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74, 2 (2022), 89–100
June 2022

research paper
оригинални научни рад

ITERATIVE METHOD FOR FINDING ZEROS OF MONOTONE MAPPINGS AND FIXED POINT OF CERTAIN NONLINEAR MAPPING

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Abstract. In this article, an inertial Mann-type iterative algorithm is constructed using the so-called viscosity method of A. Moudafi, *Viscosity approximation methods for fixed-point problems*, J. Math. Anal. Appl. 241(1) (2000), 46–55. A strong convergence theorem of mean ergodic-type is proved using the sequence of the iterative algorithm for finding zeros of monotone mappings and the fixed point of a strict pseudo nonspreading mapping in a real Hilbert space. Finally, we apply our result to solve some minimization problem.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space and C is a nonempty subset of H. We denote by $x_n \rightharpoonup x$ and $x_n \to x$ weak and strong convergence of a sequence $\{x_n\}$ to x, respectively, and by $F(T)$ the set of fixed points of a mapping T. Let $T : C \to C$ be a mapping. The mapping T is said to be

– *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|, \ \forall \ x, y \in C$;

– *firmly nonexpansive* if $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \ \forall \ x, y \in C$ (see [5]);

– *quasi-nonexpansive* if $F(T)$ is nonempty and $\|Tx-p\| \leq \|x-p\|, \ \forall \ x \in C, p \in F(T)$;

– *nonspreading* if $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \ \forall \ x, y \in C$; it can be shown (see e.g. [7]) that T is nonspreading if and only $\|Tx-Ty\|^2 \leq \|x-y\|^2 + 2\langle x-Tx, y-Ty\rangle, \ \forall \ x, y \in C$;

– *k-strictly pseudo-nonspreading* (see e.g. [16]), if there exists a constant $k \in [0, 1)$ such that $\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 + 2\langle x-Tx, y-Ty\rangle, \ \forall \ x, y \in C$. It is shown in [16] that the class of k-strictly pseudo-nonspreading mappings is wider than the class of nonspreading mappings.

2020 Mathematics Subject Classification: 47H04, 47H06, 47H15, 47H17, 47J25
Keywords and phrases: Monotone operators; strict-pseudo contractive mappings; strict-pseudo nonspreading mappings; resolvents; Hilbert space.

Let C be a nonempty closed convex subset of H. A set-valued mapping $A : D(A) \subset H \to H$ is said to be monotone if for any $x, y \in D(A)$ and $x^* \in Ax, y^* \in Ay$, the following holds: $\langle x - y, x^* - y^* \rangle \geq 0$.

A monotone operator A on H is said to be maximal if A has no monotone extension, that is, its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator A on H and $r > 0$, the single-valued operator $J_r = (I + rA)^{-1} : 2^H \to D(A)$ is called the resolvent of A. It is known (see for instance [17]) that J_r is firmly nonexpansive, hence it is nonexpansive. For a constant $\alpha > 0$, a mapping $A : C \to H$ is said to be α-inverse strongly monotone if for all $x, y \in C$, $\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$.

For solving the problem of approximating fixed points of nonexpansive mappings, Mann [15] introduced in 1953 the following iteration process: $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$, where the initial guess $x_1 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $(0, 1)$. It is known that under appropriate conditions, the sequence $\{x_n\}$ converges weakly to a fixed point of T. In fact, even in real Hilbert space, Mann iteration may fail to converge strongly (see for instance [4]).

Finding a point $x^* \in F(T) \cap (A + B)^{-1}$ where T is some nonlinear operator, and A, B are monotone operators, is of interest in applications and have been studied extensively by many authors (see for instance, [9, 13, 17] and the references therein). Recently, in the case where $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an α-inverse strongly monotone mapping, and $B \subset H \times H$ is a maximal monotone operator, Takahashi et al. [17] proved a strong convergence theorem for finding a point of $F(T) \cap (A + B)^{-1}$, where $(A + B)^{-1}$ is the set of zero points of $(A + B)$.

The class of k-strictly pseudo nonspreading mappings was first introduced by Osilike and Isiogugu [16], as an important generalization of the class of nonspreading mappings. This class of mappings has been studied by many authors (see eg, [3, 16] and the references therein).

A question of interest in the study of zeroes of nonlinear maps, variational inequalities and related optimization problems is how to increase the convergence speed of iterative methods. Lots of efforts have been made in constructing iterative methods that improve and speed up the convergence of the resultant sequences. In this regard, incorporating the inertial extrapolation term in algorithms (resulting into inertial extrapolation methods) is of contemporary interest.

In 2019, Cholamjiak et al. [2] using the viscosity method of Moudafi [14], constructed and studied an inertial Mann-type algorithm for solving inclusion problem and fixed-point problem involving nonexpansive mapping in a real Hilbert space. Furthermore, they gave a numerical example to show the convergence behaviour of their algorithm. Precisely, they proved the following theorem.

Theorem 1.1 ([2]). *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be an α-inverse strongly monotone mapping, and $B : D(B) \subseteq C \to 2^H$ be a maximal monotone mapping. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for any $\lambda > 0$, $T : C \to C$ be a nonexpansive mapping. Suppose that $\Omega := F(T) \cap (A + B)^{-1}(0) \neq \emptyset$ and let $f : C \to C$ be a contraction mapping. For arbitrary $x_0, x_1 \in C$,* *let* $\{x_n\} \subset C$ *be a sequence generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \quad n \geq 1 \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S\big(\alpha_n f(x_n) + (1-\alpha_n)J^B_{\lambda_n}(y_n - \lambda_n A y_n)\big), \end{cases} \tag{1}$$

where $\{\alpha_n\} \subset (0,1)$*,* $\{\beta_n\} \subset (0,1)$*,* $\{\lambda_n\} \subset (0,2\alpha)$*,* $\{\theta_n\} \subset [0,\theta]$*,* $\theta \in [0,1)$ *satisfy the conditions:*
(C1) $\lim \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$; *(C2)* $\liminf \beta_n(1-\beta_n) > 0$;
(C3) $0 < \liminf \lambda_n \leq \limsup \lambda_n < 2\alpha$; *(C4)* $\lim \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.
Then $\{x_n\}$ *converges strongly to an element of* Ω.

Motivated by the aforementioned results, it is our aim in this article to construct an inertial-Mann-type algorithm using the viscosity method and prove mean convergence theorem of Baillon-type for solving inclusion and fixed-point problems involving k-strictly pseudo nonspreading mapping T in a real Hilbert space. Furthermore, in the case when the mapping T is nonexpansive, we adopt the example of Cholamjiak et al. [2] and compare their algorithm with ours for efficiency in terms of computer time. Our results improve and generalize those of Takahashi et al. [17], Liu et al. [11], Osilike et al. [16] and a host of other recent important results.

2. Preliminaries

In the sequel, we shall make use of the following definition and lemmas.

Definition 2.1. Let X be a normed linear space and K a nonempty subset of X. A mapping $T : K \to K$ is said to be demiclosed at $y \in K$ if for any sequence $\{x_n\} \subset K$ which converges weakly to $x \in K$, strong convergence of the sequence $\{Tx_n\}$ to y in K implies that $Tx = y$.

Lemma 2.2. *Let* H *be a real Hilbert space. Then for all* $x, y \in H$ *the following holds* $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$.

Lemma 2.3 ([5]). *Assume that* T *is a nonexpansive self-mapping of closed convex subset* C *of a real Hilbert space* H*. If* T *has a fixed point, then* $(I-T)$ *is demiclosed at zero.*

Lemma 2.4 ([16]). *Let* H *be a real Hilbert space,* C *be a nonempty closed convex subset of* H *and* $T : C \to C$ *be a* k*-strictly pseudo-nonspreading mapping.*
(i) If $F(T) \neq \emptyset$*, then* $F(T)$ *is closed and convex;* *(ii)* $I - T$ *is demiclosed at zero.*

Lemma 2.5 ([1]). *Let* C *be a nonempty closed convex subset of* H *and* $T : C \to C$ *be a* k*- strictly pseudo-nonspreading mapping with* $F(T) \neq \emptyset$*. Let* $T_\beta = \beta I + (1-\beta)T$*,* $\beta \in [k,1)$*. Then the following conclusions hold:*
(i) $F(T) = F(T_\beta)$*;*
(ii) $I - T_\beta$ *is demiclosed at zero;*

(iii) $\|T_\beta x - T_\beta y\|^2 \leq \|x-y\|^2 + \frac{2}{(1-\beta)}\langle x - T_\beta x, y - T_\beta y\rangle$;

(iv) T_β *is a quasi-nonexpansive mapping.*

LEMMA 2.6 ([13]). *Let* $A : C \to H$ *be an* α*-inverse strongly monotone mapping, and let* B *be a maximal monotone operator on* H *with* $D(B) \subset C$. *Then for any* $\sigma > 0$, *the following holds;* $(A+B)^{-1}(0) = F(J_\sigma^B(I - \sigma A))$.

LEMMA 2.7 ([17])). *Let* B *be a maximal monotone operator on* H. *Then, for any* $s, t \in \mathbb{R}$ *with* $s, t > 0$ *and for any* $x \in H$, *the following hold:*
(i) $\|J_s^B x - J_t^B x\| \leq \frac{|s-t|}{s}\|x - J_s^B x\|$. *(ii)* $F(J_s^B) = B^{-1}(0)$.

LEMMA 2.8 ([10]). *Let* $A : C \to H$ *be an* α*-inverse strongly monotone mapping. Then for any* $\sigma \in (0, 2\alpha], (I - \sigma A)$ *is nonexpansive.*

LEMMA 2.9 ([12]). *Let* $\{a_n\}$ *and* $\{c_n\}$ *be sequences of nonnegative real numbers such that* $a_{n+1} \leq (1-\delta_n)a_n + b_n + c_n$, $\forall\ n \geq 1$, *where* $\{\delta_n\}$ *is a sequence in* $(0,1)$ *and* $\{b_n\}$ *is a real sequence. Assume* $\sum_{n=1}^\infty c_n < \infty$. *Then the following results hold:*
(i) If $b_n \leq \delta_n M$ *for some* $M \geq 0$, *then* $\{a_n\}$ *is a bounded sequence.*

(ii) If $\sum_{n=1}^\infty \delta_n = \infty$ *and* $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3. Main result

We now prove the main result of this manuscript.

THEOREM 3.1. *Let* C *be a nonempty closed convex subset of* H, $A : C \to H$ *be an* α*-inverse strongly monotone mapping and* $B : D(B) \subseteq C \to 2^H$ *be a maximal monotone mapping. Let* $J_\lambda^B = (I + \lambda B)^{-1}$ *be the resolvent of* B *for any* $\lambda > 0$, $T : C \to C$ *be a* k*-strictly pseudo nonspreading mapping with* $k \in (0,1)$. *Suppose that* $\Gamma := F(T) \cap (A+B)^{-1}(0) \neq \emptyset$ *and let* $f : C \to C$ *be a contraction map. For arbitrary* $x_0, x_1 \in C$, *let* $\{x_n\} \subset C$ *be a sequence generated by*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \quad n \geq 1 \\ z_n = J_{\sigma_n}^B(I - \sigma_n A)w_n \\ y_n = \frac{1}{n}\sum_{i=0}^{n-1} T_\beta^i z_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n \end{cases} \tag{2}$$

where $T_\beta := \beta I + (1-\beta)T$, $\beta \in [k, 1)$ *and the following conditions are satisfied:*
(I) $\{\alpha_n\}$ *is a sequence in* $(0,1)$;

(II) $\alpha_n \to 0$ *as* $n \to \infty$, $\sum \alpha_n = \infty$, $\{\theta_n\} \subset [0, \theta]$ *with* $\theta \in [0,1)$.

(III) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

(IV) $\{\sigma_n\}$ *is a sequence in* $(0, \infty)$ *and there exist* $a, b \in \mathbb{R}$ *with* $0 < a \leq \sigma_n \leq b < 2\alpha$, $\forall\ n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ *constructed by algorithm* (2) *converges strongly to* $x^* = P_\Gamma f(x^*)$, *where* P_Γ *is the metric projection of* H *onto* Γ.

The following remarks explain the advantages of our proposed algorithm (2).

REMARK 3.2. 1. It is shown in Chang et al. [1], that the map T_β is quasi nonexpansive. It is well known that every nonexpansive map with a nonempty fixed point set is a quasi nonexpansive map. Consequently, our result, Theorem 3.1 is a generalization of the main result of Cholamjiak [2]. Furthermore, we need less control sequences to prove our result than in the case of result [2].

2. In the case when the map T is nonspreading and the sequence $\theta_n = 0$, $\forall\ n \geq 1$. Algorithm 2 reduces to the Algorithm considered by Kurokawa and Takahashi [8].

3. When $A = B = I$, and $\theta_n = 0$, $\forall\ n \geq 1$, we recover the result of Osilike and Isiogugu [16].

4. Our result is more general than that of Takahashi et al. [17] in the sense that our result holds for a more general class of quasi-nonexpansive maps. Also, our algorithm contains the inertial term which is known to speed up the rate of convergence of iterative algorithms.

Proof. We divide the proof into several steps.
Step 1. $\{x_n\}$ is bounded.

Observe that by Lemma 2.8, $(I-\sigma_n A)$ is nonexpansive. Also it is known that $J^B_{\sigma_n}$, $\forall\ n\geq 1$ is a nonexpansive map. For arbitrary $w\in\Gamma$, we have the following estimate:

$$\begin{aligned}\|z_n - w\| &\leqslant \|J^B_{\sigma_n}(1-\sigma_n A)w_n - J^B_{\sigma_n}(I-\sigma_n A)w\| \leqslant \|w_n - w\| \\ &= \|x_n + \theta_n(x_n - x_{n-1}) - w\| \leqslant \|x_n - w\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3}$$

Using Lemma 2.5, we obtain

$$\|T_\beta z_n - w\|^2 = \|T_\beta z_n - T_\beta w\|^2 \leqslant \|z_n - w\|^2 + \frac{2}{1-\beta}\langle z_n - T_\beta z_n, w - T_\beta w\rangle = \|z_n - w\|^2.$$

This implies that $\|T_\beta z_n - w\| \leqslant \|z_n - w\|$. Assume that $\|T^r_\beta z_n - w\| \leqslant \|z_n - w\|$ for some $r \geqslant 1$. Then

$$\begin{aligned}&\|T^{r+1}_\beta z_n - w\|^2 = \|T_\beta(T^r_\beta z_n) - T_\beta w\|^2 \\ &\leqslant \|T^r_\beta z_n - w\|^2 + \frac{2}{1-\beta}\langle T^r_\beta z_n - T^{r+1}_\beta z_n, w - T_\beta w\rangle = \|T^r_\beta z_n - w\|^2 \leqslant \|z_n - w\|^2.\end{aligned}$$

Thus $\|T^{r+1}_\beta z_n - w\| \leqslant \|z_n - w\|$ and so, by induction, we have that $\|{T_\beta}^i z_n - w\| \leqslant \|z_n - w\|$, $\forall i, n \in N$. Hence,

$$\|y_n - w\| = \|\frac{1}{n}\sum_{i=0}^{n-1}{T_\beta}^i z_n - w\| \leqslant \frac{1}{n}\sum_{i=0}^{n-1}\|{T_\beta}^i z_n - w\| \leqslant \frac{1}{n}\sum_{i=0}^{n-1}\|z_n - w\| = \|z_n - w\| \tag{4}$$

Applying (3) in (4) gives $\|y_n - w\| \leqslant \|x_n - w\| + \theta_n\|x_n - x_{n-1}\|$. From (2), we get

$$\begin{aligned}&\|x_{n+1} - w\| = \|\alpha_n f(x_n) + (1-\alpha_n)y_n - w\| = \|\alpha_n(f(x_n) - w) + (1-\alpha_n)(y_n - w)\| \\ &\leqslant \alpha_n\|f(x_n) - f(w)\| + \alpha_n\|f(w) - w\| + (1-\alpha_n)\|y_n - w\|\end{aligned}$$

$$\begin{aligned}
&\leqslant \alpha_n\|f(x_n)-f(w)\|+\alpha_n\|f(w)-w\|+(1-\alpha_n)[\|x_n-w\|+\theta_n\|x_n-x_{n-1}\|]\\
&=\alpha_n\|f(x_n)-f(w)\|+\alpha_n\|f(w)-w\|+(1-\alpha_n)\|x_n-w\|+(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|\\
&\leq \alpha_n\beta\|x_n-w\|+\alpha_n\|f(w)-w\|+(1-\alpha_n)\|x_n-w\|+(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|\\
&=(1-\alpha_n(1-\beta))\|x_n-w\|+\alpha_n(1-\beta)\Big[\frac{\|f(w)-w\|}{(1-\beta)}+\frac{(1-\alpha_n)\theta_n}{\alpha_n(1-\beta)}\|x_n-x_{n-1}\|\Big]. \quad (5)
\end{aligned}$$

Applying Lemma 2.9 (i) and condition (III) above in (5), we have that $\{\|x_n-w\|\}$ is bounded and so $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{w_n\}$, $\{z_n\}$, $\{f(x_n)\}$ and $\{{T_\beta}^n z_n\}$ are all bounded. Since $\{\|x_n-w\|\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty}\|x_{n_j}-w\|$ exists. Again, since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$, say, of $\{x_{n_j}\}$ which we still call $\{x_{n_j}\}$ such that $x_{n_j}\rightharpoonup v\in C$ as $j\to\infty$.

We prove that $v\in\Gamma$. We first prove that $v\in F(T)$. Since $\|x_{n+1}-y_n\|=\alpha_n\|f(x_n)-y_n\|$, replacing n by n_j , we have $\|x_{n_j+1}-y_{n_j}\|=\alpha_{n_j}\|f(x_{n_j})-y_{n_j}\|$. This together with condition (II) and the fact that $\{y_n\}$ is bounded, yield $\|x_{n_j+1}-y_{n_j}\|\to 0$ as $j\to\infty$. Thus, $y_{n_j}\rightharpoonup v$ as $j\to\infty$. Since T is strictly pseudo nonspreading, then for all $\xi\in C$ and for any $k=0,1,2,\cdots,n-1$, we have, using Lemma 2.5 (iii),

$$\begin{aligned}
\|T_\beta^{k+1}z_n-T_\beta\xi\|^2 &=\|T_\beta\big(T_\beta^k z_n\big)-T_\beta\xi\|^2\\
&\leq\|T_\beta^k z_n-\xi\|^2+\frac{2}{(1-\beta)}\big\langle T_\beta^k z_n-T_\beta^{k+1}z_n,\xi-T_\beta\xi\big\rangle\\
&=\|T_\beta^k z_n-T_\beta\xi+T_\beta\xi-\xi\|^2+\frac{2}{(1-\beta)}\big\langle T_\beta^k z_n-T_\beta^{k+1}z_n,\xi-T_\beta\xi\big\rangle\\
&=\|T_\beta^k z_n-T_\beta\xi\|^2+\|T_\beta\xi-\xi\|^2+2\big\langle T_\beta^k z_n-T_\beta\xi,T_\beta\xi-\xi\big\rangle\\
&\quad+\frac{2}{(1-\beta)}\big\langle T_\beta^k z_n-T_\beta^{k+1}z_n,\xi-T_\beta\xi\big\rangle. \quad (6)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|T_\beta^{k+1}z_n-T_\beta\xi\|^2-\|T_\beta^k z_n-T_\beta\xi\|^2\leq\\
&\|T_\beta\xi-\xi\|^2+2\big\langle T_\beta^k z_n-T_\beta\xi,T_\beta\xi-\xi\big\rangle+\frac{2}{(1-\beta)}\big\langle T_\beta^k z_n-T_\beta^{k+1}z_n,\xi-T_\beta\xi\big\rangle. \quad (7)
\end{aligned}$$

Summing (7) from $k=0$ to $n-1$ and dividing by n, we have

$$\begin{aligned}
&\frac{1}{n}\big(\|T_\beta^n z_n-T_\beta\xi\|^2-\|y_n-T_\beta\xi\|^2\big)\leq\\
&\|T_\beta\xi-\xi\|^2+2\big\langle y_n-T_\beta\xi,T_\beta\xi-\xi\big\rangle+\frac{2}{n(1-\beta)}\big\langle y_n-T_\beta^n z_n,\xi-T_\beta\xi\big\rangle. \quad (8)
\end{aligned}$$

Replacing n with n_j in (8), we obtain

$$\begin{aligned}
&\frac{1}{n_j}\big(\|T_\beta^{n_j}z_{n_j}-T_\beta\xi\|^2-\|y_{n_j}-T_\beta\xi\|^2\big)\leq\\
&\|T_\beta\xi-\xi\|^2+2\big\langle y_{n_j}-T_\beta\xi,T_\beta\xi-\xi\big\rangle+\frac{2}{n_j(1-\beta)}\big\langle y_{n_j}-T_\beta^{n_j}z_{n_j},\xi-T_\beta\xi\big\rangle. \quad (9)
\end{aligned}$$

Letting $j\to\infty$ in (9) and given the fact that $\{z_n\}$ and $\{T_\beta^n z_n\}$ are bounded, we obtain $0 \leq \|T_\beta\xi - \xi\|^2 + 2\langle v - T_\beta\xi, T_\beta\xi - \xi\rangle$. In particular, for $\xi = v$ we have

$$0 \leq \|T_\beta v - v\|^2 + 2\langle v - T_\beta v, T_\beta v - v\rangle = \|T_\beta v - v\|^2 - 2\|T_\beta v - v\|^2.$$

This implies that $v = T_\beta v$. That is, $v \in F(T_\beta)$ and by Lemma 2.5 (i), we have $v \in F(T)$.

Step 2. We prove that $\|Aw_{n_j} - Aw\| \to 0$ as $j \to \infty$.

From (2) and using the fact that A is α-inverse strongly monotone and convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|x_{n+1}-w\|^2 = \|\alpha_n f(x_n)+(1-\alpha_n)(y_n-w)\|^2\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+(1-\alpha_n)\|y_n-w\|^2 \leqslant \alpha_n\|f(x_n)-w\|^2+(1-\alpha_n)\|z_n-w\|^2\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+(1-\alpha_n)\|J^B_{\sigma_n}(1-\sigma_nA)w_n-J^B_{\sigma_n}(1-\sigma_nA)w\|^2\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+(1-\alpha_n)\|w_n-w-(\sigma_nAw_n-\sigma_nAw)\|^2\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+\|w_n-w-\sigma_n(Aw_n-Aw)\|^2\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+\|w_n-w\|^2-2\sigma_n\langle w_n-w, Aw_n-Aw\rangle+{\sigma_n}^2\|Aw_n-Aw\|^2\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+\|w_n-w\|^2-2\alpha\sigma_n\|Aw_n-Aw\|^2+{\sigma_n}^2\|Aw_n-Aw\|^2\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+\|x_n-w\|^2+\theta_n\|x_n-x_{n-1}\|^2-2\alpha\sigma_n\|Aw_n-Aw\|^2+{\sigma_n}^2\|Aw_n-Aw\|^2.
\end{aligned}$$

$$(2\alpha\sigma_n-{\sigma_n}^2)\|Aw_n-Aw\|^2 \leqslant \alpha_n\|f(x_n)-w\|^2+\|x_n-w\|^2+\theta_n\|x_n-x_{n-1}\|^2-\|x_{n+1}-w\|^2.$$

Passing to subsequence, we have $(2\alpha\sigma_{n_j} - {\sigma_{n_j}}^2)\|Aw_{n_j} - Aw\|^2 \leqslant \alpha_{n_j}\|f(x_{n_j}) - w\|^2 + \|x_{n_j} - w\|^2 + \theta_{n_j}\|x_{n_j} - x_{n_j-1}\|^2 - \|x_{n_j+1} - w\|^2$. Applying conditions (II) and (IV), and the fact that $\lim_{j\to\infty}\|x_{n_j} - w\|$ exists, we obtain: $\|Aw_{n_j} - Aw\| \to 0$ as $j \to \infty$.

Step 3. We prove that $\|z_{n_j} - w_{n_j}\| \to 0$ as $j \to \infty$.

Since J^B_λ is firmly nonexpansive, we obtain

$$\begin{aligned}
\|z_n - w\|^2 =&\|J^B_{\sigma_n}(1 - \sigma_nA)w_n - J^B_{\sigma_n}(1 - \sigma_nA)w\|^2\\
\leqslant&\langle z_n - w, (I - \sigma_nA)w_n - (I - \sigma_nA)w\rangle\\
=&\frac{1}{2}\{\|z_n - w\|^2 + \|(I - \sigma_nA)w_n - (I - \sigma_nA)w\|^2\\
&- \|z_n - w - [(I - \sigma_nA)w_n - (I - \sigma_nA)w]\|^2\}\\
\leqslant&\frac{1}{2}\{\|z_n - w\|^2 + \|w_n - w\|^2 - \|z_n - w - (I - \sigma_nA)w_n + (I - \sigma_nA)w\|^2\}\\
=&\frac{1}{2}\{\|z_n - w\|^2 + \|w_n - w\|^2 - \|z_n - w_n\|^2\\
&- 2\sigma_n\langle z_n - w_n, Aw_n - Aw\rangle - {\sigma_n}^2\|Aw_n - Aw\|^2\}
\end{aligned}$$

This implies that

$$\frac{1}{2}\|z_n-w\|^2 \leqslant \frac{1}{2}\{\|w_n-w\|^2-\|z_n-w_n\|^2-2\sigma_n\langle z_n-w_n, Aw_n-Aw\rangle-{\sigma_n}^2\|Aw_n-Aw\|^2\}.$$

So, $\|z_n-w\|^2 \leqslant \|w_n-w\|^2-\|z_n-w_n\|^2-2\sigma_n\langle z_n-w_n, Aw_n-Aw\rangle-{\sigma_n}^2\|Aw_n-Aw\|$.

From (2),

$$\|x_{n+1} - w\|^2 \leqslant\alpha_n\|f(x_n) - w\|^2 + (1 - \alpha_n)\|y_n - w\|^2$$

$$\begin{aligned}
&\leqslant \alpha_n\|f(x_n)-w\|^2+(1-\alpha_n)\|z_n-w\|^2\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+(1-\alpha_n)[\|w_n-w\|^2-\|z_n-w_n\|^2\\
&\quad -2\sigma_n\langle z_n-w_n, Aw_n-Aw\rangle-{\sigma_n}^2\|Aw_n-Aw\|^2]\\
\|z_n-w_n\|^2 &\leqslant \alpha_n\|f(x_n)-w\|^2+\|w_n-w\|^2-\|x_{n+1}-w\|^2\\
&\quad -2\sigma_n\langle z_n-w_n, Aw_n-Aw\rangle-{\sigma_n}^2\|Aw_n-Aw\|^2]\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+\|w_n-w\|^2-\|x_{n+1}-w\|^2-{\sigma_n}^2\|Aw_n-Aw\|^2\\
&\leqslant \alpha_n\|f(x_n)-w\|^2+\|x_n-w\|^2+\theta_n\|x_n-x_{n-1}\|^2\\
&\quad -\|x_{n+1}-w\|^2-{\sigma_n}^2\|Aw_n-Aw\|^2 \qquad (10)
\end{aligned}$$

Passing to subsequence, applying condition (1) and (2) together with the fact that $\lim_{j\to\infty}\|x_{n_j}-w\|$ exists and conclusion of **Step 2.** in (10), we obtain $\|z_{n_j}-w_{n_j}\|\to 0$ as $j\to\infty$. Observe that $\|w_{n_j}-x_{n_j}\|=\theta_{n_j}\|x_{n_j}-x_{n_j-1}\|\to 0$ as $j\to\infty$.

Step 4. We show that $v\in(A+B)^{-1}(0)$.

By condition (IV), there exists a subsequence $\{\sigma_{n_j}\}$ of $\{\sigma_n\}$ such that $\sigma_{n_j}\to\sigma\in[a,b]$. Applying Lemma 2.7, we have

$$\begin{aligned}
&\|J_\sigma^B(I-\sigma A)w_n-z_n\|\leqslant\|J_\sigma^B(I-\sigma A)w_n-J_\sigma^B(I-\sigma_nA)w_n\|+\|J_\sigma^B(I-\sigma_nA)w_n-z_n\|\\
&\leqslant\|(I-\sigma A)w_n-(I-\sigma_nA)w_n\|+\|J_\sigma^B(I-\sigma_nA)w_n-J_{\sigma_n}^B(I-\sigma_nA)w_n\|\\
&\leqslant|\sigma_n-\sigma|\|Aw_n\|+\frac{|\sigma_n-\sigma|}{\sigma}\|J_\sigma^B(I-\sigma_nA)w_n-(I-\sigma_nA)w_n\|\\
&\leqslant|\sigma_n-\sigma|\|Aw_n\|+\frac{|\sigma_n-\sigma|}{\sigma}K \qquad (11)
\end{aligned}$$

for some $K>0$ such that $K:=\sup_n\|J_\sigma^B(I-\sigma_nA)w_n-(I-\sigma_nA)w_n\|$. Replacing n with n_j in (11) and using boundedness of $\{Aw_n\}$, we get as $j\to\infty$ that

$$\|J_\sigma^B(I-\sigma A)w_{n_j}-z_{n_j}\|\to 0 \qquad (12)$$

But
$$\begin{aligned}
\|J_\sigma^B(I-\sigma A)w_n-w_n\|&\leqslant\|J_\sigma^B(I-\sigma A)w_n-z_n+z_n-w_n\|\\
&\leqslant\|J_\sigma^B(I-\sigma A)w_n-z_n\|+\|z_n-w_n\| \qquad (13)
\end{aligned}$$

Replacing n with n_j in (13), using **Step 3.** and conclusion (12) we get $\|J_\sigma^B(I-\sigma A)w_{n_j}-w_{n_j}\|\to 0$ as $j\to\infty$.

Thus from Lemma 2.3, we have that $v\in F\big(J_\sigma^B(I-\sigma A)\big)$. Since by Lemma 2.6, $F\big(J_\sigma^B(I-\sigma A)\big)=(A+B)^{-1}(0)$, it implies that $v\in(A+B)^{-1}(0)$.

Step 5. We show that $x_n\to P_\Gamma f(v)$ as $n\to\infty$.

Without loss of generality, there exists a subsequence $\{x_{n_{j_i}+1}\}$ of $\{x_{n_j+1}\}$ which we shall call $\{x_{n_{j_i}+1}\}$ such that

$$\lim_{n\to\infty}\sup\langle u-P_\Gamma f(v),x_{n+1}-P_\Gamma f(v)\rangle=\lim_{n\to\infty}\langle f(v)-P_\Gamma f(v),x_{n_j+1}-P_\Gamma f(v)\rangle.$$

Since P is the metric projection of H onto Γ and $x_{n_j+1}\to v\in\Gamma$, we have

$$\lim_{j\to\infty}\langle f(v)-P_\Gamma f(v),x_{n_j+1}-P_\Gamma f(v)\rangle=\langle f(v)-P_\Gamma f(v),v-P_\Gamma f(v)\rangle\leqslant 0.$$

Hence, $\lim_{n\to\infty}\sup\langle f(v)-P_\Gamma f(v), x_{n+1}-P_\Gamma f(v)\rangle \leqslant 0$. Using Lemma 2.2, we have

$$\|x_{n+1}-P_\Gamma f(v)\|^2 = \|\alpha_n(f(v))-P_\Gamma f(v)+(1-\alpha_n)(y_n-P_\Gamma f(v))\|^2$$
$$\leqslant (1-\alpha_n)^2\|y_n-P_\Gamma f(v)\|^2+2\alpha_n\langle f(v)-P_\Gamma f(v), x_{n+1}-P_\Gamma f(v)\rangle$$
$$\leqslant (1-\alpha_n)^2[\|x_n-P_\Gamma f(v)\|^2+\theta_n\|x_n-x_{n-1}\|^2]+2\alpha_n\langle f(v)-P_\Gamma f(v), x_{n+1}-P_\Gamma f(v)\rangle$$
$$\leqslant (1-\alpha_n)^2\|x_n-P_\Gamma f(v)\|^2+(1-\alpha_n)^2\theta_n\|x_n-x_{n-1}\|^2+2\alpha_n\langle f(v)-P_\Gamma f(v), x_{n+1}-P_\Gamma f(v)\rangle$$
$$\leqslant (1-\alpha_n)\|x_n-P_\Gamma f(v)\|^2+\theta_n\|x_n-x_{n-1}\|^2+2\alpha_n\langle f(v)-P_\Gamma f(v), x_{n+1}-P_\Gamma f(v)\rangle.$$

Hence, we obtain from Lemma 2.9 (ii), that $x_n \to P_\Gamma f(v)$ as $n\to\infty$.
This completes the proof. □

For T a nonexpansive mapping with nonempty fixed point set, we have the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of H, $A : C \to H$ be an α-inverse strongly monotone mapping, and $B : D(B) \subseteq C \to 2^H$ a maximal monotone mapping. Let $J_\lambda^B = (I+\lambda B)^{-1}$ be the resolvent of B for any $\lambda > 0$, $T : C \to C$ be a nonexpansiove mapping. Suppose that $\Omega := F(T)\cap(A+B)^{-1}(0) \neq \emptyset$, and let $f : C \to C$ be a contraction mapping. For arbitrary $x_0, x_1, \in C$, and let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \ n \geq 1 \\ z_n = J_\lambda^B(I-\sigma_n A)w_n \\ y_n = \frac{1}{n}\sum_{i=1}^{n-1} T^i z_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n \end{cases} \tag{14}$$

where the following conditions are satisfied:
(I) $\{\alpha_n\}$ is a sequence in $(0,1)$,

(II) $\alpha_n \to 0, n\to\infty, \sum\alpha_n = \infty$ $\{\theta_n\} \subset [0,\theta]$, $\theta \in [0,1)$,

(III) $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$,

(IV) $\{\sigma_n\}$ is a sequence in $(0,\infty)$ and there exist $a, b \in \mathbb{R}$ with $0 < a \leq \sigma_n \leq b < 2\alpha$, $\forall\ n \in \mathbb{N}$.
Then the sequence $\{x_n\}$ constructed by algorithm (2) converges strongly to $x^* = P_\Gamma f(x^*)$, where P is the metric projection of H onto Γ.

4. Numerical example

Example 4.1. Solve the following minimization problem.

$$\min_{x\in\mathbb{R}^3} \|x\|_2^2 + (3,5,-1)x + 9 + \|x\|_1, \ x = (y_1,y_2,y_3) \in \mathbb{R}^3$$

and the fixed point problem of the function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x) = (-2 - y_1, -4 - y_2, -y_3)$. Now for each $x \in \mathbb{R}^3$, we set $F(x) = \|x\|_2^2 + (3,5,-1)x + 9$ and $G(x) = \|x\|_1$.

It is easy to check that F is convex and differentiable on $\mathbb{R}^3$ with 2-Lipstchiz continuous gradient. Also, G is convex and lower semi-continuos but not differentiable on $\mathbb{R}^3$. For $r > 0$ by using the soft thresholding operator (see [6]) and the proximity operator, we obtain that

$$\begin{aligned}\big(I + rB\big)^{-1}(x) =&\Big(\max\{|y_1|-r, 0\} \operatorname{sign}(y_1), \max\{|y_2|-r, 0\} \operatorname{sign}(y_2),\\ &\max\{|y_3|-r, 0\} \operatorname{sign}(y_3)\Big).\\ (I-rA)(x) =&(I-r\nabla F)(x)\\ =&\Big(\max\left\{\frac{|x_1-3|-r}{2}, 0\right\} \operatorname{sign}(x_1-3), \max\left\{\frac{|x_2-5|-r}{2}, 0\right\} \operatorname{sign}(x_2-5),\\ &\max\left\{\frac{|x_3+1|-r}{2}, 0\right\} \operatorname{sign}(x_3+1)\Big),\end{aligned}$$

where $\operatorname{sign}(\cdot)$ is the signum function of $\alpha \in \mathbb{R}$.

Put $A = \nabla f$ and $B = \partial G$ in both algorithm (1) and algorithm (14) with $\alpha_n = \frac{1}{100n+1}$, $\beta_n = \frac{3n}{100n+40}$, $\lambda_n = 0.0001$, $\theta = 0.5$, $\epsilon_n = \frac{1}{(n+1)^3}$, $\theta_n = \bar{\theta_n}$, stopping criterion is $E_n = \|x_n - J_{\lambda_n}(I - \nabla F)x_n\| + \|x_n - Sx_n\| < 10^{-3}$,

$$\bar{\theta_n} = \begin{cases} \min\{\frac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise} \end{cases}.$$

In order to compare the iterative algorithm (1) with the algorithm (14) (in terms of convergence and the CPU time) we consider different choices of x_0 and x_1 for the two algorithms.

Case 1. $x_0=(1, 2, -1)$, $x_1=(1, 5, 1)$; **Case 2.** $x_0=(0, -2, 2)$, $x_1=(2, 0, -3)$;

Case 3. $x_0=(-5, 4, 6)$, $x_1=(3, -5, -9)$; **Case 4.** $x_0=(1, 2, 3)$, $x_1=(8, 7, 3)$.

Table 1: Numerical results in comparison with Algorithm (1) and Algorithm (14)

		Algorithm (14)	Algorithm (1)
Case 1.	CPU time (sec)	0.015	0.071
	No. of Iteration	6	11
Case 3.	CPU time (sec)	0.016	0.080
	No. of Iteration	6	10
Case 3.	CPU time (sec)	0.002	0.022
	No. of Iteration	5	9
Case 4.	CPU time (sec)	0.009	0.052
	No. of Iteration	5	11

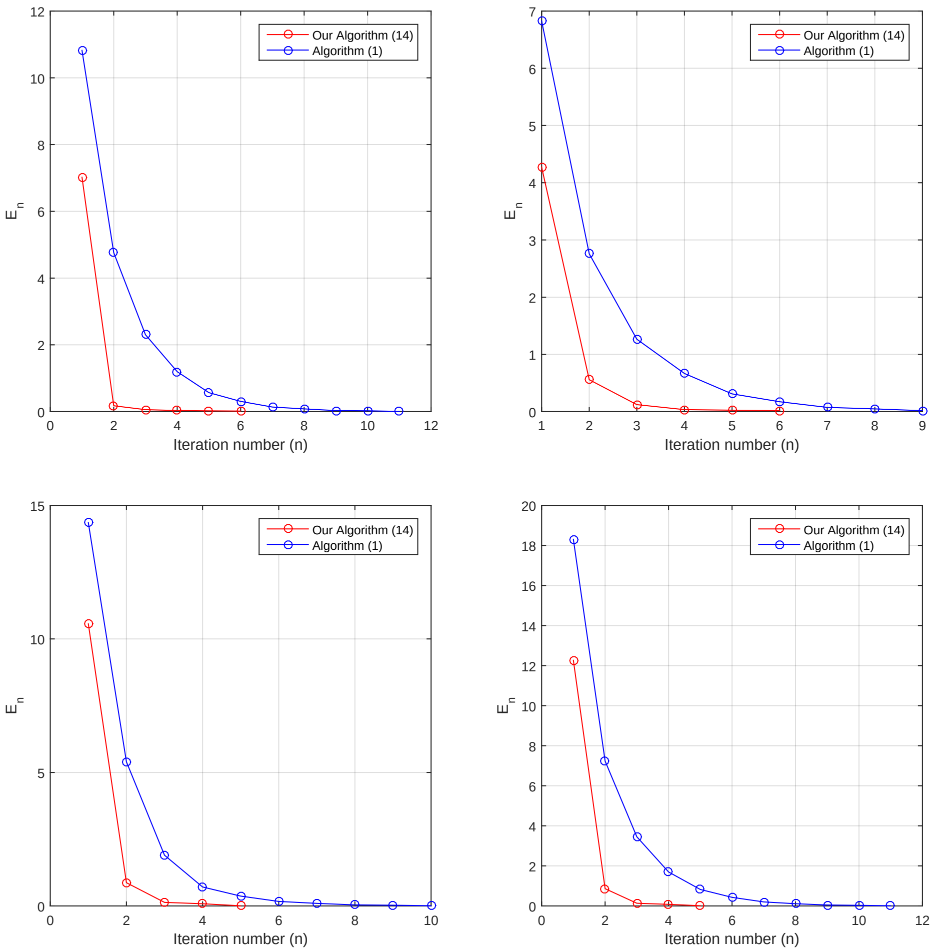

Figure 1: E_n vs Iteration numbers (n): Top Left: **Case 1.**; Top Right: **Case 2.**; Bottom Left: **Case 3.**; Bottom Right: **Case 4.**.

Acknowledgement. The research of the second author is wholly supported by the National Research Foundation (NRF) South Africa (S& F-DSI/NRF Free Standing Postdoctoral Fellowship; Grant Number: 120784). The second author also acknowledges the financial support from DSI/NRF, South Africa Center of Excellence in Mathematical and Statistical Sciences (COE-MaSS) Postdoctoral Fellowship.

Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to NRF nor CoE-MaSS.

We thank immensely the anonymous reviewers for insightful comments that improved the quality of the article.

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(received 23.07.2020; in revised form 27.07.2021; available online 22.02.2022)

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