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ON ALGEBROID FUNCTIONS WITH UNIFORM SCHWARZIAN DERIVATIVE

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Abstract. The question of determining under which conditions the Schwarzian derivative of an algebroid function turns out to be a uniform meromorphic function in the plane is considered. In order to do this the behaviour of the Schwarzian derivative of an algebroid function $w(z)$ around a ramification point is analyzed. It is concluded that in case of a uniform Schwarzian derivative $S_w(z)$, this meromorphic function presents a pole of order two at the projection of the ramification point, with a rational coefficient γ_{-2}, where $0 < \gamma_{-2} < 1$. A class of analytic algebroid functions with uniform Schwarzian derivative is presented and the question arises whether it contains all analytic algebroid functions with this property.

1. Introduction

Given a meromorphic function $f(z)$ in a domain Ω of the complex plane $\mathbb{C}$, the Schwarzian derivative $Sf(z)$ is defined by

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

In the case that $f(z)$ is locally injective then $Sf(z)$ is analytic.

Here we shall consider the wider setting than meromorphic functions formed by the algebroid functions. An algebroid function $w(z)$ of order k is a k-valued function $w(z)$ in the entire complex plane $\mathbb{C}$, or more generally in a finite disc $D(0, R)$, determined by an equation of the form

$$F(z, w) = A_k(z)w^k + A_{k-1}(z)w^{k-1} + \cdots + A_0(z) = 0,$$

where $A_0(z), A_1(z), \ldots, A_k(z)$ are uniform meromorphic functions either in $\mathbb{C}$ or more generally in $D(0, R)$. By a uniform function we mean an 1-valued function.

The multivalued function $w = w(z)$ can be considered as a uniform function on the associated Riemann surface X_F defined by

$$X_F = \{(z, w) \mid F(z, w) = 0\}.$$

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The multivalued function $w(z)$ has, for each z, different branches $w_i(z)$, $i = 1, \ldots, k$, which can coincide in a discrete set of points called ramification points. The surface X_F can be described in terms of the branches by

$$X_F = \{(z, w_j(z)) \mid z \in \mathbb{C}, j = 1, \ldots, k\},$$

in such a way that X_F becomes an $n-$sheeted covering of the complex plane by the canonical projection

$$P: \ X_F \to \mathbb{C}, \qquad (z, w_j) \mapsto z,$$

and the algebroid function $w(z)$ becomes uniform on X_F through

$$w: \ X_F \to \widehat{\mathbb{C}}, \qquad (z, w_j(z)) \mapsto w_j(z).$$

Now, given a point $p = (z, w_i(z)) \in X_F \backslash R_F$, $R_F = \{p \in X_F \mid p$ ramification point of $X_F\}$, we define the Schwarzian derivative S_w of $w(z)$ at p by

$$S_w(p) = \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2.$$

It is clear that S_w is a well-defined uniform meromorphic function considered on $X_{F \backslash P(R_F)}$ as a function of the local parameter z but in general it is not uniform considered as a function on $\mathbb{C}$.

2. A theorem on differential equations

We shall restrict ourselves to analytic algebroid functions, that is, the algebroid functions $w(z)$ that have no poles, or equivalently, all the coefficients $A_j(z)$ of $F(z, w)$ are analytic functions in $\mathbb{C}$. We present the following theorem which is an extension of a well-known theorem on differential equations in the plane, see I. Laine [6].

THEOREM 2.1. *Let $A(z)$ be analytic in $\mathbb{C} \setminus S$, where $S \subset \mathbb{C}$ is a discrete set. Then for any two linearly independent local solutions v_1, v_2 of the second order differential equation*

$$v'' + A(z)v = 0, \tag{1}$$

in a disc $D(z_0, \epsilon)$ such that $D(z_0, \epsilon) \cap S = \emptyset$, their quotient $w = v_1/v_2$ is a locally injective analytic function which satisfies

$$S_w(z) = \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 = 2A(z). \tag{2}$$

Conversely, if a locally injective analytic function element $(w(z), D(z_0, \epsilon))$ is given, with $A(z)$ defined by (2), then $A(z)$ is analytic and we can find two linearly independent local solutions v_1, v_2 of (1) such that $w = v_1/v_2$.

The function element $(w(z), D(z_0, \epsilon))$ can be continued analytically without restriction to $\mathbb{C} \setminus S$ so that we get in general a complete analytic multivalent function $w(z)$ with locally injective analytic branches $w_i(z)$ such that given two different branches $w_i(z), w_j(z)$ in a small neighbourhood $D(z_1, \epsilon)$, $D(z_1, \epsilon) \cap S = \emptyset$ there exists *a Möbius transformation T such that $w_j(z) = T \circ w_i(z)$ for z in $D(z_1, \epsilon)$. The set of Möbius transformations obtained in this way is a group G of Möbius transformations and when it turns out to be finite then the obtained complete multivalued function $w(z)$ is an algebroid function of order equal to $ord(G)$.*

Conversely, if we start from an analytic algebroid function $w(z)$ of order k for which all the branches $w_i(z)$, $i = 1, \ldots, k$ are related through the Möbius transformation T of a finite group of order k, then the Schwarzian derivative S_w is a uniform function $2A(z)$ analytic in $\mathbb{C} \setminus S$ where S is the set of the projections of the ramification points of $w(z)$.

Proof. Given two linearly independent local analytic solutions $v_1(z), v_2(z)$ of (1), they can be continued without restriction in $\mathbb{C} \setminus S$ (see Herold [5, page 33]). Therefore, there exist multivalued extensions to $\mathbb{C} \setminus S$ of $v_1(z), v_2(z)$, linearly independent local solutions of (1), and therefore their quotient $w(z)$ can be extended to a multiple valued solution of (2) in $\mathbb{C} \setminus S$. See I. Laine [6, Theorem 6.1.].

The fact that the quotient $w = v_1/v_2$ satisfies (2) is obtained by calculation.

Given two branches w_i, w_j in a disc $D(z_1, \epsilon)$ outside S, we must have $S_{w_i} = S_{w_j} = 2A(z)$, and again by [6, Remark next to Theorem 6.1], w_i, w_j can be written as quotients of local linearly independent solutions of (1)

$$w_i = \frac{v_{1i}}{v_{2i}}, w_j = \frac{v_{1j}}{v_{2j}},$$

so that for some constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ it holds

$$w_j = \frac{v_{1j}}{v_{2j}} = \frac{\alpha_1 v_{1i} + \alpha_2 v_{2i}}{\beta_1 v_{1i} + \beta_2 v_{2i}} = T \circ w_i,$$

where $T(z)$ is the Möbius transformation

$$T(z) = \frac{\alpha_1 z + \alpha_2}{\beta_1 z + \beta_2}.$$

Now, if we consider the set of all the transformations obtained in this way then we get a group G. In the case that G is finite, the number of different branches at p must be finite and independent of p, and precisely equal to $ord(G)$. In fact, the composition of two Möbius transformations T, T_1 obtained in this way should lead to a new branch $w_k = (T_1 \circ T) \circ w_i$ of w solution of (2) obtained by analytic continuation of w_i. The order of the algebroid function obtained is equal to the number of different branches at a certain point p and this is clearly equal to the number of different Möbius transformations of G, that is $ord(G)$.

The converse statement also follows from [6, Remark next to Theorem 6.1]. □

3. An example

Example 3.1. Let $S = \{a_n\}_{n \in \ltimes}$ be a sequence of distinct points tending to infinity and ordered in such a way that $|a_n| \leq |a_{n+1}|$, that is, ordered according to increasing moduli.

We can form, by the Weierstrass Product Theorem, a function $L(z)$ with zeros at the points $\{a_n\}$ and with no other zeros. This function is given in the form

$$L(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right),$$

where $E_p\left(\frac{z}{a}\right)$ denotes an elementary Weierstrass product. We consider the algebroid function $w(z)$ of order k determined by the equation $F(z,w) = w^k - L(z) = 0$.

The associated Riemann surface X_F is a covering of $\mathbb{C}$ with ramifications over the points $\{a_n\}$, and given a disc $D(a_i,\epsilon)$ excluding the a_n' with $n \neq i$, the different branches w_l, $l = 1, 2, \ldots, k$ at a point $z \in D(a_i,\epsilon)$ are obtained as

$$w_l(z) = e^{\frac{2\pi l i}{k}} \cdot \left(\prod_{k=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)^{\frac{1}{k}}\right). \tag{3}$$

Here, we have fixed inside the parenthesis a particular branch of

$$\prod_{k=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)^{\frac{1}{k}},$$

in such a way that when we consider a function element

$$(w_l(z), D(z_i,\rho_i)),$$

where

$$D(z_i,\rho_i) \subset D(a_i,\epsilon) \quad \text{and} \quad a_i \notin D(z_i,\rho_i),$$

and where the function $w_l(z) = L(z)^{1/k}$ assumes the values in (3) for $z \in D(z_i,\rho_i)$, and we continue this element analytically along a circle $C(a_i, |z_i - a_i|)$, we get a function element $(w_{l-1}(z), D(z_i,\rho_i))$, where w_{l-1} assumes the value $w_{l-1}(z)$ in (3), since the factor $\left(1 - \frac{z}{a_i}\right)^{1/k}$ suffers in $L(z)$ an argument variation equal to $-\frac{2\pi}{k}$ leaving the rest of the product invariant.

We conclude that the function element $(w_l(z), D(z_i,\rho_i))$ gives rise by analytic continuation around the ramification point a_i to the function element

$$(w_{l-1}(z) = T \circ w_l(z), D(z_i,\rho_i)),$$

where $T(z) = e^{-2\pi i}z$.

We remark that the Möbius transformation T associated to a_i is the same for any other a_j.

We conclude that the associated group G is in this case the finite cyclic group generated by T: $G = \{I, T, T^2, \ldots, T^{k-1}\}$.

Problem 3.1. Are all the analytic algebroid functions of order k with uniform Schwarzian derivative of the type described in Example 3.1?

4. The associated group G

Let $w(z)$ be an algebroid function with uniform Schwarzian derivative $S_w(p) = 2A(z)$ for every $p \in X_F$, such that $P(z) = z$, where $A(z)$ is an analytic function in $\mathbb{C}\backslash S$ with

$S \subset \mathbb{C}$ a discrete set. The group G associated to $w(z)$ is generated in the following way.

Let us enumerate the points $a_n \in S$, assuming that we do this according to increasing moduli and in case of equality of moduli according to increasing arguments. For each a_n we consider a circle $C(a_n, \epsilon_n)$ of radius sufficiently small so that they are mutually disjoint. Inside $C(a_n, \epsilon_n)$ we fix a disc $D(z_n, \rho_n)$ with $\rho_n < \epsilon_n - d(z_n, a_n)$, so that $a_n \notin D(z_n, \rho_n)$. By analytic continuation of a finite function element $(w(z), D(z_n, \rho_n))$ around a circle $C(a_n, |z_n - a_n|)$ we get a new function element $(w_1(z), D(z_n, \rho_n))$ in such a way that there exists a Möbius transformation $T_{n,i}$ for which

$$w_1 = T_{n,i} \circ w, \tag{4}$$

where $T_{n,i}$ is associated to a ramification point $p_{n,i}$ over a_n. That is, we denote by $p_{n,1}, p_{n,2}, \ldots, p_{n,k_n}$ the ramifications points over a_n, i.e. $P(p_{n,i}) = a_n$, $i = 1, \ldots, k_n$.

Then we can continue analytically both functions elements w, w_1 over $\mathbb{C} \setminus S$ and all the pairs of function elements obtained in this way by analytic continuation will always satisfy the relation (4).

Now we proceed in the same way for each $n \in \mathbb{N}$ and obtain a sequence of $\{T_{n,i}\}_{n\in\mathbb{N}, i=1,\ldots,k_n}$ of Möbius transformations.

THEOREM 4.1. *The group associated to the algebroid function $w(z)$ is the group generated by the described sequence of Möbius transformations $\{T_{n,i} \mid n \in \mathbb{N}, i = 1, \ldots, k_n\}$.*

Proof. Let T be a Möbius transformation for which $T \circ w(z)$ is a function element of the algebroid function starting from a given function element $(w(z), (D(z_0, \epsilon)))$.

The function element $T \circ w(z)$ must be obtained by analytic continuation from $w(z)$ along a closed path $\gamma \subset \mathbb{C} \setminus S$.

Since γ is compact, it is contained in a disc $D(0, R)$ and $a_n \notin D(0, R)$ for $n > N$ with N sufficiently large. Then we consider paths $\gamma_1, \ldots, \gamma_N$ in $D(0, R) \setminus \{a_1, \ldots, a_N\}$ such that $\gamma_n = \beta_n \sim C(a_n, \epsilon_n)$, where the β_n's are mutually disjoint Jordan arcs joining z_0 and $C(a_n, \epsilon_n)$. We know from basic Algebraic Topology that the γ_n, $n = 1, \ldots, N$ form a set of generators of the fundamental group π_1 of $D \setminus \{a_1, \ldots, a_N\}$ so that the path γ is homotopic to a path of the form $l_1 \sim l_2 \sim \ldots \sim l_T$ where each path l_j is one of the paths γ_n or γ_n^{-1} and the paths l_j, l_k can coincide for $j \neq k$.

If we continue analytically the original function element $(w, D(z_0, \epsilon))$ along one of the γ_n, we arrive to the function element $(T_{n,i} \circ w, D(z_0, \epsilon))$ and similarly if we continue the function element $(w, D(z_0, \epsilon))$ along the path γ_i^{-1} we get to the function element $\left(T_{n,i}^{-1} \circ w, D(z_0, \epsilon)\right)$. Therefore the analytic continuation of $(w, D(z_0, \epsilon))$ along the path $l_1 \sim l_2 \sim \ldots \sim l_T$, will yield a function element $(w_1, D(z_0, \epsilon))$ where $w_1 = T_{l_1,\ldots,l_T} \circ w$, and $T_{l_1,\ldots,l_T}$ is obtained as a product of $T_{n,i}$'s and $T_{n,i}$'s with $n = 1, 2, \ldots, N$, $i = 1, 2, \ldots, k_n$.

By the Monodromy Theorem, since γ and $l_1 \sim l_2 \sim \ldots \sim l_T$ are homotopic, the following function elements are equal: $(T \circ w\ ,\ D(z_0, \epsilon)) = (T_{l_1,\ldots,l_T} \circ w\ ,\ D(z_0, \epsilon))$, that is T is in the group generated by $\{T_{n,i}\}$. $\square$

5. The singularities of $A(z)$ at S

Let $w = w(z)$ be an algebroid function of order n with ramifications over a discrete set of points $S \subset \mathbb{C}$ and assume that its Schwarzian derivative $S_w(p)$ at a point $p \in X_F$ satisfies

$$S_w(p) = \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 = \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 = 2A(z),$$

where $z = P(p)$, that is, the value of the Schwarzian derivative is the same for all the branches $w_1, \dots, w_n$ and it only depends on $P(p)$.

In this section we shall study the behaviour of $A(z)$ at a point $a \in S$, that is, at a point which is the projection of some ramification point $p \in R_F$ of X_F. At such a point the branches group in cycles, the branches corresponding to such a cycle have power series expansions of the form (see K. Hensel und Landsberg [4, Kap.5, page 77])

$$w_i(z) = c_{-l}(z-a)^{-l/k} + c_{-(l-1)}(z-a)^{-(l-1)/k} + \cdots + c_0 + c_1(z-a)^{1/k} + \dots$$

where $l < k$. Then we obtain

$$w_i'(z) = -\frac{l}{k}c_{-l}(z-a)^{-l/k-1} - \frac{(l-1)}{k}(z-a)^{-(l-1)/k-1} + \cdots,$$

$$w_i''(z) = \frac{l}{k}\left(\frac{l}{k}+1\right)c_{-l}(z-a)^{-l/k-2} + \frac{(l-1)}{k}\left(\frac{l-1}{k}+1\right)(z-a)^{-(l-1)/k-2} + \cdots,$$

$$w_i'''(z) = -\frac{l}{k}\left(\frac{l}{k}+1\right)\left(\frac{l}{k}+2\right)c_{-l}(z-a)^{-l/k-3} - \frac{(l-1)}{k}\left(\frac{l-1}{k}+1\right)\left(\frac{l-1}{k}+2\right)(z-a)^{-(l-1)/k-3} + \cdots,$$

and we conclude that, for p in a neighbourhood of this branch point, it holds

$$\begin{aligned} S_w(p) =& \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 \\ =& \frac{-\frac{l}{k}\left(\frac{l}{k}+1\right)\left(\frac{l}{k}+2\right)c_{-l}(z-a)^{-l/k-3}\cdot\left[1+O(z-a)^{1/k}\right]}{-\frac{l}{k}c_{-l}(z-a)^{-l/k-1}\cdot\left[1+O(z-a)^{1/k}\right]} \\ &- \frac{3}{2}\cdot\left(\frac{-\frac{l}{k}\left(\frac{l}{k}+1\right)c_{-l}(z-a)^{-l/k-2}\cdot\left[1+O(z-a)^{1/k}\right]}{-\frac{l}{k}c_{-l}(z-a)^{-l/k-1}\cdot\left[1+O(z-a)^{1/k}\right]}\right)^2 \\ =& \left(\frac{l}{k}+1\right)\left(\frac{l}{k}+2\right)(z-a)^{-2}(1+o(1)) - \frac{3}{2}\cdot\left(\frac{l}{k}+1\right)^2(z-a)^{-2}(1+o(1)) \\ =& \left(\frac{l}{k}+1\right)\left[\left(\frac{l}{k}+2\right) - \frac{3}{2}\left(\frac{l}{k}+1\right)\right](z-a)^{-2} + o(z-a)^{-2}. \end{aligned}$$

That is, at a point $a \in S$, since $\left(\frac{l}{k}+2\right) - \frac{3}{2}\left(\frac{l}{k}+1\right) = -\frac{1}{2}\left(\frac{l}{k}-1\right)$, we obtain the coefficient of $(z-a)^{-2}$ at the Laurent expansion, namely $\frac{1}{2}\left(1-\left(\frac{l}{k}\right)^2\right)$, and therefore that of $A(z)$ will be $\frac{1}{4}\left(1-\left(\frac{l}{k}\right)^2\right)$, where $\frac{l}{k}\neq\pm1$, since we are assuming p to be a ramification point.

We conclude that the singularity of the Schwarzian derivative of an algebroid function $w=w(z)$ at a point $a\in\mathbb{C}$, which is the projection of a ramification point of X_F, should be a pole of order 2 and the coefficient γ_{-2} of the term $(z-a)^{-2}$ should be of the particular form $\gamma_{-2}=\frac{1}{4}\left(1-\left(\frac{l}{k}\right)^2\right)$, $l,k\in\mathbb{N}$, $l<k$, what implies $\gamma_{-2}\in\mathbb{Q}$, $0<\gamma_{-2}<1$.

Remark 5.1. It follows that for an algebroid function with uniform Schwarzian derivative, the cycles associated to all the ramification points with the same projection should have the same length l.

Example 5.2. The multivalued function $w(z)=z^{i\sqrt{3}}=e^{i\sqrt{3}\ln z}$ has the successive derivatives

$$w'(z)=i\sqrt{3}\cdot z^{i\sqrt{3}-1}\ ,\ \ w''(z)=i\sqrt{3}\cdot\left(i\sqrt{3}-1\right)\cdot z^{i\sqrt{3}-2},$$
$$w'''(z)=i\sqrt{3}\cdot\left(i\sqrt{3}-1\right)\cdot\left(i\sqrt{3}-2\right)\cdot z^{i\sqrt{3}-3},$$

so that we obtain for the Schwarzian derivative

$$\begin{aligned}S_w(z)&=\left(i\sqrt{3}-1\right)\cdot\left(i\sqrt{3}-2\right)\cdot z^{-2}-\frac{3}{2}\left(i\sqrt{3}-1\right)^2\cdot z^{-2}\\&=\left(i\sqrt{3}-1\right)\left[i\sqrt{3}-2-\frac{3}{2}\left(i\sqrt{3}-1\right)\right]\cdot z^{-2}=\left(i\sqrt{3}-1\right)\left(-\frac{i\sqrt{3}}{2}-\frac{1}{2}\right)\cdot z^{-2}\\&=-\frac{1}{2}\left(i\sqrt{3}-1\right)\left(i\sqrt{3}+1\right)\cdot z^{-2}-\frac{1}{2}(-3-1)=2z^{-2}.\end{aligned}$$

In this case $\gamma_{-2}=2$ and the Schwarzian derivative comes from the infinitely many valued $z^{i\sqrt{3}}$ which is clearly not an algebroid.

Example 5.3. One further example is yielded by the algebroid equation $w^6=z^2(z-1)^3$. In this case the corresponding algebroid function $w=w(z)$ gives rise to a uniform Schwarzian derivative $S_w(z)=2A(z)$ with two poles at $z=0$ and $z=1$ and whose respective developments around these points have the corresponding terms $\frac{\gamma_{-2}}{z^2}=\frac{1/3}{z^2}$, $\frac{\gamma_{-2}}{z^2}=\frac{1/2}{z^2}$, that is $\gamma_{-2}=\frac{1}{3}$, $\gamma_{-2}=\frac{1}{2}$ at the points $z=0$, $z=1$ respectively.

6. The reciprocal question

In Section 5 we have shown that given an algebroid function $w(z)$ for which the Schwarzian derivative $S_w(p)$ at a given point $p\in X_F$ depends only on the projection $z=P(p)$, say $S_w(p)=2A(z)$, the projection $a=P(p)$ of a branch point $p\in X_F$, the function $A(z)$, has a pole of order 2 with Laurent expansion of the form

$$A(z) = \frac{\frac{1}{4}\left(1-\left(\frac{l}{k}\right)^2\right)}{(z-a)^2} + \frac{b_1}{z-a} + b_2 + b_3(z-a) + \dots \tag{5}$$

Now we consider the *reciprocal* question—given a meromorphic function $A(z)$ with Laurent expansion of type (5) at its poles, can we find an algebroid function $w(z)$ with Schwarzian derivative $S_w(p)$ satisfying

$$S_w(p) = 2A(z), \tag{6}$$

where $z = P(p)$?

The question can be solved locally making use of the following lemma which is an adaptation to our situation of [6, Lemma 6.6] and bearing in mind Theorem 2.1.

LEMMA 6.1. *Suppose that* $h(z)$ *is analytic in*

$$B(z_0, R) = \{z \mid |z-z_0| < R\},$$

where $R > 0$ *and consider the differential equation*

$$f'' + \frac{h(z)}{(z-z_0)^2} \cdot f = 0, \tag{7}$$

in $B(z_0, R)$*. Let* ρ_1, ρ_2 *be the roots of* $\rho(\rho-1) + h(z_0) = 0$*, assuming that* $\rho_1 > \rho_2$*,* $\rho_1 - \rho_2 < 1$*. Then* (7) *admits in some disc* $B(z_0, r)$*,* $r \leq R$ *two linearly independent solutions* f_1, f_2 *of the form*

$$\begin{cases} f_1(z) = (z-z_0)^{\rho_1} \sum_{i=0}^{\infty} c_i^1 (z-z_0)^i, & c_0^1 \neq 0 \\ f_2(z) = (z-z_0)^{\rho_2} \sum_{i=0}^{\infty} c_i^2 (z-z_0)^i, & c_0^2 \neq 0. \end{cases} \tag{8}$$

Proof. It is clear that two solutions as in (8) should be linearly independent.

First of all we remark that the functions $f_1(z)$ and $f_2(z)$ will be multivalued so as their quotient is $\frac{f_1(z)}{f_2(z)} = (z-z_0)^{\rho_1-\rho_2} \cdot g(z)$, $g(z)$ analytic.

The idea of the proof is in [6], we shall proceed in the same way for both values ρ_1, ρ_2 and obtain the corresponding coefficients c_i^1, c_i^2 in a recursive way. Let ρ be one of the values and c_i the corresponding coefficients.

That is, we are looking for a function with power series expansion of the form

$$f(z) = (z-z_0)^{\rho} \sum_{i=0}^{\infty} c_i (z-z_0)^i. \tag{9}$$

We assume the Taylor expansion of $h(z)$ to be

$$h(z) = \sum_{i=0}^{\infty} b_i (z-z_0)^i. \tag{10}$$

Substituting (9) and (10) into (7) we obtain

$$(\rho+n)(\rho+n-1) + \sum_{i=0}^{n} b_i c_{n-i} = 0, \qquad \text{for } n = 0, 1, \dots \tag{11}$$

We introduce the notation

$$\begin{cases} \varphi_0(\rho) = \rho(\rho-1) + b_0 = \rho(\rho-1) + h(z_0), \\ \varphi_i(\rho) = b_i, \text{ for } i \in \mathbb{N} \setminus \{0\}. \end{cases} \tag{12}$$

From (11), making use of the notation (12), we obtain

$$\begin{cases} c_0\varphi_0(\rho) = 0 \\ c_1\varphi_0(\rho+1) + c_0\varphi_1(\rho) = 0 \\ \cdots\cdots\cdots\cdots \\ c_n\varphi_0(\rho+n) + c_{n-1}(\rho+n-1) + \cdots + c_1\varphi_{n-1}(\rho+1) + c_0\varphi_n(\rho) = 0. \end{cases} \tag{13}$$

Once we have fixed a value $c_0 \neq 0$, the first equality gives the indicial equation which has the roots ρ_1, ρ_2 which by our hypotheses satisfy $\rho_1 - \rho_2 < 1$ and therefore $\varphi_0(\rho_1 + n) \neq 0$, $\varphi_0(\rho_2 + n) \neq 0$, for every $n \in \mathbb{N} \setminus \{0\}$ and therefore (13) determines recursively the coefficients c_i.

The convergence of the series obtained in this way is proved in I. Laine [6]. □

In our case the indicial equation is

$$\rho(\rho-1) + \frac{1}{4}\left(1 - \left(\frac{l}{k}\right)^2\right) = 0, \quad \text{that is} \quad \rho(\rho-1) + \frac{1}{4} = \frac{1}{4}\left(\frac{l}{k}\right)^2,$$

which can be written as

$$\left[\left(\rho - \frac{1}{2}\right) + \frac{1}{2}\right] \cdot \left[\left(\rho - \frac{1}{2}\right) - \frac{1}{2}\right] + \frac{1}{4} = \frac{1}{4}\left(\frac{l}{k}\right)^2,$$

whence

$$\left(\rho - \frac{1}{2}\right)^2 = \frac{1}{4}\left(\frac{l}{k}\right)^2,$$

that is

$$\rho_1 = \frac{1}{2} + \frac{1}{2} \cdot \frac{l}{k}, \quad \rho_2 = \frac{1}{2} - \frac{1}{2} \cdot \frac{l}{k},$$

so that

$$\rho_1 - \rho_2 = \frac{l}{k} < 1,$$

and hence we get that the hypotheses of Lemma 6.1 are satisfied. Finally, we obtain a local solution of (6) of the form $w(z) = (z - z_0)^{l/k} \cdot g(z)$, where $g(z)$ is analytic and $g(z_0) \neq 0$.

7. A partial answer to the problem in Section 3

The following theorem yields a partial answer to the problem in Section 3. More precisely, it describes the analytic algebroid functions of a given order k with uniform Schwarzian derivative under the additional assumption of the existence of a ramification point of maximal order k.

Example 5.3 shows that this additional hypothesis is not satisfied by every algebroid function with uniform Schwarzian derivative.

THEOREM 7.1. *Let $w = w(z)$ be an algebroid function of order k with a uniform Schwarzian derivative $2A(z)$, that is*

$$S_w(p) = 2A(z), \tag{14}$$

for $p \in X$, X—the associated Riemann surface, with $z = P(p)$, and such that $w(z)$ has a ramification point a of maximal order k. Then the algebroid function $w(z)$ is of the form

$$w(z) = T \circ (e(z) \cdot w_L(z)), \tag{15}$$

where $e(z)$ is an entire function, T is a Möbius transformation and $w_L(z)$ will be an algebroid function as described in Example 3.1, i.e. $w_L(z)$ is defined by an equation of the form

$$w^k - L(z) = 0, \text{ where } L(z) = \prod_{i=1}^{\infty} E_{p_i}\left(\frac{z}{a_i}\right).$$

Proof. Let $D(a, r_a)$ be a disc centered at a, where a is a ramification point of order k of $w(z)$ and therefore a pole of order 2 of $2A(z)$ with coefficient γ_{-2} of $(z-a)^{-2}$ in its Laurent expansion around a equal to $\frac{1}{2}\left(1 - \left(\frac{1}{k}\right)^2\right)$. Let us assume that $D(a, r_a)$ does not contain any other ramification point of $w(z)$, that is, it does not contain any other pole of $A(z)$. By the results in Section 6 and as a consequence of Lemma 6.1 there exists a solution $w_a(z)$ of (14), which we can assume to be defined in $D(a, r_a)$, of the form $w_a(z) = (z-a)^{\frac{1}{k}} \cdot g_a(z)$, where $g_a(z)$ is a uniform analytic function with $g_a(a) \neq 0$.

We can assume that $w(a) = 0$ just by application of a Möbius transformation and then since both functions $w(z)$ and $w_a(z)$ have at a a ramification point of order k, the function $w \circ w_a^{-1}(z)$ is a uniform analytic function $T(z)$ in $D(a, r_a)$. It should be a Möbius transformation since both functions have the same Schwarzian derivative. Therefore, by application again of a Möbius transformation, we can assume that

$$w(z) = (z-a)^{\frac{1}{k}} \cdot g_a(z) \tag{16}$$

in $D(a, r_a)$, where $g_a(z)$ is a uniform analytic function with $g_a(a) \neq 0$.

Let us enumerate the rest of ramification points a_n according to increasing moduli and in case of equality according to increasing argument. For each a_n there exists a function element $(w_{a_n}(z), D(a_n, r_{a_n}))$ solution of (14) and such that

$$w_{a_n}(z) = (z-a_n)^{\frac{l_n}{k}} \cdot g_{a_n}(z), \tag{17}$$

where $g_{a_n}(z)$ is a uniform analytic function with $g_{a_n}(a_n) \neq 0$.

We take now a non-selfintersecting path γ_1 outside the set of projections of ramification points of $w(z)$, that is, outside the set of poles of $A(z)$, joining a point $\zeta \in D^*(a, r_a)$, the punctured disc $D(a, r_a) \setminus \{a\}$, where $\zeta \neq a$, and a point $\zeta_1 \in D^*(a_1, r_{a_1}) = D^*(a_1, r_{a_1}) \setminus \{a_1\}$, $\zeta_1 \neq a_1$. Let $\Delta(\zeta, r_\zeta) \subset D^*(a, r_a)$ be a disc centered at ζ and let $\Delta(\zeta_1, r_{\zeta_1}) \subset D^*(a_1, r_{a_1})$ be a disc centered at ζ_1. In $\Delta(\zeta, r_\zeta)$, we can consider the function element $(w(z), \Delta(\zeta, r_\zeta))$ and we can continue it analytically along γ_1 up to ζ_1 and obtain clearly the function element $(w(z), \Delta(\zeta_1, r_{\zeta_1}))$. We can further continue this function element inside $D^*(a_1, r_{a_1})$. On the other hand, a k/l_1-valued function $w_{a_1}(z)$ is defined in $D^*(a_1, r_{a_1})$ by (17), that is $w_{a_1}(z) = (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z)$.

Since the function element $(w(z), \Delta(\zeta_1, r_{\zeta_1}))$ and the function element $(w_{a_1}(z), \Delta(\zeta_1, r_{\zeta_1}))$ have the same Schwarzian derivative, they are related by a Möbius transformation, say $T_{a_1}(z)$, in such a way that

$$w(z) = T_{a_1} \circ w_{a_1}(z), \tag{18}$$

in $\Delta(\zeta_1, r_{\zeta_1})$.

By analytic continuation of $w_{a_1}(z)$ to the whole $D^*(a_1, r_{a_1})$ we deduce that the relation (18) is valid in the whole $D^*(a_1, r_{a_1})$. Both functions $w(z)$ and $w_{a_1}(z)$ are k/l_1-valued functions there, and the relation $w(a_1) = T_{a_1}(0)$ holds.

Let $T_{a_1}(z) = \frac{\alpha z+\beta}{\gamma z+\lambda}$, so that

$$w(z) = \frac{\alpha (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \beta}{\gamma (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \lambda}, \tag{19}$$

for any $z \in D^*(a_1, r_{a_1})$, assuming the values

$$w(z) = \frac{\alpha \cdot e^{\frac{2\pi s}{k} i} (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \beta}{\gamma \cdot e^{\frac{2\pi s}{k} i} (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \lambda}, \; s = l_1, 2l_1, \ldots, k-l_1. \tag{20}$$

In particular, this will happen for any $z \in \Delta(\zeta_1, r_{\zeta_1})$ so that each of these values should be the analytic continuation along γ_1 of different branches of $w(z)$ in $\Delta(\zeta, r_\zeta)$; but all these branches differ by a k-root of unity, so that the corresponding continuations (20) should also differ by a k-root of unity.

We conclude that a relation of the type

$$e^{\frac{2\pi t}{k} i} \cdot w(z) = \frac{\alpha \cdot e^{\frac{2\pi s}{k} i} (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \beta}{\gamma \cdot e^{\frac{2\pi s}{k} i} (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \lambda}, \tag{21}$$

should also hold for $z \in \Delta(\zeta_1, r_{\zeta_1})$ and we obtain from (19) and (21)

$$e^{\frac{2\pi t}{k} i} \cdot \frac{\alpha (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \beta}{\gamma (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \lambda} = \frac{\alpha \cdot e^{\frac{2\pi s}{k} i} (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \beta}{\gamma \cdot e^{\frac{2\pi s}{k} i} (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \lambda}, \tag{22}$$

and working out this equality,

$$\begin{aligned} &\left(e^{\frac{2\pi t}{k} i} \cdot \alpha (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + e^{\frac{2\pi t}{k} i} \cdot \beta\right) \cdot \left(\gamma \cdot e^{\frac{2\pi s}{k} i} (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \lambda\right) \\ &= \left(\gamma (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \lambda\right) \cdot \left(\alpha \cdot e^{\frac{2\pi s}{k} i} (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + \beta\right). \end{aligned} \tag{23}$$

From (23) we obtain $e^{\frac{2\pi t}{k} i} \cdot \beta \cdot \lambda = \beta \cdot \lambda$, and since $e^{\frac{2\pi t}{k} i} \neq 1$, we conclude that either $\beta = 0$ or $\lambda = 0$.

Let us assume first $\lambda = 0$; then we obtain from (22)

$$e^{\frac{2\pi t}{k} i} \cdot \frac{\beta}{\gamma} (z-a_1)^{-\frac{l_1}{k}} \cdot g_{a_1}(z)^{-1} + e^{\frac{2\pi t}{k} i} \cdot \frac{\alpha}{\gamma} = \frac{\beta}{\gamma} \cdot e^{-\frac{2\pi s}{k} i} \cdot \frac{\beta}{\gamma} (z-a_1)^{-\frac{l_1}{k}} \cdot g_{a_1}(z)^{-1} + \frac{\alpha}{\gamma},$$

whence we conclude $e^{\frac{2\pi t}{k} i} \cdot \frac{\alpha}{\gamma} = \frac{\alpha}{\gamma}$, and arguing as above we obtain $\frac{\alpha}{\gamma} = 0$ and also

$\alpha = 0$. In this case we should have $T_{a_1}(z) = \frac{\beta}{\gamma z}$, so that for $z \in D^*(a_1, r_{a_1})$

$$w(z) = T_{a_1} \circ \left((z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) \right) = \frac{\beta}{\gamma} \cdot (z-a_1)^{-\frac{l_1}{k}} \cdot g_{a_1}(z)^{-1}. \tag{24}$$

If we assume $\beta = 0$ we obtain from (23)

$$e^{\frac{2\pi(t+s)}{k}i} \cdot \alpha \cdot \gamma \cdot (z-a_1)^{\frac{2l_1}{k}} \cdot g_{a_1}(z)^2 + e^{\frac{2\pi t}{k}i} \cdot \alpha \cdot \lambda \cdot (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z)$$
$$= e^{\frac{2\pi s}{k}i} \cdot \alpha \cdot \gamma \cdot (z-a_1)^{\frac{2l_1}{k}} \cdot g_{a_1}(z)^2 + e^{\frac{2\pi s}{k}i} \cdot \alpha \cdot \lambda \cdot (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z),$$

and cancelling the factor $(z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z)$ we come to

$$e^{\frac{2\pi(t+s)}{k}i} \cdot \alpha \cdot \gamma \cdot (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + e^{\frac{2\pi t}{k}i} \cdot \alpha \cdot \lambda$$
$$= e^{\frac{2\pi s}{k}i} \cdot \alpha \cdot \gamma \cdot (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) + e^{\frac{2\pi s}{k}i} \cdot \alpha \cdot \lambda,$$

whence, on one hand we should have $e^{\frac{2\pi t}{k}i} \cdot \alpha \cdot \lambda = e^{\frac{2\pi s}{k}i} \cdot \alpha \cdot \lambda$. so that $t = s$. On the other hand, $e^{\frac{2\pi(t+s)}{k}i} \cdot \alpha \cdot \gamma \cdot (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) = e^{\frac{2\pi s}{k}i} \cdot \alpha \cdot \gamma \cdot (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z)$, whence it must follow $\alpha \cdot \gamma = 0$ and it is clear that from $\beta = 0$, it cannot happen $\alpha = 0$ so that we conclude $\gamma = 0$.

We should have in this case $T_{a_1}(z) = \frac{\alpha z}{\lambda}$, so that for $z \in D^*(a_1, r_{a_1})$,

$$w(z) = T_{a_1} \circ \left((z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z) \right) = \frac{\alpha}{\lambda} \cdot (z-a_1)^{\frac{l_1}{k}} \cdot g_{a_1}(z). \tag{25}$$

The possibility (24) should be excluded since we are assuming that $w(z)$ has no poles and therefore in $D^*(a_1, r_{a_1})$, (25) must be the right relation.

We conclude from this fact and (16) that the analytic continuation of $w(z) = (z-a_1)^{\frac{1}{k}} \cdot g_a(z)$ should also vanish at a_1, that is, the analytic continuation of $g_a(z)$ to $D^*(a_1, r_{a_1})$ vanishes at a_1 and can be factorized as

$$g_a(z) = (z-a_1)^{\frac{l_1}{k}} \cdot e_{a_1}(z), \tag{26}$$

where $e_{a_1}(z)$ is a uniform analytic function in $D(a_1, r_{a_1})$ with $e_{a_1}(a_1) \neq 0$.

We have obtained a cycle of k/l_1 branches of $w(z)$ at $D^*(a_1, r_{a_1})$ as analytic continuations of corresponding k/l_1 branches of $w(z)$ at $D^*(a, r_a)$. If we now start with one of the remaining branches of $w(z)$ in $\Delta(\zeta, r_\zeta)$, that is, one of the branches not corresponding with any of the branches of the obtained cycle in $D^*(a_1, r_{a_1})$, and proceed again by analytic continuation along γ_1, we should obtain a new and disjoint cycle of k/l_1 branches at $D^*(a_1, r_{a_1})$ of $w(z)$. And proceeding in this way until we have all the branches of $w(z)$ in $D^*(a, r_a)$, we should finally have the k-branches of $w(z)$ in $D^*(a, r_a)$, corresponding by analytic continuation along γ_1 with the k branches of $w(z)$ in $D^*(a_1, r_{a_1})$, grouped in cycles of k/l_1 branches each.

By the representation (26) and by (16), $w(z) = (z-a)^{\frac{1}{k}} \cdot (z-a_1)^{\frac{l_1}{k}} \cdot e_{a_1}(z)$, will be valid in every simply connected domain excluding the remaining ramification points, that is, a_n for $n \geq 2$ $A(\gamma_1, a, a_1)$ and such that

$$A(\gamma_1, a, a_1) \supset \gamma_1^* \cup D^*(a, r_a) \cup D^*(a_1, r_{a_1}).$$

Now, we may start at any point $z \in A(\gamma_1, a, a_1)$, considering a particular branch of $w(z)$, and proceed by analytic continuation along a non-selfintersecting arc γ_2 joining

z and a point $\zeta_2 \in D^*(a_2, r_{a_2})$. Arguing with γ_2, $w(z)$ and $w_{a_2}(z)$ as we did before with γ_1, $w(z)$ and $w_{a_1}(z)$, we conclude that $w(z)$ can be represented in every simply connected $A(\gamma_1, \gamma_2, a, a_1, a_2)$ excluding the remaining ramification points, that is, a_n for $n \geq 3$ and such that

$$A(\gamma_1, \gamma_2, a, a_1, a_2) \supset \gamma_2^* \cup D^*(a_2, r_{a_2}) \cup A(\gamma_1, a, a_1),$$

in the form $w(z) = (z-a)^{\frac{1}{k}} \cdot (z-a_1)^{\frac{l_1}{k}} \cdot (z-a_2)^{\frac{l_2}{k}} \cdot e_{1,2}(z)$, where $e_{1,2}(z)$ is a uniform analytic function in $A(\gamma_1, \gamma_2, a, a_1, a_2)$.

We have ordered the a_n's according to increasing moduli and now proceeding inductively we obtain for a given $n_0 \in \mathbb{N}$ a representation of $w(z)$ of the form

$$w(z) = (z-a)^{\frac{1}{k}} \cdot (z-a_1)^{\frac{l_1}{k}} \cdot (z-a_2)^{\frac{l_2}{k}} \cdot \dots \cdot (z-a_{n_0})^{\frac{l_{n_0}}{k}} \cdot e_{1,2,\dots,n_0}(z), \tag{27}$$

in every simply connected domain containing the points $a_1, a_2, \dots, a_{n_0}$ and excluding a_n for $n > n_0$ and where $e_{1,2,\dots,n_0}(z)$ is a uniform analytic function in that domain.

Now let $r > 0$, such that $r > |a|$ and $r \neq |a_n|$, for every $n \in \mathbb{N}$ and let $a_1, a_2, \dots, a_{n_0}$ be all the a_n's with $|a_n| < r$. Then the representation (27) is also valid in $D(0,r)$.

Finally, if we consider an algebroid equation of the form

$$w^k = L(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right),$$

where the sequence $\{a_n\}_{n\in\mathbb{N}}$ is formed by the a_n's but each a_n repeated l_n times, we obtain an algebroid function $w_L(z)$ of order k as that in the example. If we consider the restriction of this algebroid function to $D(0,r)$, we obtain a function of the form

$$w_L(z) = (z-a)^{\frac{1}{k}} \cdot (z-a_1)^{\frac{l_1}{k}} \cdot (z-a_2)^{\frac{l_2}{k}} \cdot \dots \cdot (z-a_{n_0})^{\frac{l_{n_0}}{k}} \cdot g(z),$$

where $g(z)$ is a uniform non-vanishing analytic function in $D(0,r)$. Therefore, we obtain

$$\frac{w(z)}{w_L(z)} = \frac{e_{1,2,\dots,n_0}(z)}{g(z)},$$

that is

$$w(z) = w_L(z) \cdot e(z), \tag{28}$$

where $e(z)$ is a uniform analytic function in $D(0,r)$.

Since $r > 0$ can be taken arbitrarily large, we conclude that the relation (28) is true in the entire plane and $e(z)$ is an entire function.

Since we modified $w(z)$ by a Möbius transformation initially, we conclude the relation $w(z) = T \circ (w_L(z) \cdot e(z))$ holds, which is (15). □

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