MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
72, 3 (2020), 232–242
September 2020

research paper
оригинални научни рад

COMMON FIXED POINTS OF GENERALIZED CONTRACTIVE MAPPINGS IN UNIFORM SPACES

Boshra Hosseini and Alireza Kamel Mirmostafaee

Abstract. In order to establish some common fixed point theorems on Hausdorff uniform spaces endowed with a graph we will define a new kind of generalized contraction for self-mappings. A few related examples are also provided to support our main results. Finally an application of our results in b-metric spaces is exhibited.

1. Introduction

Following [6], a pair (X, υ) is called a uniform space, if X is a nonempty set and υ is a special kind of filter on $X \times X$ satisfying the following conditions:
(υ_1) for each $U \in \upsilon$, $\Delta = \{(x, x) : x \in X\} \subseteq U$,

(υ_2) $U \in \upsilon$ and $U \subseteq W \subseteq X \times X$ implies $W \in \upsilon$,

(υ_3) $U \in \upsilon$ and $W \in \upsilon$ implies $U \cap W \in \upsilon$,

(υ_4) $U \in \upsilon$ implies $U^{-1} \in \upsilon$,

(υ_5) if $U \in \upsilon$, then there exists $V \in \upsilon$ with $V \circ V \subseteq U$. (The composition of two subsets V and U of $X \times X$ is defined by $V \circ U = \{(x, z) : \exists y \in X : (x, y) \in V, (y, z) \in U\}$).
A uniform space (X, υ) is said to be Hausdorff if the intersection of all members of υ reduces to the diagonal Δ of X. This guarantees the uniqueness of limits of sequences.

Knill [10] was the first who extended the notion of contractive mapping in uniform spaces. Later, a few mathematicians studied various types of fixed point theorems in non-metrizable spaces (e.g. [1–4, 7, 12, 14, 16, 17]). Aamri and El Moutawakil [1] introduced the concept of an A-distance and an E-distance to prove some common fixed point theorems for contractive and expansive maps in uniform spaces. In 2004, Ran and Reurings [13] obtained a generalization of Banach's fixed point theorem for continuous self-mappings on a complete metric space endowed with a partial ordering. Jachymski [9] noted that every partially ordered metric space $(X, d, \preceq)$ can be

2010 Mathematics Subject Classification: 47H10, 54E15, 05C40, 54H25
Keywords and phrases: Fixed points; uniform spaces; graph; connectivity.

considered as a special case of a metric space (X,d) endowed with a directed graph G, where $V(G)=X$ and $E(G)=\{(x,y)\in X\times X : x\preceq y\}$. This observation, motivated a few mathematicians to extend and unify some fixed point theorems in metric spaces endowed with a graph (e.g. [5, 8, 11, 15]).

The aim of this paper is to obtain common fixed point theorems for two self-mappings on a Hausdorff uniform space endowed with a graph when the space is equipped with an A-distance. More precisely, we obtain a general result for existence and uniqueness of common fixed points for two generalized contractive self-mappings. Our main results generalize [1, Theorem 3.1] and lead to some applications in b-metric spaces.

2. Preliminaries

In this section we introduce the concepts that we will use in the rest of the paper. We start with the following definition.

Definition 2.1 ([1]). Let (X,υ) be a uniform space. A function $\rho : X\times X\to\mathbb{R}^{\geq 0}$ is called an A-distance, if for any $U\in\upsilon$ there exists $\delta>0$ such that if $\rho(z,x)\leq\delta$ and $\rho(z,y)\leq\delta$ for some $z\in X$, then $(x,y)\in U$. If ρ also satisfies $\rho(x,y)\leq\rho(x,z)+\rho(z,y)$ for each $x,y,z\in X$, then ρ is called an E-distance.

Example 2.2. Let (X,d) be a metric space, then the metric d is an E-distance for the uniformity generated by the metric.

Example 2.3. Consider $X=[0,+\infty)$ with the uniformity generated by the Euclidean metric. Then $\rho(x,y)=\max\{x,y\}$ is an E-distance defined on X.

The following examples show that there are A-distances which are not E-distances.

Example 2.4. Let X be a nonempty set and $d : X\times X\to\mathbb{R}^{\geq 0}$ be such that
(i) $d(x,y)=d(y,x)$, (ii) $d(x,y)<\varepsilon$ and $d(y,z)<\varepsilon$ implies that $d(x,z)<2\varepsilon$.
Define $\upsilon=\{V_\varepsilon : \varepsilon>0\}$ in which $V_\varepsilon=\{(x,y)\in X^2 : d(x,y)<\varepsilon\}$. Then υ defines a uniformity on X and d is an A-distance on (X,υ). For example if $X=\{a,b,c\}$ and $d : X\times X\to\mathbb{R}^{\geq 0}$ is a symmetric function which is defined by $d(a,b)=3$, $d(b,c)=2$, $d(a,c)=6$, $d(a,a)=d(b,b)=d(c,c)=0$, then it is easy to verify that conditions (i) and (ii) hold, (X,υ_d) is a uniformity and d is an A-distance on X. Note that $d(a,c)\nleq d(a,b)+d(b,c)$. Therefore d is not an E-distance.

Example 2.5. Let X be a nonempty set and $d : X\times X\to[0,\infty)$ for some $s>1$ satisfies the following properties:
(i) $d(x,y)=0$ iff $x=y$, (ii) $d(x,y)=d(y,x)$, (iii) $d(x,z)\leq s[d(x,y)+d(y,z)]$ for all $x,y,z\in X$. Then (X,d) is called a b-metric space.

We may consider (X,d) as a Hausdorff uniform space with the uniformity υ_d generated by $U_\varepsilon=\{(x,y) : d(x,y)<\varepsilon\}$ for $\varepsilon>0$. Let $U\in\upsilon_d$, then there is $\varepsilon>0$ such that $U_\varepsilon\subseteq U$. Let $\delta=\frac{\varepsilon}{2s}$, then $d(z,x)<\delta$ and $d(z,y)<\delta$ imply that $d(x,y)\leq$

$s(d(x,z)+d(z,y)) < 2s\delta = \varepsilon$. Hence $(x,y) \in U_\varepsilon$ if $d(z,x) < \delta$ and $d(z,y) < \delta$. This means that d ia an A-distance. However, the triangle inequality is not true. Therefore d is not an E-distance.

We also need the following notions.

DEFINITION 2.6 ([1]). Let (X,υ) be a uniform space endowed with an A-distance ρ.
(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called ρ-Cauchy if $\lim_{n,m\to\infty} \rho(x_n,x_m) = 0$. Two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are said to be ρ-Cauchy equivalent if each of them is ρ-Cauchy and $\lim\limits_{n\to\infty} \rho(x_n,y_n) = 0$.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be ρ-convergent to a point $x \in X$, if $\lim\limits_{n\to\infty} \rho(x_n,x) = 0$.

(iii) X is called S-complete if every ρ-Cauchy sequence in X is ρ-convergent.

(iv) $f : X \to X$ is called ρ-continuous if $\lim\limits_{n\to\infty} \rho(x_n,x) = 0$ implies $\lim\limits_{n\to\infty} \rho(fx_n,fx) = 0$.

(v) For $A \subseteq X$ define $\operatorname{diam}(A) = \sup\{\rho(x,y) : x,y \in A\}$. A is said to be ρ-bounded if $\operatorname{diam}(A) < \infty$.

The following lemma implies uniqueness of limit of ρ-convergent sequences in Hausdorff uniform spaces.

LEMMA 2.7 ([14]). *Let (X,υ) be a Hausdorff uniform space and ρ be an A-distance on X. Let $\{x_n\}$ be an arbitrary sequence in X. Then for each $x,y,z \in X$, the following conditions hold.*
(a) If $\lim\limits_{n\to\infty} \rho(x_n,y) = 0$ and $\lim\limits_{n\to\infty} \rho(x_n,z) = 0$ then $y = z$. Especially if $\rho(x,y) = 0$ and $\rho(x,z) = 0$, then y=z.

(b) If $\lim\limits_{n,m\to\infty} \rho(x_n,x_m) = 0$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X,υ).

Let (X,υ) be a uniform space and G be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G by the pair $(V(G),E(G))$. If G is such a graph, we say that X is endowed with the graph G.

By G^{-1} we denote the conversion of a graph G. That is $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}$. The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Under this convention $E(\widetilde{G}) = E(G) \cup E(G^{-1})$.

A graph G is called connected if there is a path between any two vertices of it. G is weakly connected if $\widetilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule: *yRx if and only if there is a path in G from x to y.* Clearly G_x is connected.

3. Results

We denote by Ψ the set of all functions $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, which are non-decreasing, $\psi(0) = 0$, $\psi(r) > 0$ for each $r > 0$ and $\lim_{n\to\infty} \psi^n(r) = 0$. It follows from the definition that $\psi(r) < r$ for all $\psi \in \Psi$ and $r > 0$.

In this section, we obtain some results on existence of common fixed points for two generalized contractive mappings in uniform spaces endowed with an A-distance ρ, which may not satisfy the triangle's inequality. In order to achieve this goal, we need to the following definition.

Definition 3.1. Let (X, υ) be a Hausdorff uniform space endowed with a graph G and A-distance ρ, $\psi \in \Psi$ and $f, g : X \to X$. We say that f is a (ρ, ψ, G)-contraction with respect to g if the following statements hold:
(i) For each $x \in X$ there exists $y \in [x]_{\widetilde{G}}$ such that $fx = gy$.
(ii) f and g are G-invariant, i.e., $(x, y) \in E(G)$ implies that $(fx, fy), (gx, gy) \in E(G)$.
(iii) If $x \in X$ and $y \in [x]_{\widetilde{G}}$, then $\rho(fx, fy) \leq \psi(\rho(gx, gy))$.

Example 3.2. Let (X, d) be a b-metric space and let $f : X \to X$ be a mapping such that for some $0 \leq \alpha < 1$ satisfies $d(fx, fy) \leq \alpha d(x, y)$ for all $x, y \in X$. Define graph G_0 with $V(G_0) = X$ and $E(G_0) = X \times X$ and define function $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ by $\psi(r) = \alpha r$ for each $r \in \mathbb{R}^{\geq 0}$. Then f is a (d, ψ, G_0)-contraction with respect to $g = I$, where I is a identity mapping on X.

Example 3.3. Let $X = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{\frac{-1}{2^n} : n \in \mathbb{N}\} \cup \mathbb{Z} \setminus \{0\}$. For each $x, y \in X$ define $\rho(x, y) = |x - y|^2$. Then (X, ρ) satisfies conditions (i)–(iii) in Example 2.5 for $s = 2$, so ρ is a b-metric on X. Thus ρ defines a Hausdorff uniformity υ_ρ on X. By Example 2.5, ρ is an A-distance on (X, υ_ρ). Define graph G by $V(G) = X$ and

$$E(G) = \Delta(X) \cup \Big\{(n+1, n) : n \in \mathbb{N}\Big\} \cup \Big\{(-n-1, -n) : n \in \mathbb{N}\Big\} \cup \Big\{\Big(\frac{1}{2^n}, \frac{1}{2^{n+1}}\Big) : n \in \mathbb{N}\Big\} \cup \Big\{\Big(\frac{-1}{2^n}, \frac{-1}{2^{n+1}}\Big) : n \in \mathbb{N}\Big\} \cup \Big\{(x, -x) : x \in X\Big\} \cup \Big\{\Big(-1, -\frac{1}{2}\Big), \Big(1, \frac{1}{2}\Big)\Big\}.$$

Then G is weakly connected. Let $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be defined by $\psi(r) = \frac{r}{3}$ which belongs to Ψ and let $f, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{1}{2^{n+1}} & \text{if } x = n \text{ for some } n \in \mathbb{N} \\ \frac{-1}{2^{n+1}} & \text{if } x = -n \text{for some } n \in \mathbb{N} \\ \frac{1}{2^{n+2}} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ \frac{-1}{2^{n+2}} & \text{if } x = \frac{-1}{2^n} \text{ for some } n \in \mathbb{N} \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{1}{2^{n}} & \text{if } x = n \text{ for some } n \in \mathbb{N} \\ \frac{-1}{2^{n}} & \text{if } x = -n \text{for some } n \in \mathbb{N} \\ \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ \frac{-1}{2^{n+1}} & \text{if } x = \frac{-1}{2^n} \text{ for some } n \in \mathbb{N}. \end{cases}$$

We show that f is a (ρ, ψ, G)-contraction with respect to g.
(i) G is weakly connected and

$$f(X) = \Big\{\pm\frac{1}{4}, \pm\frac{1}{8}, \pm\frac{1}{16}, \ldots, \pm\frac{1}{2^n}, \ldots\Big\} \subseteq g(X) = \Big\{\pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm\frac{1}{16}, \ldots, \pm\frac{1}{2^n}, \ldots\Big\}.$$

Thus for each $x \in X$ there exists $y \in [x]_{\widetilde{G}} = X$ such that $fx = gy$.

(ii) For each $(x, y) \in E(G)$ we have $(fx, fy), (gx, gy) \in E(G)$.

(iii) For each $x \in X$ and $y \in [x]_{\widetilde{G}} = X$, $\rho(fxfy) \leq \psi(\rho(gx, gy))$.
Note that (X, υ_ρ) is not S-complete. Since $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$ is a ρ-Cauchy sequence in X and there is no element in X to which $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$ converges.

The following lemma is direct consequence of Definition 3.1.

Lemma 3.4. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and A-distance ρ. Assume that $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ belongs to Ψ and $f, g : X \to X$. Suppose that f is a (ρ, ψ, G)-contraction with respect to g. Then f is also (ρ, ψ, G^{-1}) and $(\rho, \psi, \widetilde{G})$-contraction with respect to g.*

Remark 3.5. Let (X, υ) be a Hausdorff uniform space endowed with a graph G and an A-distance ρ and let $\psi \in \Psi$. Assume that $f, g : X \to X$ be such that f is a (ρ, ψ, G)-contraction with respect to g. Let $x_0 \in X$. Definition 3.1(i) implies that there exists $x_1 \in [x_0]_{\widetilde{G}}$ such that $fx_0 = gx_1$. Again there exists $x_2 \in [x_1]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$ such that $fx_1 = gx_2$. By continuing this procedure, we can obtain a sequence $\{fx_n\}$ such that for each $n \in \mathbb{N}$, $x_n \in [x_0]_{\widetilde{G}}$ and $fx_{n-1} = gx_n$.

In what follows, whenever $x_0 \in X$, $\{fx_n\}$ will be the sequence described above.

Definition 3.6. Let $f, g : X \to X$. The mapping f is called orbitally bounded with respect to g at $x_0 \in X$ if for every choice $x_n \in [x_0]_{\widetilde{G}}$ with $fx_{n-1} = gx_n$, the set $\mathrm{orb}(x_0, f, g) = \{x_0, fx_0, fx_1, \cdots\}$ is ρ-bounded. f is called orbitally bounded with respect to g if it is orbitally bounded with respect to g at each point of X.

Example 3.7. Let X, ρ, G, f and g be as was described in Example 3.3. Trivially X is not ρ-bounded. For each arbitrary element $x_0 \in X$ we have

$$\mathrm{diam}(\mathrm{orb}(x_0, f, g)) = \sup\{\rho(fx_i, fx_j), \rho(x_0, fx_i) : i, j \in \mathbb{N}\} \leq (x_0)^2.$$

Thus f is orbitally bounded with respect to g.

In order to state the main results of this section, we need some auxiliary results.

Lemma 3.8. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and A-distance ρ. Assume that $\psi \in \Psi$ and $f, g : X \to X$. Let f be a (ρ, ψ, G)-contraction with respect to g and let f be orbitally bounded with respect to g at $x_0, y_0 \in X$. If $[x_0]_{\widetilde{G}} = [y_0]_{\widetilde{G}}$, then the corresponding sequences $\{fx_n\}$ and $\{fy_n\}$, where $fx_{n-1} = gx_n$ and $fy_{n-1} = gy_n$ for all $n \in \mathbb{N}$, are ρ-Cauchy equivalent.*

Proof. Since for each $n \in \mathbb{N}$ we have $x_n \in [x_{n-1}]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$, it follows that

$$\begin{aligned}\rho(fx_n, fx_{n+m}) &\leq \psi(\rho(gx_n, gx_{n+m})) = \psi(\rho(fx_{n-1}, fx_{n+m-1})) \leq \psi^2(\rho(gx_{n-1}, gx_{n+m-1})) \\ &= \psi^2(\rho(fx_{n-2}, fx_{n+m-2})) \leq \ldots \leq \psi^n(\rho(fx_0, fx_m)) \leq \psi^n(\mathrm{diam}(\mathrm{orb}(x, f, g))),\end{aligned}$$

for all $n, m \in \mathbb{N}$. Hence $\lim_{n,m\to\infty} \rho(fx_n, fx_{n+m}) = 0$. By Lemma 2.7(b), $\{fx_n\}$ is a ρ-Cauchy sequence. Similarly, one can see that $\{fy_n\}$ is also ρ-Cauchy. Moreover, since for each $n \in \mathbb{N}$, $[y_n]_{\widetilde{G}} = [x_n]_{\widetilde{G}} = [x]_{\widetilde{G}} = [y]_{\widetilde{G}}$, we have

$$\begin{aligned}\rho(fx_n, fy_n) &\leq \psi(\rho(gx_n, gy_n) = \psi(\rho(fx_{n-1}, fy_{n-1})) \leq \psi^2(\rho(gx_{n-1}, gy_{n-1})) \\ &= \psi^2(\rho(fx_{n-2}, fy_{n-2})) \leq \ldots \leq \psi^n(\rho(fx, fy)) \xrightarrow{n\to\infty} 0.\end{aligned}$$

Therefore $\{fx_n\}$ and $\{fy_n\}$ are ρ-Cauchy equivalent. □

The next result states that in a Hausdorff uniform space (X, υ), endowed with a graph G and an A-distance ρ with $\rho(x, x) = 0$ for all $x \in X$, under certain circumstances, the condition of weak connectedness of G is equivalent to two other conditions.

Lemma 3.9. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and A-distance ρ. Assume that $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ belongs to Ψ and $f, g : X \to X$. Let f be a (ρ, ψ, G)-contraction with respect to g. If $\rho(x, x) = 0$, for each $x \in X$, then the following conditions are equivalent.*
(a) G is weakly connected.

(b) If f is orbitally bounded with respect to g at $x, y \in X$, then the sequences $\{fx_n\}$ and $\{fy_n\}$ are ρ-Cauchy equivalent, where $x_0 = x$, $y_0 = y$, $fx_{n-1} = gx_n$, $fy_{n-1} = gy_n$, $x_n \in [x_{n-1}]_{\widetilde{G}}$ and $y_n \in [y_{n-1}]_{\widetilde{G}}$ for each $n \in \mathbb{N}$.

(c) f and g have at most one common fixed point.

Proof. $(a) \Rightarrow (b)$ follows immediately from Lemma 3.8.

Let (b) hold. If a_0 and b_0 are distinct common fixed points of f and g, by Definition 3.1(i), there exists $a_1 \in [a_0]_{\widetilde{G}}$ such that $fa_0 = ga_1$. If $\rho(a_0, fa_1) \neq 0$, we have $\rho(a_0, fa_1) = \rho(fa_0, fa_1) \leq \psi(\rho(ga_0, ga_1)) = \psi(\rho(a_0, a_0)) = 0$, which is a contradiction. Therefore $\rho(a_0, fa_1) = 0$. By Lemma 2.7(a), $fa_1 = a_0$. Fix some $n \in \mathbb{N}$ and let $fa_i = a_0$ for $i \leq n$. There is $a_{n+1} \in [a_n]_{\widetilde{G}}$ such that $fa_n = ga_{n+1}$. If $fa_{n+1} \neq a_0$, then by Lemma 2.7(a), $\rho(a_0, fa_{n+1}) \neq 0$. Therefore we have $\rho(a_0, fa_{n+1}) = \rho(fa_0, fa_{n+1}) \leq \psi(\rho(ga_0, ga_{n+1})) = \psi(\rho(ga_0, fa_n)) = 0$, which is a contradiction. Therefore $fa_n = a_0$ for each n. Similarly, one can show that there is a sequence $\{b_n\}$ such that $b_{n+1} \in [b_n]_{\widetilde{G}} = [b_0]_{\widetilde{G}}$, $fb_n = gb_{n+1}$ and $fb_n = b_0$, for each $n \in \mathbb{N}$. By our assumption, $\{fa_n\}$ and $\{fb_n\}$ are ρ-Cauchy equivalent. Since for each $n \in \mathbb{N}$, $fa_n = a_0$ and $fb_n = b_0$, by Lemma 2.7(a), $a_0 = b_0$. Thus (c) holds.

If (c) is true but G is not weakly connected, i.e., $\widetilde{G}$ is disconnected, then for some $a_0 \in X$, both sets $[a_0]_{\widetilde{G}}$ and $X \setminus [a_0]_{\widetilde{G}}$ are nonempty. Fix $b_0 \in X \setminus [a_0]_{\widetilde{G}}$ and define $f, g : X \to X$ by

$$fx = \begin{cases} a_0 & \text{if } x \in [a_0]_{\widetilde{G}} \\ b_0 & \text{if } x \in X \setminus [a_0]_{\widetilde{G}} \end{cases}$$

and $gx = x$ for all $x \in X$. Trivially $\text{fix}\{f, g\} = \{a_0, b_0\}$. It is enough to show that f is a (ρ, ψ, G)-contraction with respect to g.
(i) Let $x \in X$. Then either $x \in [a_0]_{\widetilde{G}}$ or $x \in X \setminus [a_0]_{\widetilde{G}}$. Hence either $fx = a_0$ or $fx = b_0$. If $fx = a_0$, then $a_0 \in [x]_{\widetilde{G}}$ and $fx = ga_0 = a_0$. If $fx = b_0$ then $b_0, x \in X \setminus [a_0]_{\widetilde{G}}$, so $[b_0]_{\widetilde{G}} = [x]_{\widetilde{G}}$. Thus $b_0 \in [x]_{\widetilde{G}}$ and $fx = gb_0 = b_0$.

(ii) Let $(x, y) \in E(G)$, then either $x, y \in [a_0]_{\widetilde{G}}$ or $x, y \in X \setminus [a_0]_{\widetilde{G}}$. By the definition either $fx = fy = a_0$ or $fx = fy = b_0$ in both cases $(fx, fy) \in E(G)$, also $(gx, gy) = (x, y) \in E(G)$.

(iii) Fix $x \in X$ and $y \in [x]_{\widetilde{G}}$. Then we have two following cases: 1) $x, y \in [a_0]_{\widetilde{G}}$; 2) $x, y \in X \setminus [a_0]_{\widetilde{G}}$. In the first case, we get $\rho(fx, fy) = \rho(a_0, a_0) = 0 \leq \psi(\rho(gx, gy))$, and in the second case, we have $\rho(fx, fy) = \rho(b_0, b_0) = 0 \leq \psi(\rho(gx, gy))$ for any arbitrary $\psi \in \Psi$. □

We also need the following result.

Lemma 3.10. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and an A-distance ρ. Let f and g be self-mappings on X and $\psi \in \Psi$ be such that f is a (ρ, ψ, G)-contraction with respect to g. Assume that $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$, for some $x_0 \in X$. Let $\widetilde{G}_{x_0}$ be the component of $\widetilde{G}$ containing x_0. Then $[x_0]_{\widetilde{G}}$ is both f and g-invariant and $f|_{[x_0]_{\widetilde{G}}}$ is a $(\rho, \psi, \widetilde{G}_{x_0})$-contraction with respect to $g|_{[x_0]_{\widetilde{G}}}$.*

Moreover, for arbitrary $y_0, z_0 \in [x_0]_{\widetilde{G}}$, if f is orbitally bounded with respect to g at y_0 and z_0, the sequences $\{fy_n\}$ and $\{fz_n\}$ are ρ-Cauchy equivalent, where $fy_n = gy_{n-1}$ and $fz_n = gz_{n-1}$ for each $n \geq 1$.

Proof. Let $x \in [x_0]_{\widetilde{G}}$. We will show that $fx, gx \in [x_0]_{\widetilde{G}}$. By our assumption, there exists a path $\{r_i\}_{i=0}^N$ in $\widetilde{G}$ from x_0 to x, i.e., $r_0 = x_0$, $r_N = x$ and $(r_{i-1}, r_i) \in E(\widetilde{G})$ for all $1 \leq i \leq N$.

By Definition 3.1(ii), we get $(fr_{i-1}, fr_i) \in E(\widetilde{G})$ for all $1 \leq i \leq N$. It means that $\{fr_i\}_{i=0}^N$ is a path in $\widetilde{G}$ from $fr_0 = fx_0$ to $fr_N = fx$. It follows that $fr_N = fx \in [fx_0]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$. Similarly one can see that $gx \in [x_0]_{\widetilde{G}}$. Thus $[x_0]_{\widetilde{G}}$ is both f and g-invariant.

Now, we will show that $f|_{[x_0]_{\widetilde{G}}}$ is a $(\rho, \psi, \widetilde{G}_{x_0})$-contraction with respect to $g|_{[x_0]_{\widetilde{G}}}$.
(i) Let $y_0 \in [x_0]_{\widetilde{G}}$. Since f is a (ρ, ψ, G)-contraction with respect to g, by Definition 3.1(i), there exists $y_1 \in [y_0]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$ such that $fy_0 = gy_1$.

(ii) $(x, y) \in E(\widetilde{G}_{x_0})$ implies $(x, y) \in E(\widetilde{G})$. Thus $(fx, fy), (gx, gy) \in E(\widetilde{G})$. In order to show that $(fx, fy), (gx, gy) \in E(\widetilde{G}_{x_0})$, we note that if $(x, y) \in E(\widetilde{G}_{x_0})$, then $x, y \in [x_0]_{\widetilde{G}}$. By the above argument $fx, fy, gx, gy \in [x_0]_{\widetilde{G}}$. Therefore (fx, fy) and (gx, gy) are in $E(\widetilde{G}_{x_0})$.

(iii) Since $E(\widetilde{G}_{x_0}) \subseteq E(\widetilde{G})$ and f is a $(\rho, \psi, \widetilde{G})$-contraction with respect to g, we get $\rho(fx_0, fy_0) \leq \psi(\rho(gx_0, gy_0))$, for all $y_0 \in [x_0]_{\widetilde{G}}$.

Now, let $y_0, z_0 \in [x_0]_{\widetilde{G}}$ be such that f is orbitally bounded with respect to g at y_0 and z_0. Since $[y_0]_{\widetilde{G}} = [z_0]_{\widetilde{G}}$, by Lemma 3.8, the sequences $\{fy_n\}$ and $\{fz_n\}$ are ρ-Cauchy equivalent, where $fy_n = gy_{n-1}$ and $fz_n = gz_{n-1}$ for each $n \geq 1$. □

Now, we are ready to state of the main result of this section which gives some sufficient conditions for the existence and uniqueness of a common fixed point for self-mappings f and g where f is a (ρ, ψ, G)-contraction with respect to g on a Hausdorff uniform space (X, υ).

Theorem 3.11. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and an A-distance ρ, such that $\rho(x, x) = 0$ for all $x \in X$. Let $\psi \in \Psi$, X be S-complete and the triple (X, ρ, G) have the following property.*
(*) For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X with $\lim_{n \to \infty} \rho(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, there exists a subsequence $\{x_{k_n}\}_{n \in \mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ for each $n \in \mathbb{N}$.

Assume that $f,g : X \to X$ are commuting ρ-continuous mappings on X such that f is a (ρ,ψ,G)-contraction with respect to g and f is orbitally bounded with respect to g. Define $X_{f,g} = \{x_0 \in X : fx_0, gx_0 \in [x_0]_{\widetilde{G}}$ and $(gx_n, fx_n) \in E(G)$ for all $n \in \mathbb{N}\}$, where $fx_{n-1} = gx_n, \quad x_n \in [x_{n-1}]_{\widetilde{G}}$ for each $n \in \mathbb{N}$. Then for each $x \in X_{(f,g)}$, the mappings $f|_{[x]_{\widetilde{G}}}$ and $g|_{[x]_{\widetilde{G}}}$ have a unique common fixed point. In particular, if $X_{(f,g)} \neq \emptyset$ and G is weakly connected, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X_{(f,g)}$, then $fx_0, gx_0 \in [x_0]_{\widetilde{G}}, (gx_n, fx_n) = (fx_{n-1}, fx_n) \in E(G)$ for each $n \in \mathbb{N}$. Since f is orbitally bounded with respect to g at each point of X, Lemma 3.8 implies that for all $y_0 \in [x_0]_{\widetilde{G}}$, the sequences $\{fx_n\}_{n\in\mathbb{N}}$ and $\{fy_n\}_{n\in\mathbb{N}}$ are ρ-Cauchy equivalent where $fx_{n-1} = gx_n$ and $fy_{n-1} = gy_n$, for each $n \in \mathbb{N}$. Since X is S-complete, there is $u \in X$ such that $\lim_{n\to\infty} \rho(fx_n, u) = 0$. Since for each $n \in \mathbb{N}$, $fx_{n-1} = gx_n$, we get $\lim_{n\to\infty} \rho(fx_n, u) = \lim_{n\to\infty} \rho(gx_n, u)$. Therefore $\lim_{n\to\infty} \rho(gx_n, u) = 0$. By our assumption f and g are ρ-continuous, hence $\lim_{n\to\infty} \rho(gfx_n, gu) = \lim_{n\to\infty} \rho(fgx_n, fu) = 0$. Since $fg = gf$, we have $\lim_{n\to\infty} \rho(fgx_n, fu) = \lim_{n\to\infty} \rho(fgx_n, gu) = 0$, and by Lemma 2.7(a), $gu = fu$. We will show that fu is a common fixed point of f and g. Since $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$, by Lemma 3.10, $[x_0]_{\widetilde{G}}$ is both f and g-invariant. Moreover, for each $n \in \mathbb{N}$, $x_n \in [x_0]_{\widetilde{G}}$, therefore $fx_n \in [x_0]_{\widetilde{G}}$, for all n.

On the other hand $\lim_{n\to\infty} \rho(fx_n, u) = 0$ and $(fx_{n-1}, fx_n) \in E(G)$, for all $n \in \mathbb{N}$. Therefore by (*) there exists a subsequence $\{fx_{k_n}\}_{n\in\mathbb{N}}$ such that $(fx_{k_n}, u) \in E(G)$ for all $n \in \mathbb{N}$. Hence $(ffx_{k_n}, fu) \in E(G)$ for all $n \in \mathbb{N}$. Since for each n, $ffx_{k_n} \in [x_0]_{\widetilde{G}}$, there is a finite sequence $r_0 = x_0, r_1, r_2, \ldots, r_{M-1} = ffx_{k_1}, r_M = fu$ such that $(r_{i-1}, r_i) \in E(\widetilde{G})$. It means $fu \in [x_0]_{\widetilde{G}}$. By applying a similar argument, we see that $u \in [x_0]_{\widetilde{G}}$. Thus $[fu]_{\widetilde{G}} = [u]_{\widetilde{G}}$. If $\rho(fu, ffu) \neq 0$, we have $\rho(fu, ffu) \leq \psi(\rho(gu, gfu)) = \psi(\rho(fu, ffu)) < \rho(fu, ffu)$ which is a contradiction. On the other hand $\rho(fu, fu) = 0$, by Lemma 2.7(a). Hence $ffu = fu$ and $gfu = fgu = ffu = fu$. Therefore fu is a common fixed point of f and g. Since $\widetilde{G}_{x_0}$ is weakly connected, by Lemma 3.9, fu is a unique common fixed point of f and g.

If G is weakly connected then $[x]_{\widetilde{G}} = X$. Therefore $f = f|_{[x]_{\widetilde{G}}}$ and $g = g|_{[x]_{\widetilde{G}}}$ have a unique common fixed point. □

In 2004, Aamri and El Moutawakil [1] investigated the existence and uniqueness of common fixed point for two self-mappings on a Hausdorff uniform space as follows.

Theorem 3.12 ([1, Theorems 3.1 and 3.2]). *Let (X,υ) be a Hausdorff uniform spaces and ρ be an A-distance on X. Suppose X is ρ-bounded and S-complete. Suppose that $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ satisfies $\psi(t) > 0$ and $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$. Let f and g be commuting ρ-continuous or $\tau(\upsilon)$-continuous self mappings of X such that*
(i) $f(X) \subseteq g(X)$, (ii) $\rho(f(x), f(y)) \leq \psi(\rho(g(x), g(y)))$, for all $x, y \in X$.
Then f and g have a common fixed point. Moreover if ρ is an E-distance, then f and g have a unique common fixed point.

Let X be ρ-bounded and f be a (ρ,ψ,G)-contraction with respect to g. Then trivially f is orbitally bounded with respect to g. Thus Theorem 3.11 is a refinement of Theorem 3.12.

The following result shows that one can replace ρ-continuity of f by continuity of the A-distance ρ in Theorem 3.11.

THEOREM 3.13. *Let (X, υ) be a Hausdorff uniform space endowed with a graph G and a continuous A-distance ρ such that $\rho(x,x) = 0$ for all $x \in X$ and $\psi \in \Psi$. Let X be S-complete and the triple (X, ρ, G) have the property (*).*

Assume that f and g are commuting mappings on X such that f is a (ρ, ψ, G)-contraction with respect to g. Let g be ρ-continuous and let f be orbitally bounded with respect to g. Define $X_{f,g} = \{x_0 \in X : fx_0, gx_0 \in [x_0]_{\widetilde{G}}$ and $(gx_n, fx_n) \in E(G)$ for all $n \in \mathbb{N}\}$, where $fx_{n-1} = gx_n, \quad x_n \in [x_{n-1}]_{\widetilde{G}}$ for each $n \in \mathbb{N}$.

Then for each $x \in X_{(f,g)}$, the mappings $f|_{[x]_{\widetilde{G}}}$ and $g|_{[x]_{\widetilde{G}}}$ have a unique common fixed point. In particular, if $X_{(f,g)} \neq \emptyset$ and G is weakly connected, then f and g have a unique common fixed point.

Proof. By applying the same argument as in the beginning of the proof of Theorem 3.11, we can find a sequence $\{x_n\}_{n\geq 0}$ and $u \in X$ such that $fx_0, gx_0 \in [x_0]_{\widetilde{G}}$, $(gx_n, fx_n) \in E(G)$, $fx_{n-1} = gx_n$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} \rho(fx_n, u) = \lim\limits_{n\to\infty} \rho(gx_n, u) = 0$.

Since $gx_0 \in [x_0]_{\widetilde{G}}$ and for each $n \geq 0$, $(gx_n, gx_{n+1}) \in E(G)$, by (*) there exists a subsequence $\{gx_{k_n}\}$ of $\{gx_n\}$ such that $(gx_{k_n}, u) \in E(G)$ for each $n \in \mathbb{N}$. Hence $[gx_{k_n}]_{\widetilde{G}} = [u]_{\widetilde{G}}$, for each $n \in \mathbb{N}$. Thus

$$\rho(fgx_{k_n}, fu) \leq \psi(\rho(ggx_{k_n}, gu)) \leq \rho(ggx_{k_n}, gu), \tag{1}$$

Since $\lim_{n\to\infty} \rho(gx_{k_n}, u) = 0$ and g is ρ-continuous, Definition 2.6(iv) implies that $\lim_{n\to\infty} \rho(ggx_{k_n}, gu) = 0$. By (1) we get $\lim_{n\to\infty} \rho(fgx_{k_n}, fu) = 0$.

On the other hand, ρ-continuity of g implies that $\lim_{n\to\infty} \rho(gfx_n, gu) = 0$. Since f and g are commuting, $\lim_{n\to\infty} \rho(fgx_n, gu) = 0$. By Lemma 2.7(a), $fu = gu$.

The equality $[fx_n]_{\widetilde{G}} = [u]_{\widetilde{G}}$, for each $n \in \mathbb{N}$ together with continuity of ρ and ρ-continuity of g implies that

$$\lim_{n\to\infty} \rho(ffx_n, fu) \leq \lim_{n\to\infty} \psi(\rho(gfx_n, gu)) \leq \lim_{n\to\infty} \rho(gfx_n, gu) = \rho(gu, gu) = 0.$$

It means that the sequence $\{ffx_n\}$ is ρ-convergent to fu. The rest of the proof is similar to the end part of the proof of Theorem 3.11. □

COROLLARY 3.14. *Let (X, d) be a complete b-metric space endowed with a graph G with the following property.*
(**) For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $\lim\limits_{n\to\infty} d(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ for each $n \in \mathbb{N}$.
Let d be continuous, $\psi \in \Psi$ and $f, g : X \to X$ be commuting mappings such that g is continuous and f is orbitally bounded with respect to g and the following conditions holds.
(i) For each $x \in X$ there exists $y \in [x]_{\widetilde{G}}$ such that $fx = gy$.
(ii) For each $x, y \in X$, if $(x, y) \in E(G)$ then $(fx, fy), (gx, gy) \in E(G)$.
(iii) For each $x \in X$ and each $y \in [x]_{\widetilde{G}}$, we have $d(fx, fy) \leq \psi(d(gx, gy))$.

Define $X_{f,g} = \{x_0 \in X : fx_0, gx_0 \in [x_0]_{\widetilde{G}}$ *and* $(gx_n, fx_n) \in E(G)$ *for all* $n \in \mathbb{N}\}$, *where* $fx_{n\text{-}1} = gx_n$, $x_n \in [x_{n\text{-}1}]_{\widetilde{G}}$ *for each* $n \in \mathbb{N}$. *The mappings* $f|_{[x]_{\widetilde{G}}}$ *and* $g|_{[x]_{\widetilde{G}}}$ *have a unique common fixed point forfor each* $x \in X_{(f,g)}$. *In particular, if* $X_{(f,g)} \neq \emptyset$ *and* G *is weakly connected, then* f *and* g *have a unique common fixed point.*

Proof. By Example 2.5, d generates a Hausdorff uniformity on X and, with respect to it, d is an A-distance for X. Conditions (i)-(iii) imply that f is a (d, ψ, G)-contraction with respect to g. Thus the result follows from Theorem 3.13. □

EXAMPLE 3.15. Let $X = \left\{\frac{1}{n} : n \geq 1\right\} \cup \left\{\frac{-1}{n} : n \geq 1\right\} \cup \{0\}$. For $x, y \in X$ define $d(x, y) = |x - y|^2$. Then d is a b-metric on X. Indeed (X, d) satisfies conditions (1)-(3) in Example 2.5 for $s = 2$. Thus d defines a Hausdorff uniformity υ_d on X. By Example 2.5, d is an A-distance on (X, υ_d).

Let $\{x_n\}$ be a Cauchy sequence in X. It means that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $m, n > N_0$ implies that $d(x_m, x_n) < \varepsilon$. Therefore either $x_n = x$, for some $x \in X$ and for large enough n, or $x_n \to 0$ as $n \to \infty$. Thus X is complete.

Define graph G, by $V(G) = X$ and

$$E(G) = \Delta(X) \cup \left\{\left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{-1}{2}, \frac{-1}{3}\right)\right\} \cup \left\{\left(\frac{1}{n}, \frac{1}{n+1}\right) : n \geq 4\right\}$$
$$\cup \left\{\left(\frac{-1}{n}, \frac{-1}{n+1}\right) : n \geq 4\right\} \cup \left\{\left(\frac{1}{n}, 0\right) : n \geq 1\right\} \cup \left\{\left(\frac{-1}{n}, 0\right) : n \geq 1\right\}.$$

Then G is weakly connected. Assume that $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is defined by $\psi(r) = \frac{r}{2}$ which belongs to Ψ and let $f, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{1}{3} & \text{if } x = 1 \\ \frac{-1}{3} & \text{if } x = -1 \\ 0 & \text{if } x \neq 1, -1 \end{cases} \quad \text{and} \quad gx = \begin{cases} x & \text{if } x = 0, 1, -1, \frac{1}{3}, \frac{-1}{3} \\ \frac{1}{1+n} & \text{if } x = \frac{1}{n}, n > 1, n \neq 3 \\ \frac{-1}{1+n} & \text{if } x = \frac{-1}{n}, n > 1, n \neq 3 \end{cases}.$$

Then $fgx = gfx$ for all $x \in X$, and $f(X) = \left\{0, \frac{1}{3}, \frac{-1}{3}\right\} \subseteq g(X) = \left\{0, \pm 1, \pm\frac{1}{3}, \pm\frac{1}{5}, \pm\frac{1}{6}, \ldots\right\}$. Moreover, f is orbitally bounded with respect to g at each point of X.

Assume that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(x_n, x) = 0$ for some $x \in X$. By the definition of d, we get $\lim_{n\to\infty} |x_n - x|^2 = 0$.

Hence $\lim_{n\to\infty} |x_n - x| = 0$. It means for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $n \geq N_\varepsilon$ implies that $|x_n - x| < \varepsilon$. Hence either $x_n = x$ for large enough n or $x = 0$. In both cases we get $\lim_{n\to\infty} d(gx_n, gx) = 0$, thus g is continuous.

Also, the triple (X, d, G) satisfies the property (**) of Corollary 3.14. One can easily check that the following conditions hold.

(i) G is weakly connected and $f(X) \subseteq g(X)$. Thus for each $x \in X$ there exists $y \in [x]_{\widetilde{G}} = X$ such that $fx = gy$.

(ii) For each $(x, y) \in E(G)$ we have $(fx, fy), (gx, gy) \in E(G)$.

(iii) For each $x \in X$ and $y \in [x]_{\widetilde{G}} = X$, $d(fxfy) \leq \psi(d(gx, gy))$.

Therefore f is (ρ, ψ, G)-contraction with respect to g. Moreover $0 \in X_{f,g} \neq \emptyset$. Since G is weakly connected $[0]_{\widetilde{G}} = X$. By Corollary 3.14, f and g have a unique common fixed point on $[0]_{\widetilde{G}} = X$, i.e. $x = 0$.

Acknowledgement. The authors would like to thank the anonymous reviewers for their careful reading of the manuscript and useful suggestions.

References

[1] M. Aamri, D. El Moutawakil, *Common fixed point theorems for E-contractive or E-expansive maps in uniform spaces*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **20(1)** (2004), 83–91.

[2] S. P. Acharya, *Some results on fixed points in uniform spaces*, Yokohama Math. J. **22** (1974), 105–116.

[3] S. P. Acharya, *Some extensions of Banach's contraction principle in uniform spaces*, Indian J. Math. **17(2)** (1975), 93–100.

[4] A. Aghanians, K. Nourouzi, *Fixed points for Kannan type contractions in uniform spaces endowed with a graph*, Nonlinear Anal. Model. **21(1)** (2016), 103–113.

[5] F. Bojor, *Fixed point of $\varphi-$contraction in metric spaces endowed with a graph*, An. Univ. Craiova, Ser. Mat. Inform. **37(4)** (2010), 85–92.

[6] N. Bourbaki, *Éléments de mathématique. Fasc. II. Livre III: Topologie générale. Chapitre 1: Structures topologiques. Chaptir 2: Structures uniformes*, Quatrième édition. Actualités Scientifiques et Industrielles, No. 1142. Hermann. Paris, 1965.

[7] P. Chaoha, S. Songsa-ard, *Fixed points in uniform spaces*, Fixed Point Theory Appl.,(2014), Article ID **134**, 13 pp.

[8] G. Gwóźdź-Łukawska, J. Jachymski, *IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem*, J. Math. Anal. Appl. **356(2)** (2009), 453–463.

[9] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. **136(4)** (2008), 1359–1373.

[10] R. J. Knill, *Fixed points of uniform contractions*, J. Math. Anal. Appl. **12**, (1965), 449–455.

[11] A. K. Mirmostafaee, *Coupled fixed points for mappings on a b-metric space with a graph*, Math. Vesnik **69(3)** (2017), 214–225.

[12] A. K. Mirmostafaee, *Fixed point theorems for set-valued mappings in b-metric spaces*, Fixed Point Theory **18(1)** (2017), 305–314.

[13] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132(5)** (2004), 1435–1443.

[14] J. Rodríguez-Montes, J. A. Charris, *Fixed points for W-contractive or W-expansive maps in uniform spaces: toward a unified approach*, Southwest J. Pure Appl. Math. **1** (2001), 93–101 (electronic).

[15] M. Samreen, T. Kamran, N. Shahzad, *Some fixed points theorems in b-metric space endowed with graph*, Abstr. Appl. Anal. (2013), Article ID **967132**, 9 pp.

[16] N. A. Secelean, D. Wardowski, *New fixed point tools in non-metrizable spaces*, Results Math. **72**, no. 1-2, (2017), 919–935.

[17] E. Tarafdar, *An approach to fixed-point theorems on uniform spaces*, Trans. Amer. Math. Soc. **191**, (1974), 209–225.

(received 22.12.2018; in revised form 10.09.2019; available online 23.12.2019)

Department of Pure Mathematics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad 91775, Iran

E-mail: bhosseini1395@gmail.com

Center of Excellence in Analysis on Algebraic Structures, Department of Pure Mathematics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad 91775, Iran

E-mail: mirmostafaei@ferdowsi.um.ac.ir