A NEW CLASS OF FINSLER METRICS

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Abstract. In this paper, we construct a new class of Finsler metrics which are not always \((\alpha, \beta)\)-metrics. We obtain the spray coefficients and Cartan connection of these metrics. We have also found a necessary and sufficient condition for them to be projective. Finally, under some suitable conditions, we obtain many new Douglas metrics from the given one.

1. Introduction

\((\alpha, \beta)\)-metrics form a rich class of Finsler metrics. They are computable and the patterns offer references for more study in Finsler spaces. Then, introducing new Finsler metrics which are not \((\alpha, \beta)\)-metrics helps us to evaluate the patterns. There are some classes of Finsler metrics which are not always \((\alpha, \beta)\)-metrics such as generalized \((\alpha, \beta)\)-metrics [15] or spherically symmetric Finsler metrics [17].

Here we are going to consider the Finsler metrics given by
\[
\bar{F} = F\phi(s),
\]
where \(F\) is a Finsler metric, \(s = \frac{\beta}{F}, \beta = b_iy^i, \|\beta\|_F < b_0\) and \(\phi(s)\) is a positive \(C^\infty\) function on \((-b_0, b_0)\). We call them \((F, \beta)\)-metrics. These metrics are not always \((\alpha, \beta)\)-metrics even if \(F\) is an \((\alpha, \beta)\)-metric.

Let \(\bar{F} = \alpha + \gamma\) be a Randers metric, where \(\alpha\) is a Riemannian metric and \(\gamma\) is a 1-form on \(M\). Put
\[
\bar{F} = \frac{(F + \beta)^2}{F} = \frac{(\alpha + \gamma + \beta)^2}{\alpha + \beta} = \frac{\alpha(1 + s + \bar{s})^2}{1 + s},
\]
where \(s = \frac{\beta}{\alpha}\) and \(\bar{s} = \frac{\gamma}{\alpha} \neq s\). Here \(\bar{F} = \alpha\Psi(s, \bar{s})\) is a Finsler metric but not \((\alpha, \beta)\)-metric. Whereas, for any 1-form \(\beta\) on \(M\), \(\bar{F} = F + \beta = \alpha + \beta + \gamma\) is a Randers metric.

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Let $F$ be a projectively flat Finsler metric such as the generalized Berwald’s metric
\[
F = \frac{(1 + (a, x))(1 - |x|^2)|y|^2 + (x, y)^2 + (x, y)) + (1 - |x|^2)(a, y))^2}{(1 - |x|^2)^2(1 - |x|^2)|y|^2 + (x, y)^2},
\]
where $a \in \mathbb{R}^n$ is a constant vector. It is locally projectively flat with constant flag curvature $K = 0$ \[11\]. For any closed 1-form $\beta$ such that $\beta = \frac{(a, y)}{1 + (a, x)}$, metric $\bar{F} = F + \beta$ is also a projectively flat Finsler metric and $\beta$ is closed form (see Theorem 1.1). One could consider the above metrics as a change of a given Finsler metric. Various Finsler changes have been extensively studied and they have numerous applications.

In 1971, Matsumoto introduced the transformation of Finsler metric $\bar{F}(x, y) = F(x, y) + \beta(x, y)$, where $\beta(x, y) = b_i(x)y^i$ is a 1-form \[9\]. In 1984, Shibata \[12\] introduced the transformation of Finsler metric $\bar{F}(x, y) = f(F, \beta)$, where $\beta(y) = b_i(x)y^i$, $b_i(x)$ are components of a covariant vector in $(\mathbb{M}^m, F)$ and $f$ is positively homogenous function of degree 1 in $F$ and $\beta$. This change of metric is called a $\beta$-change.

In 1980, while studying the conformal transformation of Finsler spaces, H. Izumi \[8\] introduced the concept of an $h$-vector $b_i$. The $h$-vector $b_i$, as well as the function of coordinates $x^i$ itself, are also dependent on $y^i$. The $h$-vector $b_i$ is $\nu$-covariant constant with respect to the Cartan connection and satisfies $FC_i^b b_i = \rho h_{ij}$, where $\rho$ is a non-zero scalar function, $C_i^b$ are components of Cartan tensor and $h_{ij}$ are components of angular metric tensor. We will prove the following theorem.

**Theorem 1.1.** An $(F, \beta)$-metric $\bar{F} = F(\phi(s))$, where $s = \frac{\beta}{\phi}$, $\beta(x, y) = b_i(x)y^i$ with $h$-vector $b_i$, is projectively flat if and only if
\[
2(\phi - s\phi' + \rho\phi')h_i^\gamma G_{\gamma}^r + 2F\phi's_{i0} + \phi\thetaj_i = 0, \tag{2}
\]
where $r_{ij} = \frac{1}{2}(b_{ij} + b_{ji})$, $s_{ij} := \frac{1}{2}(b_{ij} - b_{ji})$, $s_{i0} = s_{ij}y^j$, $\Theta = (\phi - s\phi' + \rho\phi')r_{00} - 2F\phi's_{00}$, $\lambda = \phi - s\phi' + \rho\phi' + (2s^2 - s^3)\phi''$, $h_{ij} = g_{ij} - \ell_i\ell_j$ and $m_i = b_i - \ell_i$.

One could easily show that the above theorem is satisfied for every $(F, \beta)$-metric with $\beta(y) = b_i(x)y^i$ just by putting $\rho = 0$, with $\beta$ not being necessarily an $h$-vector.

In this paper, we study the $(F, \beta)$-metric with $F$ being an $m$-root Finsler metric. Let $(\mathbb{M}, F)$ be a Finsler manifold of dimension $n$, $TM$ its tangent bundle and $(x^i, y^i)$ the coordinates in a local chart on $TM$. Let $F$ be a scalar function on $TM$ defined by $F = \sqrt{A}$, where $A$ is given by $A := a_{i_1 \ldots i_m}(x)y^{i_1}y^{i_2} \ldots y^{i_m}$ with $a_{i_1 \ldots i_m}$ symmetric in all its indices. Then $F$ is called an $m$-root Finsler metric.

Theorem 1.1 includes all known results about projective changes of Finsler metrics \[10, 13, 14\]. For instance, we get the following two corollaries which were stated as theorems in the respected papers.

**Corollary 1.2** \([2, 10]\). Let $F = \sqrt{A}$ ($m > 3$) be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Then Randers change $\bar{F} = F + \beta$ with $\beta = b_i(x)y^i$ is locally projectively flat if and only if it is locally Minkowski.
Corollary 1.3 ([13]). Let $F = \sqrt{A}$ $(m > 3)$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Then Matsumoto change $\bar{F} = \frac{F^2}{f - \beta}$ with $\beta = b_i(x)y^i$ is locally projectively flat if and only if $\frac{\partial A}{\partial x^i} = 0$ and $b_i = \text{constant}$.

Finally, one could easily conclude that the following also holds.

Corollary 1.4. Let $F = \sqrt{A}$ $(m > 3)$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Then $(F, \beta)$-metric $F = F\phi(\frac{x}{F})$ with $\beta = b_i(x)y^i$ is locally projectively flat if and only if

$$-m(m - 1)\lambda(\phi - s\phi')y_iA_0A^{1 - \frac{m}{2}} + m\lambda(\phi - s\phi')(A_{0i} - A_{\lambda i})A^{1 - \frac{m}{2}} + 2\phi'(\lambda s_{i0} - \phi''s_0m_1)A^m + \phi''(\phi - s\phi')r_{00}m_i = 0,$$

where $\lambda = \phi - s\phi' + (b^2 - s^2)\phi''$, $r_{00}$, $s_0$ and $s_0$ are represented as (20), $A_{0i}$, $A_{x^i}$ and $A_0$ are defined in (46).

A Finsler metric is called Douglas metric if the Douglas tensor $D = 0$. The Douglas curvature was introduced by J. Douglas in 1927 [4]. In the same paper it was proven that Douglas and Weyl tensors are invariant under projective changes. Roughly speaking, a Douglas metric is a Finsler metric having the same geodesics as a Riemannian metric. Hence, in this paper we are going to obtain the conditions under which the change $\bar{F} = F\phi(s)$ of Douglas space becomes a Douglas space. Then we will prove the following.

Theorem 1.5. Let $(M, F)$ be a Douglas space. An $(F, \beta)$-metric $F = F\phi(\frac{x}{F})$ with $h$-vector $b_i$ is Douglas if and only if

$$H^{ij} := \frac{F\phi'}{\phi - s\phi' + \rho\phi'}(s_i^0y^j - s_i^jy^0) + \frac{\phi''[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi's_0]}{2(\phi - s\phi' + \rho\phi')\lambda}(b^jy^i - b^iy^j),$$

where $r_{ij}$, $s_i^0$ and $s_0$ are represented as (20), are homogeneous polynomial in $y^i$ of degree 3.

By the above theorem one could obtain many Douglas metrics from a given one. For example, the following corollary introduces some new Douglas metrics from a given $m$-root Finsler metric of Douglas type.

Corollary 1.6. (i) Let $F = \sqrt{A}$ $(m > 3)$ be an $m$-root Finsler metric of Douglas type. Then Randers change $\bar{F} = F + \beta$ with $\beta = b_i(x)y^i$ is of Douglas type if and only if $s_{ij} = 0$.

(ii) Let $F = \sqrt{A}$ $(m > 3)$ be an $m$-root Finsler metric of Douglas type. Then Matsumoto change $\bar{F} = \frac{F^2}{f - \beta}$ with $\beta = b_i(x)y^i$ is of Douglas type if and only if $b_{ij} = 0$.

2. Preliminaries

Let $M$ be a smooth manifold and $TM := \bigcup_{x \in M} T_xM$ be the tangent bundle of $M$, where $T_xM$ is the tangent space at $x \in M$. A Finsler metric on $M$ is a function
F : TM \to [0, +\infty) \text{ with the following properties}

(i) F is $C^\infty$ on $TM \setminus \{0\}$;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;

(iii) for each $x \in M$, the following quadratic form $g_y$ on $T_x M$ is positive definite, $g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[ F^2(y + su + tv) \right]_{|s,t=0}$, $u, v \in T_x M$.

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by $C_y(u, v, w) := \frac{1}{2} \frac{\partial}{\partial s} \left[ g_y + tv(u, v) \right]_{|s=t=0}$, $u, v, w \in T_x M$.

The family $C := \{C_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if F is Riemannian.

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^j)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G_i(x, y) \frac{\partial}{\partial y^i}$, where $G^i(x, y)$ are local functions on $TM_0$ given by $G^i = \frac{1}{2} g^{ij} \left( \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) y^j y^k$. $G$ is called the associated spray to $(M, F)$. The projection of an integral curve of the spray $G$ is called a geodesic in $M$.

The Cartan connection in $M$ is given as $CT = (\Gamma^i_{jk}, N^i_j, C^i_{jk})$, where

\[
\Gamma^i_{jk} = \frac{1}{2} g^{ij} \left( \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^j} \right), \quad \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^m_i \frac{\partial}{\partial y^m}, \quad N^i_j := \partial_j G^i, \quad N^i_j y^j = 2G^i.
\]

Note that $\partial_i$ and $\partial_i$ denote the derivations with respect to $x^i$ and $y^i$ respectively.

For the Cartan connection, we define $X^i_{jk} := \frac{\delta X^i}{\delta x^k} + X^i_j \Gamma^j_{rk} - X^i_r \Gamma^j_{rk}$, $X^i_{jk} := \partial_k X^i_j + X^i_r C^r_{jk} - X^i_l C^l_{jk}$, where "$\circ\circ"$ and "$\circ\circ\circ$" denote the horizontal and vertical covariant derivative of $X^i_j$. Also, the axioms $g_{ij0} = 0$ and $g_{ijk0} = 0$ hold.

The h-vector $b_i$ is $\nu$-covariant constant with respect to the Cartan connection and satisfies $FC^i_{ij} b_i = \rho h_{ij}$, where $\rho$ is a non-zero scalar function, $C^i_{ij} = g^{nu} C_{ijn}$, and $h_{ij}$ are components of angular metric tensor. Thus if $b_i$ is an h-vector then (i) $b_{i0} = 0$, (ii) $FC^i_{ij} b_i = \rho h_{ij}$. Put $c^i = g^{ij} C^j_{ij}$. Hence we obtain

\[
\rho = \frac{F}{n-1} c^i b_i, \quad (4)
\]

\[
\partial_i b_i = \frac{\rho}{F} h_{ij}, \quad (5)
\]

Since $\rho \neq 0$ and $h_{ij} \neq 0$, the h-vector $b_i$ depends not only on positional coordinates but also on directional arguments. Izumi [8] showed that $\rho$ is independent of directional arguments and that if $b_i$ is an h-vector then $* b_i := b_i - \rho \ell_i$ and $b := ||b||_F$ are independent of y.

3. $(F, \beta)$-metrics

Throughout the paper we shall use the notations $\ell_i := \partial_i F$, $\ell_{ij} := \partial_i \partial_j F$, $\ell_{ijk} := \partial_i \partial_j \partial_k F$. Let $b_i$ be an h-vector in the Finsler space $(M, F)$. Since $h_{ij} y^j = 0$, we have $\partial_i \beta = b_i$. Contracting with $y^j$ will be denoted by the subscript 0. For example, we
write \( b_{ij0} \) for \( b_{ij}y^i \).

Using (5) and the fact that \( \ell_{ij} = \ell_{ij|k} = 0 \) we have the following relations
\[
\partial_j b_i = b_{ij} + \rho N_j^r \ell_{ir} + b_i \Gamma^r_{ij}, \tag{6}
\]
\[
\partial_j \ell_i = N_j^r \ell_{ir} + \ell_i \Gamma^r_{ij}, \tag{7}
\]
\[
\partial_k \ell_{ij} = N_k^r \ell_{ijr} + \ell_{ij} \Gamma^r_{ik} + \ell_{ir} \Gamma^r_{jk}. \tag{8}
\]
For \( s = \beta/F \), by (6) and the fact that \( \partial_k F = \ell_i N^r_k \) we have
\[
\dot{\ell}_s = \frac{1}{F} m_i, \quad \partial_s = \frac{1}{F} (b_{0i} + m_r N^r_i), \tag{9}
\]
where \( m_i := b_i - s \ell_i \). Using (6) and (7) we get
\[
\partial_k m_i = (\rho - s) \ell_{ik} - \frac{1}{F} m_{ki} \ell_i, \tag{10}
\]
\[
\partial_k m_i = b_{ij} + (\rho - s) \ell_{ir} N^r_k - \frac{1}{F} m_r N^r_k \ell_i + m_r \Gamma^r_{ik} - \frac{1}{F} b_{0i} \ell_i.
\]
Differentiating equation (1) with respect to \( y^i, y^j, y^k \) and using the first equations in (9) and (10) imply that
\[
\ell_i = \phi \ell_i + \phi' m_i, \tag{11}
\]
\[
\ell_{ij} = \left( \phi - s \phi' + \rho \phi'' \right) \ell_{ij} + \frac{\phi''}{F} m_i m_j, \tag{12}
\]
\[
\ell_{ijk} = \left[ \phi - s \phi' + \rho \phi'' \right] \ell_{ijk} + \frac{\phi''}{F} (\rho - s) [ m_{ij} \ell_{ik} + m_{ij} \ell_{ik} + m_i \ell_{jk} ] + \frac{\phi''}{F^2} m_i m_j m_k
\]
\[
- \frac{\phi''}{F^2} [ m_i m_j \ell_k + m_i m_k \ell_j + m_k m_i \ell_j ]. \tag{13}
\]

**Definition 3.1.** A Finsler metric \( \bar{F} \) is called \((F, \beta)\)-metric if it has the following form \( \bar{F} = F(\phi(s)), s := \frac{\beta}{F} \), where \( F \) is a Finsler metric and \( \beta = b_i y^i \) is a 1-form on an \( n \)-dimensional manifold \( M \), \( \phi(s) \) is a positive \( C^\infty \) function on \((-b_0, b_0)\) and \( \|\phi\|_F < b_0 \).

\((F, \beta)\)-metric \( \bar{F} \) is called \((F, \beta)\)-metric with \( h\)-vector if \( b_i := b_i(x, y) \) is an \( h\)-vector on \((M, F)\).

**Lemma 3.2.** For any Finsler metric \( F \) and 1-form \( \beta = b_i y^i \) with \( h\)-vector \( b_i \) on manifold \( M \) with \( \|\beta\|_F < b_0 \), \( \bar{F} = F(\phi(s)) \) is a Finsler metric if and only if the positive \( C^\infty \) function \( \phi = \phi(s) \) satisfies
\[
\phi(s) - s \phi'(s) + \rho \phi''(s) > 0, \quad \phi(s) - s \phi'(s) + \rho \phi''(s) + b^2 - s^2) \phi''(s) > 0, \tag{14}
\]
when \( n \geq 3 \) or \( \phi(s) - s \phi'(s) + \rho \phi''(s) + (b^2 - s^2) \phi''(s) > 0, \) when \( n = 2 \), where \( s = \frac{\beta}{F} \) and \( b \) is arbitrary numbers with \( |s| \leq b < b_0 \) and \( \rho \) is given by (4).

**Proof.** The case \( n = 2 \) is similar to \( n \geq 3 \), so we only prove the proposition for \( n \geq 3 \). It is easy to verify that \( \bar{F} \) is a function with regularity and positive homogeneity. In the following we will verify strong convexity. Direct computations yield the fundamental tensor \( g_{ij} = \frac{1}{2} \partial_i \partial_j F^2 \) as follows
\[
g_{ij} = \eta g_{ij} + \eta_0 b_i b_j + \eta_1 (\ell_i b_j + \ell_j b_i) + \eta_2 \ell_i \ell_j, \tag{15}
\]
Using [11, Lemma 1.1.1], we obtain
\[
\rho_{\bar{\ell}} \in \text{tensor} \bar{\varphi}
\]
where \((\cdot)\) denotes the horizontal covariant derivative with respect to the Cartan connection of \(F\). Moreover, we define \(r_{ij} := r_{i}y_{j}, r_{i} := b'r_{ij}, r_{0} := r_{j}y_{i}, r_{00} = r_{ij}y_{i}y_{j}, s_{i0} := s_{ij}y_{i}, s_{j} := b's_{ij}, s_{0} := g^{ij}s_{j0}. Then} \(\partial_{k}s_{ij} = \frac{1}{2}(\rho_{j}\ell_{ik} - \rho_{i}\ell_{jk}), \partial_{k}s_{0i} = \frac{1}{2}\rho_{0}\ell_{ik} + s_{ik}, \text{where \(\rho_{i} = \partial_{i}\rho \text{ and } \rho_{0} = b_{i}y^{k}.}
\]

4. Spray coefficients of \((F, \beta)\)-metrics

In this section we are going to calculate the spray coefficients of \((F, \beta)\)-metrics. First assume that \(\beta\) is a 1-form with \(h\)-vector.

Differentiating (11) with respect to \(x^{j}\) and using (7) and the second equations
in (9) and (10) yield
\[ \partial_i \ell_i = \phi \left[ \ell_i N_j^r + \ell_r \Gamma_{ij}^r \right] + \frac{\phi''}{F} \left[ b_{0ij} + m_r N_j^r \right] m_r + \phi' \left[ b_{0ij} + (\rho - s) \ell_i N_j^r + m_r \Gamma_{ij}^r \right]. \] (21)
Next, we deal with \( \ell_{ij} \), that is \( \partial_j \ell_i = \ell_i \phi_j + \ell_r \phi_{ij} \). Let us define
\[ D_{jk}^i := \Gamma_{jk}^i - \Gamma_j^i, \quad D_j^i := D_{jk}^i y^k = N_j^i - N_j^{i'}, \quad D^i := D_j^i y^j = 2G^i - 2G^i. \] (22)
Then \( \partial_j \ell_i = \ell_i (D_j^i + N_j^i) + \ell_r (D_{ij}^i + \Gamma_{ij}^r). \)
Contracting (24) and (25) by \( \phi \), we have
\[ 2 \phi' r_{ij} = \ell_i D_j^i + \ell_j D_i^i + 2 \ell_j D_i^j - \frac{\phi''}{F} \left[ m_r b_{0ij} + m_r b_{0ji} \right], \] (24)
\[ 2 \phi' s_{ij} = \ell_i D_j^i - \ell_j D_i^i - \frac{\phi''}{F} \left[ m_r b_{0ij} - m_r b_{0ji} \right]. \] (25)
Contracting (24) and (25) by \( y^i \) implies that
\[ 2 \phi' r_{i0} = \ell_i D^r + 2 \ell_r D_i^r - \frac{\phi''}{F} r_{00} m_i, \] (26)
\[ 2 \phi' s_{i0} = \ell_i D^r - \frac{\phi''}{F} r_{00} m_i. \] (27)
Subtracting (27) from (26) yields
\[ \phi' (r_{i0} - s_{i0}) = \ell_i D^r. \] (28)
Contracting (28) by \( y^i \) leads to
\[ \phi' r_{00} = \ell_r D^r. \] (29)
To obtain the spray coefficients of \( \bar{F} \), first we must prove the following lemma.

**Lemma 4.1.** The system of algebraic equations
\( (i) \) \( \ell_i A^r = B_i, \quad (ii) \) \( \ell_r A^r = B \),
has a unique solution \( A^r \) for given \( B \) and \( B_i \) such that \( B_i y^i = 0 \). The solution is given by
\[ A^i = \frac{F}{\phi - s \phi' + \rho \phi''} B^i \left[ B - \frac{F}{\lambda} \phi' B_i y^i \right] \ell_i - \frac{F \phi'' (B_i B_j')}{\lambda (\phi - s \phi' + \rho \phi'')} m_i, \]
where \( B^i = g^i B_i \) and \( m^i = g^i m_i \).

**Proof.** Contracting (12) by \( b^i \) implies that
\[ \ell_i b^i = \frac{\lambda}{F} m_r, \] (30)
where \( \lambda := \phi - s \phi' + \rho \phi'' + (b^2 - s^2) \phi'' \).
Then contracting equation (i) by \( b^i \) and using (30), we get the following
\[ \frac{\lambda}{F} m_r A^r = B_i b^i. \] (31)
Substituting (11) in equation (ii) yields \( \phi\ell_r A^r + \phi^s m_r A^s = B \). Putting (31) in this equation we get
\[
\ell_r A^r = \frac{1}{\phi} \left( B - \frac{F}{\lambda} \phi^i B_i B_r \right). \tag{32}
\]
Substituting (12) in equation (i) and using the fact that \( \ell_r = \frac{1}{\rho} (g_{ir} - \ell_i \ell_r) \), we get
\[
g_{ir} A^r = \frac{F}{\phi - s \phi' + \rho \phi'} B_i + (\ell_r A^r) \ell_i - \frac{\phi''}{\phi - s \phi' + \rho \phi'} (m_r A^r) m_i.
\]
Contracting this equation using \( g^{ij} \) and using (31) and (32) complete the proof. \( \square \)

Now, we are able to obtain the spray coefficients of \( \bar{F} \).

By contracting (27) by \( b^r \) and using the above relations, we get \( \frac{1}{\lambda} m_r D^r = 2 \rho' s_0 + \frac{\phi''}{\lambda} r_{00} (b^2 - s^2) \). The equations (27) and (29) constitute a system of algebraic equations in \( \ell_r, D^r \) and \( m_r, D^r \) whose solution from Lemma 4.1 is given by
\[
D^i = \frac{F}{\phi - s \phi' + \rho \phi'} B^i + \frac{1}{\phi} \left( B - \frac{F}{\lambda} \phi^i B_i b^r \right) \ell^i - \frac{F \phi''}{\lambda (\phi - s \phi' + \rho \phi')} B_r b^r m^i,
\]
where \( B^i = 2 \rho' s_0 + \frac{\phi''}{\lambda} r_{00} m^i \), \( B = \phi' r_{00} \), \( B_r b^r = 2 \rho' s_0 + \frac{\phi''}{\lambda} (b^2 - s^2) r_{00} \). Since \( D^i = 2 \bar{G}^i - G^i \), we get the following theorem.

**Theorem 4.2.** Let \( \bar{F} \) be an \((F, \beta)\)-metric with h-vector \( b_i \). Then the spray coefficients of \( \bar{F} \) are given by
\[
2 \bar{G}^i = 2 G^i + \frac{2 F \phi'}{\phi - s \phi' + \rho \phi'} s_0 + \left[ \frac{\phi' (\phi - s \phi' + \rho \phi')'}{(\phi - s \phi' + \rho \phi')^2} \right] \left( \phi' r_{00} - 2 F' \phi' s_0 \right) \ell^i
+ \frac{\phi'' \left( (\phi - s \phi' + \rho \phi') r_{00} - 2 F' \phi' s_0 \right)}{(\phi - s \phi' + \rho \phi') \lambda} b^i.
\tag{33}
\]

**Corollary 4.3.** Let \( \bar{F} \) be an \((F, \beta)\)-metric. Then the spray coefficients of \( \bar{F} \) are given by
\[
2 \bar{G}^i = 2 G^i + \frac{2 F \phi'}{\phi - s \phi' + \rho \phi'} s_0 + \left[ \frac{\phi' (\phi - s \phi' + \rho \phi')'}{(\phi - s \phi' + \rho \phi')^2} \right] \left( \phi' r_{00} - 2 F' \phi' s_0 \right) \ell^i
+ \frac{\phi'' \left( (\phi - s \phi' + \rho \phi') r_{00} - 2 F' \phi' s_0 \right)}{(\phi - s \phi' + \rho \phi') \lambda} b^i.
\tag{34}
\]

### 5. Cartan connection of \((F, \beta)\)-metrics

Here the Cartan connection coefficients of \((F, \beta)\)-metrics are calculated. Differentiating (12) with respect to \( x^k \) and using (9) and (10), we get
\[
\partial_k \bar{\ell}_{ij} = [\phi - s \phi' + \rho \phi'] \partial_k \ell_{ij} + \frac{\phi''}{F} (p - s) [b_{0i} + m_r N^r_k] \ell_{ij} + \phi' p_k \ell_{ij} + \frac{\phi''}{F^2} [b_{0ij} + m_r N^r_{ik}] m_i m_j
- \frac{\phi''}{F^2} m_i m_j \partial_k F + \frac{\phi''}{F} m_j [b_{0ij} + (p - s) \ell_{ir} N^r_k - \frac{1}{F} m_r N^r_i \ell_i + m_r \Gamma^r_{ik} - \frac{1}{F} b_{0ij} \ell_i].
\]
+ \frac{\phi''}{F} m_i \{ b_{ijk} + (\rho - s) \ell_{kj} N_j^r - \frac{1}{F} m_r N_j^r \ell_j + m_r \Gamma_j^r - \frac{1}{F} b_{0ijk} \ell_j \}. \quad (35)

With the help of \( \ell_{ijk} = 0 \), that is \( \partial_{k} \ell_{ij} = \bar{\ell}_{ij}, \bar{N}_k^r + \bar{\ell}_{ij} \Gamma^r_{ik} + \bar{\ell}_{ij} \Gamma^r_{jk} \), and by (22) we have \( \partial_{k} \bar{\ell}_{ij} = \bar{\ell}_{ij} (D^r_k + N_k^r) + \bar{\ell}_{ij} (D^r_k + \Gamma^r_{ik}) + \bar{\ell}_{ir} (D^r_{jk} + \Gamma^r_{jk}) \). Putting the values of \( \bar{\ell}_{ir} \), \( \bar{\ell}_{rj} \) and \( \bar{\ell}_{ijr} \) from (12) and (13) in the above equation yields

\[
\partial_{k} \bar{\ell}_{ij} = \bar{\ell}_{ijr} D^r_k + \bar{\ell}_{ij} D^r_k + \bar{\ell}_{ijr} D^r_{jk} + \left\{ [\rho - s \phi' + \rho \phi'] \ell_{ijr} + \frac{\phi''}{F} (\rho - s) \right\} \left\{ m_r \ell_{ij} + m_j \ell_{ir} + m_j \ell_{ijr} \right\} + \frac{\phi''}{F^2} m_i m_j m_r - \frac{\phi''}{F^2} [m_i m_j \ell_r + m_i m_r \ell_j + m_j m_r \ell_i] \bar{N}_k^r + \frac{\phi''}{F^2} \left\{ [\rho - s \phi' + \rho \phi'] \ell_{ijr} + \frac{\phi''}{F} m_r \right\}. \quad (36)
\]

By comparing (35) and (36) and using (8) and the fact that \( \partial_{k} F = \ell_r N^r_k \) we get the following

\[
\ell_{ijr} D^r_k + \bar{\ell}_{ij} D^r_k + \bar{\ell}_{ijr} D^r_{jk} = \phi' \rho_k \ell_{ij} + \frac{\phi''}{F} (\rho - s) b_{0ijk} \ell_{ij} + \frac{\phi''}{F} \left\{ m_j b_{i0} + m_i b_{j0} \right\} - \frac{\phi''}{F^2} b_{0ijk} [m_i \ell_j + m_j \ell_i] + \frac{\phi''}{F^2} b_{0ijk} m_i m_j. \quad (37)
\]

Contracting (37) by \( g^k \) yields

\[
\ell_{ijr} D^r + \bar{\ell}_{ij} D^r + \bar{\ell}_{ijr} D^r_{jk} = \phi' \rho_k \ell_{ij} + \frac{\phi''}{F} (\rho - s) r_{0ij} \ell_{ij} + \frac{\phi''}{F} \left\{ m_j b_{i0} + m_i b_{j0} \right\} - \frac{\phi''}{F^2} r_{0ij} [m_i \ell_j + m_j \ell_i] + \frac{\phi''}{F^2} r_{0ij} m_i m_j. \quad (38)
\]

Substituting (25) in equation (38) implies that

\[
\ell_{ij} D^r_j = Q_{ij}, \quad (39)
\]

where

\[
Q_{ij} := - \frac{1}{2} \ell_{ij} D^r + \phi' s_{ij} + \frac{1}{2} \rho_0 \phi' \ell_{ij} + \frac{\phi''}{F} (m_i r_{j0} + m_j s_{i0}) + \frac{\phi''}{F^2} (\rho - s) r_{0ij} \ell_{ij} - \frac{\phi''}{F^2} r_{0ij} (m_i \ell_j + m_j \ell_i) + \frac{\phi''}{F^2} r_{0ij} m_i m_j.
\]

From (27), we see \( Q_{ij} y^i = 0 \). On the other hand, the equation (28) may be written as

\[
\ell_{ij} D^r_j = Q_{ij}, \quad (40)
\]

where \( Q_{ij} := \phi' (r_{j0} - s_{i0}) \). The equations (40) and (39) constitute the system of algebraic equations whose solution from Lemma 4.1 is given by

\[
D^i_j = \frac{F}{\phi - s \phi' + \rho \phi'} Q^i_j + \frac{1}{\phi} \left( Q_j - \frac{F}{\lambda \phi' Q_{rj} b^r} \right) \ell^i - \frac{F \phi''}{\lambda (\phi - s \phi' + \rho \phi')} Q_{rj} b^r m^i,
\]

here \( Q^i_j = g^{ir} Q_{rj} \). Then by (22) we have

\[
N_j^r = N_j^r + \frac{F}{\phi - s \phi' + \rho \phi'} Q_j^r + \frac{1}{\phi} \left( Q_j - \frac{F}{\lambda \phi' Q_{rj} b^r} \right) \ell^i - \frac{F \phi''}{\lambda (\phi - s \phi' + \rho \phi')} Q_{rj} b^r m^i. \quad (41)
\]
Finally, applying Christoffel process with respect to indices \( i, j, k \) in equation (37) we obtain

\[ \bar{\ell}_{ij} D^r_{ik} = M_{ijk}, \]  

where

\[ M_{ijk} := \frac{1}{2} \left[ \bar{\ell}_{ijr} D^r_k + \bar{\ell}_{jkr} D^r_i - \bar{\ell}_{ikr} D^r_j \right] + \frac{1}{2} \phi' \left[ \ell_{kij} + \rho_i \bar{\ell}_{jk} - \rho_j \bar{\ell}_{ik} \right] \]

\[ + \frac{\phi''}{F} \left[ m_j r_{ik} + m_i s_{jk} + m_k s_{ji} \right] + \frac{\phi''}{2F} (\rho - s) \left[ b_{0jk} \ell_{ij} + b_{00} \ell_{jk} - b_{0ij} \ell_{ik} \right] \]

\[ + \frac{\phi''}{2F^2} \left[ b_{0|k} m_i m_j + b_{0|j} m_k m_j - b_{0ij} m_i m_k \right] \]

\[ - \frac{\phi''}{2F^2} \left[ b_{0|k} (m_i \ell_j + m_j \ell_i) + b_{0|j} (m_j \ell_k + m_k \ell_j) - b_{0ij} (m_i \ell_k + m_k \ell_i) \right]. \]

Moreover, by (38) we get \( M_{ijk} b^i = 0 \). Besides, the equation (24) may be written as

\[ \bar{\ell}_{ij} D^r_{ik} = M_{ijk}, \]

where \( M_{ij} := \phi' r_{ij} - \frac{1}{2} \bar{\ell}_{ir} D^r_j - \frac{1}{2} \bar{\ell}_{jr} D^r_i + \frac{\phi''}{2F} [m_i b_{0|k} + m_k b_{0|i}] \). Applying Lemma 4.1 to equations (42) and (43) implies that

\[ D^r_{jk} = \frac{F}{\phi - s \phi' + \rho \phi'} M^i_{jk} + \frac{1}{\phi} (M_{jk} - \frac{F}{\lambda} \phi' M_{rjk} b^r) \bar{\ell}^i - \frac{F \phi''}{\lambda (\phi - s \phi' + \rho \phi')} M_{rjk} b^r m^i, \]

where \( M^i_{jk} = g^{ir} Q_{rjk} \). Then by (22) we get

\[ \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{F}{\phi - s \phi' + \rho \phi'} M^i_{jk} + \frac{1}{\phi} (M_{jk} - \frac{F}{\lambda} \phi' M_{rjk} b^r) \bar{\ell}^i - \frac{F \phi''}{\lambda (\phi - s \phi' + \rho \phi')} M_{rjk} b^r m^i. \]

**Theorem 5.1.** Let \( \bar{C}^i = (\bar{\Gamma}^i_{jk}, \bar{N}^i_{jk}, \bar{C}^i_{jk}) \) be the Cartan connection for the Finsler space \( (M, \bar{F}) \) where \( \bar{F} \) is an \( (F, \beta) \)-metric with \( h \)-vector \( b \). Then the Cartan connection is completely determined by the equations (19), (41) and (44).

### 6. Proof of Theorem 1.1

**Proof.** Suppose that \( F \) and \( \bar{F} \) be projectively related i.e. \( \bar{G}^i - G^i = Py^i \), where \( \bar{G}^i \) and \( G^i \) are the geodesic spray coefficients of \( \bar{F} \) and \( F \), respectively and \( P = P(x, y) \) is a scalar function on the slit tangent bundle \( TM \). By (22) we have \( D^i = 2Py^i \). Putting it in (33) we get

\[ 2Py^i = \frac{2F \phi'}{\phi - s \phi' + \rho \phi'} s^i + \frac{[\phi' (\phi - s \phi' + \rho \phi') - s \phi' \phi''] [\phi - s \phi' + \rho \phi'] r_{00} - 2F \phi' s_0]}{\phi (\phi - s \phi' + \rho \phi') \lambda} \bar{\ell}^i \]

\[ + \frac{\phi'' [\phi - s \phi' + \rho \phi'] r_{00} - 2F \phi' s_0]}{(\phi - s \phi' + \rho \phi') \lambda} b^i. \]
Contracting (45) by \( y_i := g_{ij} y^j \) and using the facts that \( s_i y_i = 0 \) and \( \ell^i y_i = F \), we obtain
\[
P = \frac{\phi'[(\phi - s \phi' + \rho \phi') r_{00} - 2F \phi' s_0]}{2F \lambda_0}.
\]
Now let \( \bar{F} \) be projectively flat; then one has
\[
2\bar{G}^i = 2G^i + D^i = 2 Py^i.
\]
Using the same calculations as above, by (33) one gets
\[
h_{ij} G^j + \frac{F \phi'}{\phi - s \phi' + \rho \phi'} s_{0 i} + \frac{\phi''[(\phi - s \phi' + \rho \phi') r_{00} - 2F \phi' s_0]}{2(\phi - s \phi' + \rho \phi') \lambda} m_i = 0.
\]
Conversely, putting (2) in (33) yields that
\[
G^i = G^i + \left( \frac{F \phi'}{\phi - s \phi' + \rho \phi'} s_{0 i} + \frac{\phi''[(\phi - s \phi' + \rho \phi') r_{00} - 2F \phi' s_0]}{2(\phi - s \phi' + \rho \phi') \lambda} m_i \right) g^i
\]
\[
= G^i - h_{ij} g^{ri} G^j + \frac{\phi''[(\phi - s \phi' + \rho \phi') r_{00} - 2F \phi' s_0]}{2\phi \lambda} \ell^i
\]
\[
= \left( \ell_i G^j + \frac{\phi''[(\phi - s \phi' + \rho \phi') r_{00} - 2F \phi' s_0]}{2\phi \lambda} \right) \ell^i = \dot{P} y^i.
\]
This completes the proof. \( \square \)

### 6.1 Proof of Corollary 1.2

Note that for m-root Finsler metrics we have [16]:
\[
A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i, \quad A_{0l} = A_{x^i} y^i y^r,
\]
and
\[
2G^i = A^{ri}(A_{0r} - A_{x^r}). \quad \text{Also, it is not hard to get } A_i = mA^{1-\frac{3}{m}} y_i, \quad \text{and } A_i' = (mA^{1-\frac{3}{m}} y^r)_{,i} = mA^{1-\frac{3}{m}} \left( \delta^r_i + (m - 2) \ell_i r^r \right).
\]
Then after some calculations we have
\[
2h_{ij} G^j = mA^{1-\frac{3}{m}} \left( A_{0i} - A_{x^i} - (m - 1) A_{0l} A^{1-\frac{3}{m}} \ell_i \right).
\]
Putting the above equations in (2) yields that
\[
-m(m - 1) A_{0i} y_i A^{1-\frac{3}{m}} + m(A_{0i} - A_{x^i}) A^{1-\frac{3}{m}} + 2s_{0i} A^{1-\frac{3}{m}} = 0.
\]
By the following lemma, the above equation yields \( A_{x^i} = 0 \) and \( s_{ij} = 0 \) for \( m \neq 5 \).

For \( m = 5 \) we get the same conclusion just by separating rational and irrational parts of equation.

**Lemma 6.1.** Let \( F = \sqrt[3]{A} \) (\( m > 2, \ m \neq 5 \)), be an m-th root Finsler metric on an open subset \( U \subset \mathbb{R}^n \). Suppose that the equation \( \Psi A^{1-\frac{3}{m}} + \Omega A^{1-\frac{3}{m}} + \Theta A^{\frac{3}{m}} = 0 \) holds, where \( \Psi, \Omega \) and \( \Theta \) are homogeneous polynomials in \( y \). Then \( \Psi = \Omega = \Theta = 0 \).

Corollaries 1.3 and 1.4 are proven in a similar manner.
7. \((F, \beta)\)-metrics of Douglas type

In [4], Douglas introduced the local functions \(D^i_{jkl}\) on \(TM_0\) defined by

\[
D^i_{jkl} := \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^k} \frac{\partial}{\partial y^l} \left( G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right).
\]

It is easy to verify that \(D := D^i_{jkl} dx^j \otimes \frac{\partial}{\partial x^k} \otimes dx^l\) is a well-defined tensor on \(TM_0\). \(D\) is called the Douglas tensor. The Finsler space \((M, F)\) is called a Douglas space if and only if \(G^i y^j - G^j y^i\) is a homogeneous polynomial of degree three in \(y^i\) [1].

7.1 Proof of Theorem 1.5

By (33) we get \(\bar{G}^i y^j - \bar{G}^j y^i = G^i y^j - G^j y^i + H^{ij}\), where

\[
H^{ij} := \frac{F \phi'}{\phi - s \phi' + \rho \phi''} (s_0^i y^j - s_0^j y^i) + \frac{\phi''}{2(\phi - s \phi' + \rho \phi'')} \lambda (b^i y^j - b^j y^i).
\]

With the help of the above definition, if \(F\) and \(\bar{F}\) are Douglas metrics then \(H^{ij}\) must be a homogeneous polynomial of degree three in \(y^j\).

By this theorem one could obtain many new Douglas metrics from a given one.

7.2 Proof of Corollary 1.6

(i) Putting \(F = \sqrt{\bar{A}}\) and \(\phi(s) = 1 + s\) in (3) yields \(2H^{ij} - (s_0^i y^j - s_0^j y^i) \sqrt{\bar{A}} = 0\). Then by separating rational and irrational parts of the above equation one gets \(s_0^i y^j = s_0^j y^i\) and thus \(s_{ij} = 0\).

(ii) Here \(F = \sqrt{\bar{A}}\) and \(\phi(s) = \frac{1}{2} s^2\); then one has \(\phi'(s) = \frac{1}{(1-s)^2}, \phi''(s) = \frac{2}{(1-s)^3}\), \(\lambda = \frac{1+2s-3s^2}{(1-s)^3}\). Putting them in (3) yields

\[
(1-2s)(1+2b^2-3s)H^{ij} - (1+2b^2-3s) \sqrt{\bar{A}} (s_0^i y^j - s_0^j y^i) - \left( (1-2s) r_{00} - 2s_0 \sqrt{\bar{A}} \right) (b^i y^j - b^j y^i) = 0.
\]

Multiplying above equation by \(A^{\frac{1}{2}}\) yields

\[
6b^2 H^{ij} + \beta \left[ 2r_{00} (b^i y^j - b^j y^i) - (4b^2 + 5) H^{ij} \right] A^{\frac{1}{2}} + \left[ (1+2b^2) H^{ij} + 3\beta (s_0^i y^j - s_0^j y^i) - r_{00} (b^i y^j - b^j y^i) \right] A^{\frac{1}{2}} + 2s_0 (b^i y^j - b^j y^i) - (1+2b^2) (s_0^i y^j - s_0^j y^i) \right] A^{\frac{1}{2}} = 0.
\]

Similar to Lemma 6.1 \((m > 3)\), one could easily get \(H_{ij} = 0, r_{00} = 0\) and \(s_{ij} = 0\), which yields \(b_{0ij} = 0\).

REFERENCES


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