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CONICS FROM THE ADJOINT REPRESENTATION OF $SU(2)$

Mircea Crasmareanu

Abstract. The aim of this paper is to introduce and study the class of conics provided by the symmetric matrices of the adjoint representation of the Lie group $SU(2) = S^3$. This class depends on three real parameters as components of a point of sphere S^2 and various relationships between these parameters give special subclasses of conics. A symmetric matrix inspired by one giving by Barning as Pythagorean triple preserving matrix and associated hyperbola are carefully analyzed. We extend this latter hyperbola to a class of hyperbolas with integral coefficients. A complex approach is also included.

1. Introduction

After more than two thousand of years the conics continue to be a versatile object of mathematics and the very recent book [5] is a veritable proof for this fact. A lot of techniques, from analytical to projective, are developed to handle these remarkable curves.

The starting point of this note is the article [3] where symmetric Pythagorean triple preserving (PTP) matrices are used to generate conics. Hence, we continue this line of research with another class of symmetric matrices of order 3, namely the ones produced by the adjoint representation of 3-dimensional matrix Lie group $SU(2) = S^3$. On this way we produce a class of remarkable conics, called $SU(2)$-*conics* and we study this special class in both Euclidean coordinates (x, y) as well as complex variable $z \in \mathbb{C}$ following the approach of [3]. Since our conics depend on three real parameters $(b, u, v) \in S^2$ we obtain various classes of $SU(2)$-conics by imposing (algebraic) conditions on these parameters. Some examples reduces the domain of definition from sphere S^2 to circles S^1 or $S^1(\frac{1}{\sqrt{2}})$ and then we get some 1-parameter families of $SU(2)$-conics, with constant invariants. Other main tools for our analysis are the Euler angles, the Hopf bundle and spherical coordinates of S^2. To any $SU(2)$-conic is associated a line in a natural way and this line is expressed for all our examples.

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A very interesting symmetric PTM matrix was considered in 1963 by F. J. M. Barning. We introduce here a variant of it which corresponds in our setting to the unit vector $\frac{1}{\sqrt{3}}(1,1,1) \in S^2$. The associated $SU(2)$-conic is a hyperbola Γ^b which in its canonical form contains a numerable set of integer (or lattice) points. We connect also its canonical form with the Pell equation $\tilde{x}^2 - 3\tilde{y}^2 = 1$ by a scaling transformation and one of its spherical coordinates with the magic angle θ_m. Two values of a canonical conic function associated to hyperbolas with the same eccentricity are computed and on this way two new transcendental numbers are obtained.

In the second section we discuss a $SU(2)$-conic Γ in terms of its Hermitian coefficients and another complex number called affix of Γ. Also, we continue the algebraic study by discussing the nature of a binary quadratic form F associated to the Γ. This form F corresponding to our Barning type matrix yields an elliptic curve with the integer points $O(0,0)$, $P_{\pm}(4,\pm 2)$ and we obtain the sum $P_{\pm} + P_{\pm}$. Also for this elliptic curve we compute its j-invariant and its associated Picard-Fuchs differential equation.

The final section concerns with an irreducible representation involving $SU(4)$ and a quartic associated to the corresponding matrix in the (real) symmetric case. We note that some hard computations are performed with WolframAlpha.

2. Conics provided by the adjoint representation of $SU(2)$

In the setting of two-dimensional Euclidean space $(\mathbb{R}^2, g_{can} = diag(1,1))$ let us consider the conic Γ implicitly defined by $f \in C^\infty(\mathbb{R}^2)$ as $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$ where f is a quadratic function of the form $f(x,y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$.

The study of Γ is based on the symmetric matrices (e means extended):

$$\Gamma := \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix} \in Sym(2), \quad \Gamma^e := \begin{pmatrix} r_{11} & r_{12} & r_{10} \\ r_{12} & r_{22} & r_{20} \\ r_{10} & r_{20} & r_{00} \end{pmatrix} \in Sym(3).$$

In fact, the algebraic invariants associated to Γ are:
$I := r_{11} + r_{22} = Tr\Gamma$, $\delta := \det \Gamma$, $\Delta := \det \Gamma^e$, $D := \delta + r_{11}r_{00} - r_{10}^2 + r_{22}r_{00} - r_{20}^2$. These invariants are preserved by the action of Euclidean group of isometries in usual matrix forms. Also, invariants of affine, equiaffine, translation and other transformations groups can be studied. It follows the necessity to search for remarkable symmetric matrices of order three.

A very useful three-dimensional matrix Lie group in both mathematics and physics is $SU(2)$ with its Lie algebra $su(2)$; in fact, $SU(2)$ is the most "simple" non-commutative compact Lie group. The adjoint representation $Ad : SU(2) \to GL(su(2)) \simeq GL(3,\mathbb{R})$,

$Ad(A)B := ABA^{-1}$, is presented, for example, in [1, p. 31]:

$$\begin{cases} z := a+bi, \quad w := u+iv \\ Ad\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = \begin{pmatrix} a^2+b^2-u^2-v^2 & -2av+2bu & 2au+2bv \\ 2av+2bu & a^2-b^2+u^2-v^2 & -2ab+2uv \\ -2au+2bv & 2ab+2uv & a^2-b^2-u^2+v^2 \end{pmatrix} mod S^3 \end{cases}$$

where, due to the isomorphism $SU(2) \simeq S^3$, we have $|z|^2+|w|^2 = a^2+b^2+u^2+v^2 = 1$; the complex numbers z, w are called *the Cayley-Klein parameters*. In fact, the image of the map Ad is $SO(3)$ as subgroup in $GL(3, \mathbb{R})$. The right 3×3 matrix is symmetric if and only if $a = \Re z = 0$ and then we arrive at the matrix:

$$X(b,u,v) := \begin{pmatrix} b^2-u^2-v^2 & 2bu & 2bv \\ 2bu & -b^2+u^2-v^2 & 2uv \\ 2bv & 2uv & -b^2-u^2+v^2 \end{pmatrix} \quad (1)$$
$$\in SO(3) \cap Sym(3), b^2+u^2+v^2 = 1$$

which is then defined on the equator $S^2 \times \{a = 0\} \subset S^3$ and which can be also written:

$$X(b,u,v) := 2\begin{pmatrix} b^2 & bu & bv \\ bu & u^2 & uv \\ bv & uv & v^2 \end{pmatrix} - I_3 = 2\begin{pmatrix} b \\ u \\ v \end{pmatrix} \cdot (b,u,v) - I_3. \quad (2)$$

This matrix X yields the following class of conics.

DEFINITION 2.1. A $SU(2)$-*conic* is a conic depending on $(b,u,v) \in S^2$ in the form:

$$\Gamma(b,u,v) : (b^2-u^2-v^2)x^2 + 4buxy + (u^2-b^2-v^2)y^2 + 4bvx + 4uvy + (v^2-b^2-u^2) = 0. \quad (3)$$

A straightforward computation yields.

PROPOSITION 2.2. *The invariants of* $\Gamma(b,u,v)$ *are*

$$\delta(b,u,v) = (b^2+u^2+v^2)(v^2-u^2-b^2) = v^2-u^2-b^2 = r_{00},$$
$$I = I(v) = -2v^2 \leq 0, \quad , D(b,u,v) = -(b^2+u^2+v^2)^2 = -1,$$
$$\Delta(b,u,v) = (b^2+u^2+v^2)^3 = 1. \quad (4)$$

Hence any $SU(2)$*-conic* Γ *is non-degenerate and its eccentricity* e_Γ *depends only of* v*:*

$$e_\Gamma(v) = \sqrt{2(1-v^2)} \in [0, \sqrt{2}]$$

and then we can express I *in terms of eccentricity:* $I = e^2 - 2$.

REMARK 2.3. (i) In [3, p. 86] the symmetric Pythagorean triple preserving (PTP on short) matrices are considered:

$$Y(r,s,w) := \begin{pmatrix} r^2-2s^2+w^2 & 2s(r-w) & r^2-w^2 \\ 2s(r-w) & 2(rw+s^2) & 2s(r+w) \\ r^2-w^2 & 2s(r+w) & r^2+2s^2+w^2 \end{pmatrix}$$

again for defining a special class of conics. We point out that a $SU(2)$-matrix X can not be equal (from the point of view of conics theory) with a PTP-matrix Y. Indeed, the first and last invariants in second case are $I(r,s,w) = (r+w)^2 \geq 0$, $\Delta(r,s,w) =$

$8(rw-s^2)^3$ and hence the equality in I implies $I=0$ which means $v=0$ (a case discussed below) respectively $r=-w$ which implies a strictly negative $\Delta(r=-w,s)$, in contraction with (4). In fact, all elements of $X(v=0)$ and $Y(r=-w)$ coincide for $b=\sqrt{2}r$ and $u=\sqrt{2}s$ with the exception of $X_{33}=-Y_{33}=-2(r^2+s^2)$.

(ii) In [6] a triple product $\{\cdots\}$ is considered on $Sym(3)$: $\{xyz\}:=xyz+zyx-Tr(xy)z$. Our matrix X being in $SO(3)\cap Sym(3)$ is idempotent i.e. $X^2=I_3$ and hence: $\{XXZ\}=2Z-3Z=-Z$ for all X and Z. In particular: $\{XXX\}=-X$.

(iii) It is well-known that a matrix $X\in SO(3)$ is completely described by its *Euler angles* (φ,ψ,θ) trough:

$$X:=\begin{pmatrix}\cos\varphi\cos\psi-\sin\varphi\sin\psi\cos\theta & -\cos\varphi\cos\psi-\sin\varphi\cos\psi\cos\theta & \sin\varphi\sin\theta\\ \sin\varphi\cos\psi+\cos\varphi\sin\psi\cos\theta & -\sin\varphi\sin\psi+\cos\varphi\cos\psi\cos\theta & -\cos\varphi\sin\theta\\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta\end{pmatrix}.$$

Then, another straightforward computation yields for $v\notin\{-1,0,1\}$ the following Euler angles of our $X(b,u,v)$:

$$\begin{cases}\cos\varphi=\frac{-usgn(v)}{\sqrt{b^2+u^2}}, \quad \sin\varphi=\frac{bsgn(v)}{\sqrt{b^2+u^2}},\\ \cos\psi=\frac{u}{\mathrm{sgn}(v)\sqrt{b^2+u^2}}, \quad \sin\psi=\frac{b}{\mathrm{sgn}(v)\sqrt{b^2+u^2}},\\ \cos\theta=2v^2-1, \quad \sin\theta=2|v|\sqrt{b^2+u^2}=2|v|\sqrt{1-v^2},\end{cases}$$

with $\mathrm{sgn}(v)$ the signum of v. We remark that θ depends only of v and hence the eccentricity depends only on θ: $e_\Gamma=e_\Gamma(\theta)=\sqrt{1-\cos\theta}$. An ellipse is called *self-complementary* if its eccentricity is $e=\frac{1}{\sqrt{2}}$ and then a $SU(2)$-ellipse is self-complementary if and only if $v=\frac{\sqrt{3}}{2}$, equivalently $\theta=\frac{\pi}{3}$.

(iv) A remarkable tool for the geometry of sphere S^3 is *the Hopf bundle* [10, p. 5]:

$$\pi:S^3\to S^2\left(\frac{1}{2}\right)\subset\mathbb{R}\times\mathbb{C}, \quad (z,w)\to\left(\frac{1}{2}(|w|^2-|z|^2),z\bar{w}\right).$$

Hence, for our setting we have the restriction to the domain of Γ:

$$\pi|_{dom\ \Gamma}:(b,u,v)\in S^2\to\left(\frac{1}{2}(u^2+v^2-b^2)=\frac{1}{2}(1-2b^2),b(v+iu)\right)\in S^2\left(\frac{1}{2}\right)$$

which we call *the restricted Hopf map.*

(v) For the sphere S^2 as domain of Γ we can use the spherical coordinates (α,β):

$$b=\sin\alpha, \quad u=\cos\alpha\cos\beta, \quad v=\cos\alpha\sin\beta\to z=i\sin\alpha, w=\cos\alpha e^{i\beta}$$

and then the defining matrix X is:

$$X(\alpha,\beta):=\begin{pmatrix}2\sin^2\alpha-1 & \sin2\alpha\cos\beta & \sin2\alpha\sin\beta\\ \sin2\alpha\cos\beta & 2\cos^2\alpha\cos^2\beta-1 & \cos^2\sin2\beta\\ \sin2\alpha\sin\beta & \cos^2\sin2\beta & 2\cos^2\alpha\sin^2\beta-1\end{pmatrix}.$$

The restricted Hopf map becomes:

$$\pi|_{dom\ \Gamma}:(b,u,v)(\alpha,\beta)\in S^2\to\frac{1}{2}(\cos2\alpha,\sin2\alpha e^{i(\frac{\pi}{2}-\beta)})\in S^2\left(\frac{1}{2}\right).$$

(vi) In [11, p. 163] two conics provided by the matrices $A,B\in Sym(3)$ are called *dual* if there exists $\lambda \in \mathbb{R}$ such that $AB = \lambda I_3$. Since our X belongs to $SO(3)$ it follows that an $SU(2)$-conic is a self-dual conic.

(vii) In the equation (2) the symmetric matrix:

$$\begin{pmatrix} b^2 & bu & bv \\ bu & u^2 & uv \\ bv & uv & v^2 \end{pmatrix} = \begin{pmatrix} b \\ u \\ v \end{pmatrix} \cdot (b, u, v)$$

occurs. According to [11, p. 165] the associated conic is the double line l provided by the equation in homogeneous coordinates $bx + uy + vz = 0$. Hence we can denote our $SU(2)$-conic as $\Gamma = 2l \cdot l^t - I_3$ with l^t being the transpose of matrix l. Let us call l as being *the associated line* of Γ. Due to the self-duality of our conic if follows that the pole of line l with respect to Γ is the (homogeneous) point $L(b, u, v)$; for $v \neq 0$ we have the real point $L(\frac{b}{v}, \frac{u}{v})$.

In other words, the equation (2) yields the following expression of our conic:

$$\Gamma(b, u, v) : 2(bx + uy + v)^2 - (x^2 + y^2 + 1) = 0.$$

In [9], given a (non-degenerate) conic c and a line e, to this pair is associated the *bitangent pencil of conics*:

$$\alpha c + \beta e^2 = 0$$

with scalars α, β not simultaneously zero; moreover, sometimes the condition $\alpha+\beta=1$. Our conic Γ is exactly of this type and satisfies $\alpha + \beta = -1 + 2 = 1$ but the initial conic c is degenerate to $x^2 + y^2 + 1 = 0$.

(viii) It is well-known that the locus of the points with orthogonal tangents to the conic Γ is *the Monge* (or director) *circle* with general equation:

$$\mathcal{C}(\Gamma) : \delta(x^2 + y^2) - 2\begin{vmatrix} r_{12} & r_{10} \\ r_{22} & r_{20} \end{vmatrix} x + 2\begin{vmatrix} r_{11} & r_{10} \\ r_{12} & r_{20} \end{vmatrix} y + \begin{vmatrix} r_{12} & r_{10} \\ r_{10} & r_{00} \end{vmatrix} + \begin{vmatrix} r_{22} & r_{20} \\ r_{20} & r_{00} \end{vmatrix} = 0.$$

For our $SU(2)$-conic it results the circle:

$$\mathcal{C}(\Gamma)(b, u, v) : (2v^2 - 1)(x^2 + y^2) - 8bv(1 - u^2)x + 8uv(b^2 - 1)y - 2v^2 = 0.$$

EXAMPLE 2.4. The vanishing $I = 0$ is equivalent with $v = \Im w = 0$; hence $b^2 + u^2 = 1$. It results the $SU(2)$-conic defined on S^1: $\Gamma^1(b, u) : (b^2 - u^2)(x^2 - y^2) + 4buxy - 1 = 0$ with the constant invariants: $I^1 = 0$, $\delta^1 = D^1 = -\Delta^1 = -1$ and then Γ^1 is an equilateral hyperbola; for $bu \neq 0$ this hyperbola has as center the origin $O(0, 0)$. Three examples are:

$$\Gamma^1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) : 2xy - 1 = 0, \quad \Gamma^1(1, 0) : x^2 - y^2 - 1 = 0, \quad \Gamma^1(0, 1) : x^2 - y^2 + 1 = 0.$$

Hence the eccentricity is $e = \sqrt{2}$. $\Gamma^1(1, 0)$ is parametrised by the hyperbolic trigonometrical functions $(\cosh t, \sinh t)$ and $\Gamma^2(0, 1)$ by $(\sinh t, \cosh t)$. This example can be handled also trigonometrically by choosing $b = \sin t$ and $u = \cos t$; hence $z = i \sin t$, $w = \cos t$ and then:

$$X(b, u) := X(t) = \begin{pmatrix} -\cos 2t & \sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SO(3).$$

The associated $SU(2)$-conic is: $\Gamma^1(t) : \cos 2t(x^2 - y^2) - 2\sin 2txy + 1 = 0$ and the restricted Hopf map is: $\pi|_{dom\ \Gamma} : (\sin t, \cos t) \in S^1 \to \frac{1}{2}(\cos 2t, i\sin 2t) \in S^2\left(\frac{1}{2}\right) \subset \mathbb{R} \times \mathbb{C}$. The spherical coordinates of $X(t)$ are $(\alpha = t, \beta = 1)$. The associated line $l = l(t)$ contains the origin O: $\sin tx + \cos ty = 0$.

Example 2.5. The vanishing of δ i.e. the reduction of the Monge circle to a line means $v^2 = b^2 + u^2 = \frac{1}{2}$ and we get two $SU(2)$-conics defined on $S^1\left(\frac{1}{\sqrt{2}}\right)$, again with constant invariants: $\Gamma^2_{\pm}(b,u) : u^2x^2 + b^2y^2 - 2buxy \mp \sqrt{2}(bx + uy) = 0$, $I^2 = D^2 = -\Delta^2 = -1$. Its Euler angles are given by: $\cos\varphi_{\pm} = -\cos\psi_{\pm} = \mp\sqrt{2}u$, $\sin\varphi_{\pm} = \sin\psi_{\pm} = \pm\sqrt{2}b$, $\theta_{\pm} = \pm\frac{\pi}{2}$. $\Gamma^2_{\pm}$ is a parabola through the origin O and some examples are:

$$\begin{cases} \Gamma^2_{\pm}\left(\frac{1}{2}, \frac{1}{2}\right) : x^2 + y^2 - 2xy \mp 2\sqrt{2}(x + y) = 0, \\ \Gamma^2_{\pm}(1,0) : -y^2 \pm 2x = 0, \quad \Gamma^2_{\pm}(0,1) : -x^2 \pm 2y = 0. \end{cases} \tag{5}$$

For $\Gamma^2_{\pm}\left(\frac{1}{2}, \frac{1}{2}\right)$ we use the trigonometrical rotation of $\frac{\pi}{4}$: $x = \frac{1}{\sqrt{2}}(\tilde{x} - \tilde{y})$, $y = \frac{1}{\sqrt{2}}(\tilde{x} + \tilde{y})$ to reduce to canonical form: $\Gamma^2_{\pm}\left(\frac{1}{2}, \frac{1}{2}\right) : \tilde{y}^2 \mp 2\tilde{x} = 0$, which means exactly $\Gamma^2_{\pm}(1,0)$. This parabola $\Gamma^2_{\pm}(1,0)$ contains the sequence $(\pm 2n^2, 2n)$ of integer points.

We can use again a trigonometrically substitution: $b = \frac{1}{\sqrt{2}}\sin t$, $u = \frac{1}{\sqrt{2}}\cos t$ and then: $\Gamma^2_{\pm}(t) : \cos^2 tx^2 + \sin^2 ty^2 - \sin 2txy \mp 2(\sin tx + \cos ty) = 0$ and the restricted Hopf map is: $\pi|_{dom\ \Gamma} : (b,u) = \frac{1}{\sqrt{2}}(\sin t, \cos t) \in S^1\left(\frac{1}{\sqrt{2}}\right) \to (u^2, b(\pm\frac{1}{\sqrt{2}} + ui)) = \frac{1}{2}(\cos^2 t, \sin t(\pm 1 + i\cos t)) \in S^2\left(\frac{1}{2}\right) \subset \mathbb{R} \times \mathbb{C}$. The spherical coordinates of $X(t)$ are $(\alpha = \frac{\pi}{4}, \beta = t)$. The degenerated Monge circles are: $\mathcal{C}(\Gamma_{\pm})(t) : \mp 2\sqrt{2}\sin t(2 - \cos^2 t)x \pm 2\sqrt{2}\cos t(\sin^2 t - 2)y - 1 = 0$.

We finish this example with the remark that in both Examples 2.4 and 2.5 although the conic is dependent of one parameter all the invariants are constants.

Example 2.6. A remarkable unimodular matrix, given by Barning in [2], was analyzed in the setting of PTP-conics from [3, p. 87]:

$$N_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} = 2Y\left(r = \sqrt{2}, s = \frac{1}{\sqrt{2}}, w = 0\right).$$

The associated PTP-conic is a hyperbola having the canonical form: $9X^2 - 3Y^2 + 1 = 0$ with eccentricity $e = \frac{2}{\sqrt{3}} > 1$. Inspired by this case we introduce *the $SU(2)$-Barning matrix*:

$$X\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \in SO(3)$$

respectively *the $SU(2)$-Barning hyperbola*:

$$\Gamma^b : x^2 - 4xy + y^2 - 4(x + y) + 1 = 0 \tag{6}$$

having the center $C(-2,-2)$. Its canonical form and eccentricity are:

$$\Gamma^b : \frac{X^2}{9} - \frac{Y^2}{3} - 1 = 0, \quad e(\Gamma^b) = \frac{\sqrt{9+3}}{3} = \frac{2}{\sqrt{3}}. \tag{7}$$

The associated (homogeneous) line is $l : x + y + z = 0$ with the pole $L(1, 1)$. With WolframAlpha we obtain that the canonical form (7) has a numerable set of integer (or lattice) points:

$$(X^n_{\pm}, Y^n_{\pm}) = \pm\frac{1}{2}\left(3[(2-\sqrt{3})^n + (2+\sqrt{3})^n], \sqrt{3}[(2-\sqrt{3})^n - (2+\sqrt{3})^n]\right), \quad n \in \mathbb{N}.$$

The scaling transformation $(\tilde{x}, \tilde{y}) = \frac{1}{3}(X, Y)$ yields the Pell equation:

$$\tilde{x}^2 - 3\tilde{y}^2 = 1 \tag{8}$$

and for a positive solution $\tilde{x}$ the numbers $n-1$, $n = 2\tilde{x}$, $n+1$ are the consecutive edges of a Heron triangle [3, p. 91].

The Euler angles of the $SU(2)$-Barning type matrix X are given by: $\varphi = \frac{3\pi}{4}$, $\psi = \frac{\pi}{4}$, $\cos\theta = -\frac{1}{3}$ and $\sin\theta = \frac{2\sqrt{2}}{3}$. Again with WolframAlpha we find: $\theta = \frac{\pi}{2} + 70.53 = 160.53$. Also, the value of the restricted Hopf map is: $\pi\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{6}(1, 2\sqrt{2}e^{\frac{\pi i}{4}}) = 0$.

The spherical coordinates of $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are $(\alpha = \arctan\frac{1}{\sqrt{2}}, \beta = \frac{\pi}{4})$ i.e. $\cos\alpha = \sqrt{\frac{2}{3}}$, $\sin\alpha = \frac{1}{\sqrt{3}}$. Hence, α is the complementary of *the magic angle* θ_m provided by $\tan\theta_m = \sqrt{2}$.

Example 2.7. A conic Γ is called *symmetric* if $r_{11} = r_{22}$. For our $SU(2)$-conic this means $u = \pm b$ and from $v^2 = 1 - 2b^2$ (and then $\cos\theta = 3 - 4b^2$) it results that symmetric $SU(2)$-conics appear in pairs depending only on $b \in [0, \frac{1}{\sqrt{2}}]$:

$$\Gamma_{\pm}(b) : (1-2b^2)(x^2+y^2) \mp 4b^2xy \mp 4b\sqrt{1-2b^2}(x \pm y) + (4b^2 - 1) = 0,\ e_{\Gamma_{\pm}}(b) = 2|b|.$$

For example, the $SU(2)$-Barning conic is a symmetrical one.

To the general $SU(2)$-matrix (1) we associate in a natural manner a linear fractional (or Möbius) function $f : P^1(\mathbb{R}) := \mathbb{R} \cup \{\infty\} \to P^1(\mathbb{R})$ by using the symmetric matrix providing δ :

$$f_X(t) := \frac{(b^2 - u^2 - v^2)t + 2bu}{(2bu)t + (u^2 - b^2 - v^2)}.$$

Its fixed (or double) points $t = f_X(t)$ are then solutions of the algebraic equation: $(bu)t + (u^2 - b^2)t - bu = 0$ and for our example $u = \pm b \neq 0$ it follows the universal (i.e. for all conics in this class) fixed points $t_{\pm} = \pm 1$.

Remark 2.8. In [12, p. 360] is defined a function, called *canonical conic function*, on the set of hyperbolas with the same eccentricity e as:

$$C_{hyperbola}(e) := \frac{1}{4(e^2-1)^2}\left[e^3 - e - \sqrt{e^2-1}\ln(e + \sqrt{e^2-1})\right].$$

For our equilateral and Barning type hyperbolas we obtain:

$$\begin{cases} C_{hyperbola}(\sqrt{2}) = \frac{1}{4}[\sqrt{2} - \ln(\sqrt{2}+1)] = 0.133209\ldots, \\ C_{hyperbola}\left(\frac{2}{\sqrt{3}}\right) = \frac{\sqrt{3}}{8}(4 - 3\ln 3) = 0.152455\ldots. \end{cases} \tag{9}$$

Both numbers are transcendental and since the number $4C_{hyperbola}(\sqrt{2})$ is already

A222362 from The On-Line Encyclopedia of Integer Sequences we propose the name *Barning hyperbolic number* for the second transcendental number above. Remark that $\frac{2}{\sqrt{3}}$ is the radius of the circumcircle for the equilateral triangle with sides of length 2.

In [4] a quaternion-inspired (non-internal) product is considered on the set $[1, +\infty)$:

$$R_1 \odot_c R_2 = \sqrt{\frac{(R_1 R_2)^2 - 1}{R_1^2 + R_2^2}} < \min\{R_1, R_2\}.$$

Hence the quaternionic product of the class of equilateral hyperbolas with the class of hyperbolas with eccentricity $e_2 = \frac{2}{\sqrt{3}}$ is the class of ellipses with eccentricity $e = \sqrt{2} \odot_c \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{2}}$.

We generalize the last example through the following definition.

Definition 2.9. A *$SU(2)$-Barning type matrix* is a symmetric 3-matrix depending on two integers:

$$B(t_1, t_2) = \begin{pmatrix} 1 & -2 & t_1 \\ -2 & 1 & t_1 \\ t_1 & t_1 & t_2 \end{pmatrix} \in Sym(3, \mathbb{Z})$$

and its *$SU(2)$-Barning type hyperbola* is: $\Gamma^B(t_1, t_2) : x^2 - 4xy + y^2 + 2t_1(x+y) + t_2 = 0$.

It follows that $\Gamma^b = \Gamma^B(-2, 1)$. The determinant of the $SU(2)$-Barning type matrix is $\det B(t_1, t_2) = -6t_1^2 - 3t_2$ and for any $k \in \mathbb{Z}$ the parabola: $P_k : detB(t_1, t_2) = -3k$ has again a numerable set of integer points: $(t_1^n, t_2^n) = (n, k - 2n^2)$, $n \in \mathbb{Z}$.

Remark that the 2-block common of all $SU(2)$-Barning type hyperbolas is:

$$\mathcal{B} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

Rank 2 symmetric hyperbolic Kac–Moody algebras $\mathcal{H}(a)$ with the Cartan matrices:

$$C_a := \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$$

with $a \geq 3$ are studied in [7]. Hence a multiple of the matrix $\mathcal{B}$ of (9) is a (symmetric) Cartan matrix $2\mathcal{B} = C_4$. The automorphic correction of $\mathcal{H}(4)$ is given in [8].

3. A complex approach to $SU(2)$-conics

The aim of this section is to study the $SU(2)$-conic Γ by using the complex structure of the plane. More precisely, with the usual notation $z = x + iy \in \mathbb{C}$ we derive the complex expression of a general conic Γ:

$$\Gamma : F(z, \bar{z}) := Az^2 + Bz\bar{z} + \bar{A}\bar{z}^2 + Cz + \bar{C}\bar{z} + r_{00} = 0 \tag{10}$$

with $A = \frac{r_{11} - r_{22}}{4} - \frac{r_{12}}{2} i \in \mathbb{C}$, $\quad 2B = r_{11} + r_{22} = I \in \mathbb{R}$, $\quad C = r_{10} - r_{20} i \in \mathbb{C}$.

It follows that the usual rotation performed to eliminate the mixed term xy has the meaning to reduce/rotate A in the real line while the translation which eliminates the term y has a similar meaning with respect to C. The inverse relationship between f and F is: $r_{11} = B + 2\Re A$, $r_{22} = B - 2\Re A$, $r_{12} = -2\Im A$, $r_{10} = \Re C$, $r_{20} = -\Im C$ with $\Re$ and $\Im$ respectively the real and imaginary part. The conic is symmetric if and only if A is pure imaginary.

The linear invariant I and the quadratic invariant δ are the trace respectively the determinant of the Hermitian matrix:

$$\Gamma^c = \begin{pmatrix} B & 2\bar{A} \\ 2A & B \end{pmatrix}$$

which is a special one, the entries of the main diagonal being equal; hence their set is the three-dimensional subspace $Sym(2)$ of the four-dimensional real linear space of 2×2 Hermitian matrices.

For our $SU(2)$-conic (3) we have the new coefficients, which we call *Hermitian*:

$$A(b,u,v) = \frac{1}{2}(b - iu)^2, \quad B(b,u,v) = -v^2, \quad C(b,u,v) = 2v(b - iu) \tag{11}$$

which are (real) 2-homogeneous functions of (b,u,v) and satisfy the quadratic relation:

$$-8AB = C^2. \tag{12}$$

Hence, a multiplication with $(-8B)$ of equation (10) gives a new relation for the $SU(2)$-conic, expressed only in B and C:

$$\Gamma(b,u,v) : (Cz)^2 - 8(B|z|)^2 + (\bar{C}\bar{z})^2 - 8B(Cz + \bar{C}\bar{z} + r_{00}(b,u,v)) = 0$$

while the square of the eccentricity is: $e^2 = \frac{4|A|}{2|A| - B\mathrm{sgn}(\Delta)}$, where sgn means the signum function and $|z|$ is the modulus of the complex number z.

Example 3.1. Both A and C from (11) are real if and only if $u = \Re w = 0$ and then $b^2 + v^2 = 1$; remark that both z, w are pure imaginary complex numbers. We get the $SU(2)$-conic defined on S^1:

$$\begin{cases} \Gamma^3(b,v) : (b^2 - v^2)x^2 - y^2 + 4bvx + (v^2 - b^2) = 0 \\ A^3(b,v) = \frac{b^2}{2}, \quad C^3(b,v) = 2bv. \end{cases}$$

Some examples are: $\Gamma^3\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) : y^2 - 2x = 0$, $\Gamma^3(1,0) : x^2 - y^2 - 1 = 0$, $\Gamma^3(0,1) = S^1 : x^2 + y^2 - 1 = 0$.

We point out that if in the examples of Section 2 all conics have the same type (elliptic, hyperbolic or parabolic) the examples above are of all types: the first is the horizontal right-oriented parabola (and $\Gamma^3\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \Gamma^2_+(1,0)$), the second is the unit hyperbola and the last is the unit circle, hence of elliptic type. Since circles are characterized by zero eccentricity the only $SU(2)$-circle corresponds to $v = 1$ and this is exactly S^1.

The triple $(x,y,z) = (A^3, B^3, C^3)$ provides a rational parametrization of the cone: $\mathcal{C} : -8xy = z^2$ or, considering (x,y,z) as homogeneous coordinates corresponding to $X = \frac{x}{z}$, $Y = \frac{y}{z}$, a rational parametrization for the equilateral hyperbola: $XY = -\frac{1}{8}$. With $b = \cos t$ and $v = \sin t$ we have the 1-parameter family of $SU(2)$-conics: $\Gamma^3(t) : \cos 2tx^2 - y^2 + 2\sin 2tx - \cos 2t = 0$ with the associated homogenous line: $l(t) : \cos tx + \sin tz = 0$. For $z = 1$ we have the vertical lines: $\cos tx + \sin t = 0$.

Example 3.2. Let us call the complex number:

$$z_\Gamma := b - iu = \sin\alpha - i\cos\alpha\cos\beta \in \mathbb{C}, \quad |z_\Gamma| = \sqrt{1-v^2} \to e_\Gamma = \sqrt{2}|z_\Gamma|$$

as being *affix of* Γ, even in the case when z_Γ does not belongs to Γ. For the example of the $SU(2)$-Barning type hyperbola (6) we have:

$$z_{\Gamma^b} = \frac{1-i}{\sqrt{3}} = \sqrt{\frac{2}{3}}e^{-\frac{i\pi}{4}} \notin \Gamma^b$$

and then the Hermitian coefficients of Γ^b are:

$$A(\Gamma^b) = \frac{1}{2}z^2_{\Gamma^b} = -\frac{i}{3}, \quad C(\Gamma^b) = \frac{2}{\sqrt{3}}z_{\Gamma^b} = \frac{2\sqrt{2}}{3}e^{-\frac{i\pi}{4}}.$$

The affix z_Γ belongs to Γ if and only if: $(b^2-u^2)^2 + v^4 + v^2 + 4v(b^2-u^2) = 4(bu)^2 + b^2 + u^2$. Searching for a symmetrical conic it results the solution $u = \frac{1}{2} = \pm b$, $v = \frac{1}{\sqrt{2}}$ (and then $\theta = \frac{\pi}{2}$) which means: $x^2 \mp 2xy + y^2 - 2\sqrt{2}(x \pm y) = 0$, $z_\pm = \frac{1}{2}(1 \mp i)$. We recover the parabolas (5).

Example 3.3. The associated binary quadratic form the matrix $\mathcal{B}$ is:

$$f(x,y) = f_{\mathcal{B}}(x,y) = x^2 - 4xy + y^2 \tag{13}$$

and its complex variant is: $F_{\mathcal{B}}(z,\bar{z}) = z\bar{z} - \frac{1}{i}(z^2 - \bar{z}^2) = iz^2 + z\bar{z} - i\bar{z}^2$, which means that $A = i \in \mathbb{C}$ and $B = 1$. The equation (12) is: $C^2 = -8AB = -8i = 2^3 e^{\frac{3\pi}{2}i}$ with solutions: $C_\pm = \pm 2^{\frac{3}{2}} e^{\frac{3\pi}{4}i}$.

Remark 3.4. Recall after [3, p. 91] that the binary quadratic form: $F(x,y) = ax^2 + 2bxy + cy^2$ has *the discriminant* $\Delta(F) := 4(b^2 - ac)$ and is called *indefinite* if $\Delta(F) > 0$. Also, an indefinite quadratic form is called *reduced* if:

$$|\sqrt{\Delta(F)} - 2|a|| < 2b < \sqrt{\Delta(F)}. \tag{14}$$

The $SU(2)$-Barning quadratic form given by (13) is indefinite with $\Delta(f_{\mathcal{B}}) = 12$ but is not reduced since the left inequality of (14) does not holds. But for $\Delta = 12$ we get another binary quadratic form, called *principal*: $F^b(x,y) := x^2 - \frac{\Delta}{4}y^2 = x^2 - 3y^2$ and we re-obtain the left-hand side of Pell equation (8).

Remark 3.5. Recall also that to every binary quadratic form F one can associate an elliptic curve:

$$E_F : y^2 = x^3 + (2ba^{-\frac{2}{3}})x^2 + (ca^{-\frac{1}{3}})x, \quad \Delta(E_F) = 16\left(\frac{c}{a}\right)^2 \Delta(F).$$

Hence a *$SU(2)$-Barning elliptic curve* is obtained from $f_{\mathcal{B}}$:

$$E_b : y^2 = x^3 - 4x^2 + x, \quad \Delta(E_b) = 16 \cdot 12 = 2^6 \cdot 3 = 192 \tag{15}$$

with integer points $O(0,0)$, $P_\pm(4, \pm 2)$; the first point belongs to the compact component of E_b while the points $P_\pm$ belong to the non-compact component of E_b. Also, with the transformation $z = x - \frac{4}{3}$ we cancel the term x^2:

$$E_b : y^2 = z^3 - \frac{13}{3}z - \frac{92}{27}.$$

It is well-known that any elliptic curve can be made into a commutative group with additive identity equal to the point at infinity [0: 1: 0]. For our E_b of (15) is simple to compute $P^{\pm} + P^{\pm}$ since these are not two-torsion points from their $2y = \pm 4 \neq 0$. We have:

$$2P^{\pm} = \frac{15}{16}\left(15, \mp\frac{191}{4}\right) = \left(\frac{225}{16}, \mp\frac{15 \cdot 191}{64}\right).$$

The j-invariant of an arbitrary elliptic curve $\mathcal{E} : y^2 = x^3 - G_2x - G_3$ is $j(\mathcal{E}) := \frac{4G_2^3}{4G_2^3 - 27G_3^2}$ and for our E_b it results $j(E_b) = \frac{13^3}{13^3 - 46^2} = \frac{2197}{81}$.

The Picard-Fuchs differential equation of $\mathcal{E}$:

$$PF(\mathcal{E}) : y'' + \frac{1}{j}y' + \frac{31j - 4}{144j^2(1-j)^2}y = 0$$

becomes for our E_b:

$$PF(E_b) : y'' + \frac{81}{2197}y' + \frac{9^5 \cdot 67783}{4^2 \cdot 2197^2 \cdot 46^4}y = 0.$$

The L-functions and Modular Forms Database http://www.lmfdb.org/ provides the minimal integral model for E_b as: $minE_b : y^2 = x^3 - x^2 - 4x - 2$.

Remark 3.6. Another elliptic curve which can be naturally considered in pair with E_b is $C_b : y^2 + y = x^3 - 4x^2 + x$, which have 4 integer points: $(0, -1)$, $(0, 0)$, $(5, -6)$, $(5, 5)$.

4. Relationships with $SU(4)$

There exists an irreducible representation $SU(2) \to SU(4)$ with the matrix representation:

$$\mu_4 \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = \begin{pmatrix} z^3 & -\sqrt{3}z^2w & \sqrt{3}zw^2 & -w^3 \\ \sqrt{3}z^2\bar{w} & (|z|^2 - 2|w|^2)z & -(2|z|^2 - |w|^2)w & \sqrt{3}\bar{z}w^2 \\ \sqrt{3}z\bar{w}^2 & (2|z|^2 - |w|^2)\bar{w} & (|z|^2 - 2|w|^2)\bar{z} & -\sqrt{3}\bar{z}^2w \\ \bar{w}^3 & \sqrt{3}\bar{z}\bar{w}^2 & \sqrt{3}\bar{z}^2\bar{w} & \bar{z}^3 \end{pmatrix}.$$

This matrix is real-symmetric i.e. the transpose of μ_4 is equal with μ_4 if and only if $\bar{w} = -w$ i.e. w is a pure imaginary complex number. So, adding the symmetry discussed in previous sections we arrive at the case of Example 3.1: $z = bi$ and $w = vi$. The matrix become:

$$\mu_4 \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = i \begin{pmatrix} -b^3 & \sqrt{3}b^2v & -\sqrt{3}bv^2 & v^3 \\ \sqrt{3}b^2v & (b^2 - 2v^2)b & (v^2 - 2b^2)v & \sqrt{3}bv^2 \\ -\sqrt{3}bv^2 & (v^2 - 2b^2)v & (2v^2 - b^2)b & \sqrt{3}b^2v \\ v^3 & \sqrt{3}bv^2 & \sqrt{3}b^2v & b^3 \end{pmatrix}$$

which is a traceless matrix. We can associate a quadric:

$$Q(b, v) : b^3(1 - x^2) + b(b^2 - 2v^2)(y^2 - z^2) + 2\sqrt{3}(b^2vxy - bv^2xz + bv^2y + b^2vz) + 2v(v^2 - 2b^2)yz + 2v^3x = 0$$

which can be studied again through its invariants.

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Faculty of Mathematics, University “Al. I.Cuza”, Iasi, 700506, România

E-mail: mcrasm@uaic.ro