}\right| \geq \frac{1}{1-r^2} - \frac{r}{1-r^2} \sec\frac{\beta}{2} - \frac{br}{1-r}.$$

Thus using Theorem 1.2 with $\theta = 0$, F is close-to-convex in $|z| < R$, where R is the least positive root of $br^2 + r\left(b + \sec\frac{\beta}{2}\right) - 1 = 0$. This completes the proof. □

The following examples illustrate our above result.

Example 2.2. Let $f_1(z) = h_1(z) + \overline{g_1(z)}$ be such that $h_1(z) + g_1(z) = \frac{z}{(1-z)}$ and $w_1(z) = \frac{g_1'(z)}{h_1'(z)} = z$. Then $h_1(z) = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{(1-z)}$ and $g_1(z) = \frac{1}{2}\frac{z}{(1-z)} - \frac{1}{4}\log\left(\frac{1+z}{1-z}\right)$. Also, let $f_2(z) = h_2(z) + \overline{g_2(z)}$ be such that $h_2(z) + g_2(z) = \frac{z}{(1-z)}$ and

$w_2(z) = \frac{g_2'(z)}{h_2(z)} = -z$. Then $h_2(z) = \frac{z-z^2/2}{(1-z)^2}$ and $g_2(z) = \frac{-z^2/2}{(1-z)^2}$.

(a) Set $\lambda = \frac{1+i}{2}$, so that $|\lambda| + |1-\lambda| = \sqrt{2}$ and $\beta = \arg\left(\frac{\lambda}{1-\lambda}\right) = \frac{\pi}{2}$. Then $F_1(z) = \frac{1+i}{2} f_1(z) + \frac{1-i}{2} f_2(z)$, is close-to-convex in the disc $|z| < r_1 \approx 0.3067$, where r_1 is the positive root of the equation $\sqrt{2}r^2 + 2\sqrt{2}r - 1 = 0$.

(b) Take $\lambda = \frac{1+i\sqrt{3}}{2}$, so that $|\lambda| + |1-\lambda| = 2$ and $\beta = \arg\left(\frac{\lambda}{1-\lambda}\right) = \frac{2\pi}{3}$. Thus $F_2(z) = \frac{1+i\sqrt{3}}{2} f_1(z) + \frac{1-i\sqrt{3}}{2} f_2(z)$, is close-to-convex in the disc $|z| < r_2 \approx 0.2247$, where r_2 is the least positive root of the equation $2r^2 + 4r - 1 = 0$.

Images of $|z| < r_1$ and $|z| < r_2$ under F_1 and F_2 are shown in Figure 1 (plotted using Mathematica).

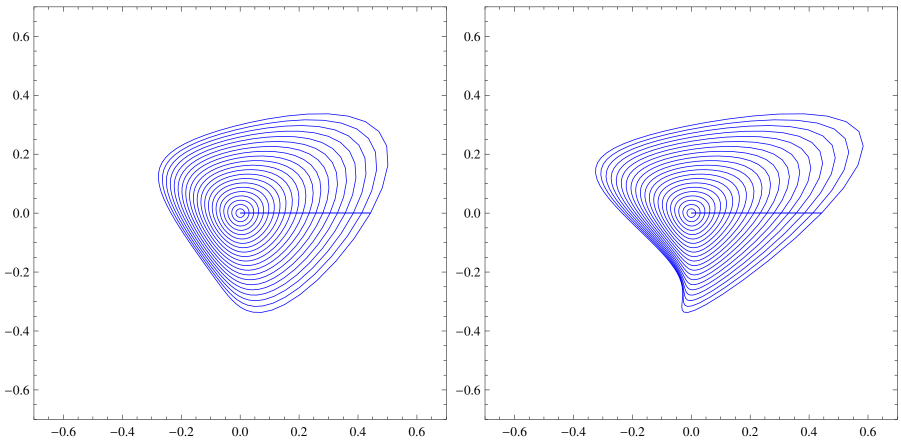

Figure 1: Images of $|z| < r_1$ under F_1 (left) and images of $|z| < r_2$ under F_2 (right)

Remark 2.3. We observe that the values of r_1 and r_2 obtained in Example 2.2, using Theorem 2.1, are not sharp.

In [3,5,20], it is proved that the linear combination $\lambda f_1(z)+(1-\lambda)f_2(z), 0 \le \lambda \le 1$ of two univalent harmonic mappings f_1 and f_2 with real constant λ is locally-univalent and sense-preserving as long as the dilatations of both harmonic mappings f_1 and f_2 are the same. In the next result we present a more general condition in the case of complex constant λ.

Theorem 2.4. *If $f_j(z) = h_j(z)+\overline{g_j(z)} \in S_{\mathcal{H}}, (j = 1, 2)$ with $g_1'(z)/h_1'(z) = g_2'(z)/h_2'(z)$. Then for a complex constant λ, $F(z) = \lambda f_1(z) + (1-\lambda)f_2(z)$ is locally-univalent and sense-preserving, provided*

$$\mathfrak{Im}(\lambda)\mathfrak{Im}(h_1'(z)\overline{h_2'(z)}) \le 0. \quad (4)$$

Proof. Let $w(z) = g_1'(z)/h_1'(z) = g_2'(z)/h_2'(z)$. Then for a complex constant λ, we have

$$\begin{aligned} F(z) &= \lambda f_1(z) + (1-\lambda) f_2(z) = \lambda h_1(z) + (1-\lambda) h_2(z) + \overline{\overline{\lambda} g_1(z) + (1-\overline{\lambda}) g_2(z)} \\ &= H(z) + \overline{G(z)} \quad \text{(say)}, \end{aligned}$$

where, $H(z) = \lambda h_1(z) + (1-\lambda)h_2(z)$ and $G(z) = \overline{\lambda} g_1(z) + (1-\overline{\lambda}) g_2(z)$. Since $|w(z)| < 1$, we have

$$\begin{aligned} |G'(z)/H'(z)| &= \left| \frac{\overline{\lambda} g_1'(z) + (1-\overline{\lambda}) g_2'(z)}{\lambda h_1'(z) + (1-\lambda) h_2'(z)} \right| \qquad (5) \\ &= |w(z)| \left| \frac{\overline{\lambda} h_1'(z) + (1-\overline{\lambda}) h_2'(z)}{\lambda h_1'(z) + (1-\lambda) h_2'(z)} \right| < \left| \frac{\overline{\lambda} h_1'(z) + (1-\overline{\lambda}) h_2'(z)}{\lambda h_1'(z) + (1-\lambda) h_2'(z)} \right|. \end{aligned}$$

Thus F is locally-univalent and sense-preserving if $\left| \frac{\overline{\lambda} h_1'(z) + (1-\overline{\lambda}) h_2'(z)}{\lambda h_1'(z) + (1-\lambda) h_2'(z)} \right| \le 1$. That is, if $\mu(\lambda) = |\lambda h_1'(z) + (1-\lambda) h_2'(z)|^2 - \left|\overline{\lambda} h_1'(z) + (1-\overline{\lambda}) h_2'(z)\right|^2 \ge 0$. After computations we have, $\mu(\lambda) = 2\mathfrak{Re}\left(2i\mathfrak{Im}(\lambda) h_1'(z)\overline{h_2'(z)}\right) = -4\mathfrak{Im}(\lambda)\mathfrak{Im}(h_1'(z)\overline{h_2'(z)})$. Thus F is locally-univalent and sense-preserving, provided $\mathfrak{Im}(\lambda)\mathfrak{Im}(h_1'(z)\overline{h_2'(z)}) \le 0$. □

Remark 2.5. In Theorem 2.4, if we set λ as a real constant then the condition (4) is trivially true.

In the case when $w_1(z) = g_1'(z)/h_1'(z) \neq w_2(z) = g_2'(z)/h_2'(z)$, Theorem 2.4 can be stated in a more general way as follows.

Theorem 2.6. *Let $f_j(z) = h_j(z) + \overline{g_j(z)} \in S_{\mathcal{H}}$ $(j = 1, 2)$ be harmonic univalent mappings and w_1, w_2 the dilatations of f_1 and f_2, respectively. For a complex constant λ, suppose that*

$$\mathfrak{Re}\left((\lambda(1-\overline{\lambda}) - \overline{\lambda}(1-\lambda) w_1(z)\overline{w_2(z)}) h_1'(z)\overline{h_2'(z)} \right) \ge 0. \qquad (6)$$

Then $F(z) = \lambda f_1(z) + (1-\lambda) f_2(z)$ is locally-univalent and sense-preserving in $\mathbb{D}$.

Proof. If $w_1(z) \neq w_2(z)$, then in view of (5), we get

$$|G'(z)/H'(z)| = \left| \frac{\overline{\lambda} w_1(z) h_1'(z) + (1-\overline{\lambda}) w_2(z) h_2'(z)}{\lambda h_1'(z) + (1-\lambda) h_2'(z)} \right|.$$

Thus F is locally-univalent and sense-preserving if

$$\psi(\lambda) = |\lambda h_1'(z) + (1-\lambda) h_2'(z)|^2 - \left|\overline{\lambda} w_1(z) h_1'(z) + (1-\overline{\lambda}) w_2(z) h_2'(z)\right|^2 > 0.$$

After a simplification, we get

$$\begin{aligned} \psi(\lambda) =& |\lambda|^2 |h_1'(z)|^2 (1 - |w_1(z)|^2) + |1-\lambda|^2 |h_2'(z)|^2 (1 - |w_2(z)|^2) \\ &+ 2\mathfrak{Re}\left((\lambda(1-\overline{\lambda}) - \overline{\lambda}(1-\lambda) w_1(z)\overline{w_2(z)}) h_1'(z)\overline{h_2'(z)} \right). \end{aligned}$$

Thus in view of (6), $\psi(\lambda) \ge 0$ and hence F is locally-univalent and sense-preserving in $\mathbb{D}$. □

Remark 2.7. In case of real constant λ, condition (6) coincides with [20, Theorem 2].

We end this paper by presenting another sufficient condition for the linear combination of some univalent harmonic mappings with complex constant λ to be convex in the horizontal direction (CHD).

THEOREM 2.8. *For $j = 1, 2$, let $f_j(z) = h_j(z) + \overline{g_j(z)} \in S_{\mathcal{H}}$ such that*

$$h_1(z) - e^{-2i\alpha}g_1(z) = \int_0^z \frac{e^{-i\alpha}d\zeta}{(1+\zeta e^{i\theta})(1+\zeta e^{-i\theta})},\ \theta \in \mathbb{R}$$

and

$$h_2(z) - e^{-2i\beta}g_2(z) = \int_0^z \frac{e^{-i\beta}d\zeta}{(1+\zeta e^{i\theta})(1+\zeta e^{-i\theta})},\ \theta \in \mathbb{R},$$

where $\alpha = \arg(\lambda)$ and $\beta = \arg(1-\lambda)$. Then $F(z) = \lambda f_1(z) + (1-\lambda)f_2(z)$ is convex in the horizontal direction provided it is locally-univalent.

Proof. Let

$$\begin{aligned} F(z) &= \lambda f_1(z) + (1-\lambda)f_2(z) \\ &= \lambda h_1(z) + (1-\lambda)h_2(z) + \left(\overline{\overline{\lambda}g_1(z) + (1-\overline{\lambda})g_2(z)}\right) = H(z) + \overline{G(z)} \text{ (say)}. \end{aligned}$$

Then, let $f(z) = H(z) - G(z) = \left(\lambda h_1(z) - \overline{\lambda}g_1(z)\right) + (1-\lambda)h_2(z) - (1-\overline{\lambda})g_2(z)$. Setting $\lambda = |\lambda|e^{i\alpha}$ and $1-\lambda = |1-\lambda|e^{i\beta}$, we get

$$\begin{aligned} f(z) =& |\lambda|e^{i\alpha}h_1(z) - |\lambda|e^{-i\alpha}g_1(z) + |1-\lambda|e^{i\beta}h_2(z) - |1-\lambda|e^{-i\beta}g_2(z), \\ =& |\lambda|e^{i\alpha}\left(h_1(z) - e^{-2i\alpha}g_1(z)\right) + |1-\lambda|e^{i\beta}\left(h_2(z) - e^{-2i\beta}g_2(z)\right). \end{aligned}$$

If $k(z) = \dfrac{z}{(1+ze^{i\theta})(1+ze^{-i\theta})}$, then

$$\begin{aligned} \mathfrak{Re}\left(z\frac{f'(z)}{k(z)}\right) =& |\lambda|\mathfrak{Re}\left(ze^{i\alpha}\frac{(h_1'(z) - e^{-2i\alpha}g_1'(z))}{k(z)}\right) \\ &+ |1-\lambda|\mathfrak{Re}\left(ze^{i\beta}\frac{(h_2'(z) - e^{-2i\beta}g_2'(z))}{k(z)}\right) = |\lambda| + |1-\lambda| > 0. \end{aligned}$$

Thus in view of Lemma 1.6, f is convex in the horizontal direction and also univalent in $\mathbb{D}$, see [10]. Subsequently, using Lemma 1.4, F is univalent and convex in the horizontal direction. □

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REFERENCES

[1] Y. Abu Muhanna, S. Ponnusamy, *Extreme point method and univalent harmonic mappings*, Complex analysis and dynamical system VI. Part 2, 223–237, Contemp. Math., 667, Israel Math. Conf. Proc. amer. Math. soc., Providence, RI, 2016.

[2] J. Clunie, T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math., **9**, (1984), 3–25.

[3] M. Dorff, J.Rolf, *Anamorphosis, mapping problems, and harmonic univalent functions*, in: Explorations in Complex Analysis, Math. Assoc. of America, Inc., Washington DC, (2012), 197–269.

[4] D. Kalaj, S. Ponnusamy, M. Vuorinen, *Radius of close-to-convexity and fully starlikeness of harmonic mappings*, Complex Var. Elliptic Equ., **59(4)**, (2014), 539–552.

[5] R. Kumar, S. Gupta, S. Singh, *Linear combinations of univalent harmonic mappings convex in the direction of the imaginary axis*, Bull. Malaysian Math. Sci. Soc., **39(2)**, (2016), 751–763.

[6] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc., **42(10)**, (1936), 689–692.

[7] T. H. MacGregor, *The univalence of a linear combination of convex mappings*, J. London Math. Soc., **44(1)**, (1969), 210–212.

[8] M. Obradović, S. Ponnusamy, *On harmonic combination of univalent functions*, Bull. Belg. Math. Soc. Simon Stevin, **19(3)**, (2012), 461–472.

[9] B-Y. Long, H-Y. Huang, *Radii of harmonic mappings in the plane*, J. Aust. Math. Soc., **102(3)**, (2017), 331–347.

[10] C. Pommerenke, *On starlike and close-to-convex functions*, Proc. London Math Soc., **13(3)**, (1963), 290–304.

[11] S. Ponnusamy, A. Sairam Kaliraj, *On harmonic close-to-convex functions*, Comput. Methods Funct. Theory, **12(2)**, (2012), 669–685.

[12] S. Ponnusamy, H. Yamamoto, H. Yanagihara, *Variability regions for certain families of harmonic univalent mappings*, Complex Var. Elliptic Equ., **58(1)**, (2013), 23–34.

[13] B. N. Rahmanov, *On the theory of univalent functions*, Dokl. Acad. Nauk, SSSR(N.S.), **82**, (1952), 341–344.

[14] M. S. Roberston, *The sum of univalent functions*, Duke Math. J., **37(3)**, (1970), 411–419.

[15] St. Ruscheweyh, K.J. Wirths, *Convex sums of convex univalent functions*, Indian J. Pure Appl. Math., **7(1)**, (1976), 49–52.

[16] H. Silverman, *Linear combinations of convex mappings*, Rocky Mountain J. Math., **5(4)**, (1975), 629–632.

[17] Y. Sun, Y-P. Jiang, Z-G. Wang, *On the convex combinations of slanted half-plane harmonic mappings*, J. Math. Anal., **6(3)**, (2015), 46–50.

[18] Y. Sun, A. Rasila, Y-P. Jiang, *Linear combinations of harmonic quasi conformal mappings convex in one direction*, Kodai Math. J., **39(2)**, (2016), 366–377.

[19] R. K. Stump, *Linear combinations of univalent functions with complex coefficients*, Canad. J. Math., **23**, (1971), 712–717.

[20] Z-G. Wang, Z-H. Liu, Y-C. Li, *On the linear combinations of harmonic mappings*, J. Math. Anal. Appl., **400(2)**, (2013), 452–459.

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