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МАТЕМАТИЧКИ ВЕСНИК
72, 3 (2020), 226–231
September 2020

research paper
оригинални научни рад

CERTAIN RESULTS ON A CLASS OF INTEGRAL FUNCTIONS REPRESENTED BY MULTIPLE DIRICHLET SERIES

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Abstract. In the present paper we obtain a condition on vector valued coefficients of multiple Dirichlet series for when the series converges in the whole complex plane. We also prove some results related to Banach algebraic structure, topological divisor of zero and more on a class of such series satisfying certain condition.

1. Introduction

Consider a multiple Dirichlet series of the form:

$$f(s_1, s_2, \ldots, s_n) = \sum_{m_1, m_2, \ldots, m_n = 1}^{\infty} a_{m_1, m_2, \ldots, m_n} e^{(\lambda_{1_{m_1}} s_1 + \lambda_{2_{m_2}} s_2 + \ldots + \lambda_{n_{m_n}} s_n)}, \tag{1}$$

where $s_j = \sigma_j + it_j$ $(\sigma_j, t_j \in \mathbb{R})$ for $j = 1, 2, \ldots, n$; $a_{m_1, m_2, \ldots, m_n}$ belong to a commutative Banach algebra over the field $\mathbb{E}$, i.e. $(\mathbb{E}, \|\cdot\|)$, having identity element w with $\|w\| = 1$. Also, $0 < \lambda_{i_1} < \lambda_{i_2} < \ldots < \lambda_{i_p} \to \infty$ as $p \to \infty$, for $i = 1, 2, \ldots, n$. For the sake of simplicity, we denote $s = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n$, $m = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$, and $\lambda_m = (\lambda_{1_{m_1}}, \lambda_{2_{m_2}}, \ldots, \lambda_{n_{m_n}}) \in \mathbb{R}^n$.

For n-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ we denote $[x] = x_1 + x_2 + \ldots + x_n$, $xy = (x_1y_1, x_2y_2, \ldots, x_ny_n)$, and $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$. Thus the series (1) can be written as

$$f(s) = \sum_{m=1}^{\infty} a_m e^{[\lambda_m s]}. \tag{2}$$

Sarkar [4] considered multiple Dirichlet series with complex coefficients and made a characterization of coefficients in the case when the series converges absolutely in the whole $\mathbb{C}^n$. He further characterized the order and type of an entire function defined by such series and expressed them in terms of its complex coefficients and exponents.

2010 Mathematics Subject Classification: 30B50, 46J15, 17A35
Keywords and phrases: Dirichlet series; Banach algebra; topological divisor of zero; continous linear functional and total set.

2. Entire function

In this section, we find a necessary and sufficient condition for the series (2) to converge absolutely in the whole $\mathbb{C}^n$.

THEOREM 2.1. *The series represented by* (2) *satisfying*

$$\limsup_{[m]\to\infty} \frac{\sum_{k=1}^{n} \log m_k}{[\lambda_m]} = 0 \tag{3}$$

converges for all $s \in \mathbb{C}^n$ *if and only if*

$$\limsup_{[m]\to\infty} \frac{\log \|a_m\|}{[\lambda_m]} = -\infty. \tag{4}$$

Proof. Let (2) represents an entire function. If the series satisfies (3), the domain of absolute convergence of the series (2) coincides with its domain of convergence (see [1]). Thus (2) converges absolutely for all $s \in \mathbb{C}^n$.

Consider $r = (r, r, \ldots, r)$ with $r > 0$. Then there exists a constant K such that $\sum_{m=1}^{\infty} \|a_m\| e^{[r\lambda_m]} < K$, that is $\|a_m\| e^{[r\lambda_m]} < K$ which implies $\log \|a_m\| < \log K - r[\lambda_m]$. Hence for arbitrary $r > 0$, we have $\limsup_{[m]\to\infty} \frac{\log \|a_m\|}{[\lambda_m]} < -r$, which gives the desired result.

Conversely, let (4) hold. Let $s \in \mathbb{C}^n$, $\sigma > 0$ such that $\Re s_1 < \sigma$, $\Re s_2 < \sigma$, $\ldots$, $\Re s_n < \sigma$. For some $\epsilon > 0$, there exists R such that whenever $[m] \geq R$ we have $\frac{\log \|a_m\|}{[\lambda_m]} < -(\sigma + \epsilon)$. Thus $\|a_m\| e^{\sigma[\lambda_m]} < e^{-\epsilon[\lambda_m]}$. In view of (3), $\sum_{m=1}^{\infty} e^{-\epsilon[\lambda_m]} < \infty$. Thus the series (2) converges absolutely for all s where $\Re s_1 < \sigma$, $\Re s_2 < \sigma$,..., $\Re s_n < \sigma$. Hence the series also converges at $(s_1{}', s_2{}', \ldots, s_n{}')$ where $\Re s_1{}' < \Re s_1$, $\Re s_2{}' < \Re s_2$, $\ldots$, $\Re s_n{}' < \Re s_n$. As $\sigma > 0$ is arbitrary, thus the series (2) converges in the whole $\mathbb{C}^n$. □

3. The class $\mathbb{M}$

Srivastava in [5] considered a class of Dirichlet series in one variable of the form $\sum_{m=1}^{\infty} a_m e^{\lambda_m s}$ where $a_m \in \mathbb{C}$ for which $\left(\frac{\lambda_m}{e}\right)^{\lambda_m} |a_m|$ is bounded and studied some growth properties of the class of such series.

Kumar, Chutani and Manocha [3] proved various results on a class of vector valued Dirichlet series in two variables of the form $\sum_{m,n=1}^{\infty} a_{m,n} e^{(\lambda_m s_1 + \lambda_n s_2)}$ for which $(\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \|a_{m,n}\|$ is bounded where $c_1, c_2 \geq 0$ and c_1, c_2 are simultaneously not zero.

In this paper, we provide a more general form of bounded condition by considering a function $\Psi : \mathbb{N}^n \to \mathbb{R}^+$ satisfying $\lim_{[m]\to\infty} \log \Psi(m)^{\frac{1}{[\lambda_m]}} = \infty$. For example, if $(\lambda_m) = (m_1, m_2, \ldots m_n)$ where m_j for $j = 1, 2 \ldots, n$ denotes a sequence of natural numbers and $\Psi(m) = [m\lambda_m]^{[\lambda_m]}$, the previous condition holds.

Throughout this paper we consider $\mathbb{M}$ to be a class of all Dirichlet series (2) with sequence of exponents (λ_m) satisfying (3) for which $\Psi(m)\|a_m\|$ is bounded. Since every element of $\mathbb{M}$ satisfies (4) thus $\mathbb{M}$ represents a class of entire functions represented by vector valued multiple Dirichlet series.

In particular, if $\Psi(m) = e^{m\lambda_m} m!$ we get the same class of Dirichlet series in one variable as the one in [2]. The aim of this paper is to extend some results from [3] to class of Dirichlet series in several variables and also prove some new results.

Operations in $\mathbb{M}$ for $f(s) = \sum_{m=1}^{\infty} a_m e^{[\lambda_m s]}$ and $g(s) = \sum_{m=1}^{\infty} b_m e^{[\lambda_m s]}$ are defined as $(f+g)(s) = \sum_{m=1}^{\infty}(a_m + b_m)e^{[\lambda_m s]}$, $\alpha f(s) = \sum_{m=1}^{\infty}(\alpha a_m)e^{[\lambda_m s]}$, $\alpha \in \mathbb{E}$ and $(f \cdot g)(s) = \sum_{m=1}^{\infty} \Psi_m a_m b_m e^{[\lambda_m s]}$.

The norm in $\mathbb{M}$ can be defined as $\|f\|^* = \sup_{[m]\geq n} \Psi(m)\|a_m\|$. Also, the identity element in $\mathbb{M}$ is $e(s) = \sum_{m=1}^{\infty} w\{\Psi(m)\}^{-1} e^{[\lambda_m s]}$. For definition of the terms used in the sequel, we refer to [6, 7].

4. Main results

LEMMA 4.1. $\mathbb{M}$ *is not a division algebra.*

Proof. We need to show that there exists an element in $\mathbb{M}$ whose inverse does not exist in $\mathbb{M}$.

Let $\alpha(s) = \sum_{m=1}^{\infty} w(m_1 m_2 \dots m_n)^{-1}\{\Psi(m)\}^{-1} e^{[\lambda_m s]}$. Clearly, $\alpha(s) \in \mathbb{M}$. Let $\beta(s) = \sum_{m=1}^{\infty} b_m e^{[\lambda_m s]}$ be the inverse of $\alpha(s)$. Then

$$(\alpha \cdot \beta)(s) = e(s) \Rightarrow \sum_{m=1}^{\infty} w(m_1 m_2 \dots m_n)^{-1} b_m e^{[\lambda_m s]} = \sum_{m=1}^{\infty} w\{\Psi(m)\}^{-1} e^{[\lambda_m s]}$$
$$\Rightarrow b_m = w(m_1 m_2 \dots m_n)\{\Psi(m)\}^{-1}.$$

However, $\beta(s) = \sum_{m=1}^{\infty} w(m_1 m_2 \dots m_n)\{\Psi(m)\}^{-1} e^{[\lambda_m s]}$ does not belong to $\mathbb{M}$. □

THEOREM 4.2. *The function* $f(s) = \sum_{m=1}^{\infty} a_m e^{[\lambda_m s]}$ *is invertible in* $\mathbb{M}$ *if and only if* $\{\{\Psi(m)\}^{-1}\|{a_m}^{-1}\|\}$ *is a bounded sequence.*

Proof. Let $g(s) = \sum_{m=1}^{\infty} b_m e^{[\lambda_m s]}$ be the inverse of $f(s)$ such that $(f \cdot g)(s) = e(s)$ Then $\Psi(m) a_m b_m = w\{\Psi(m)\}^{-1}$ which implies that $\Psi(m)\|b_m\| = \|w\{\Psi(m)a_m\}^{-1}\|$ or equivalently, $\Psi(m)\|b_m\| = \|{a_m}^{-1}\|\{\Psi(m)\}^{-1}$. Since $g(s)$ is an element of $\mathbb{M}$, it can be concluded that $\{\{\Psi(m)\}^{-1}\|{a_m}^{-1}\|\}$ is a bounded sequence.

Conversely, suppose that $\{\{\Psi(m)\}^{-1}\|{a_m}^{-1}\|\}$ is a bounded sequence. Let $g(s) = \sum_{m=1}^{\infty} w\{\Psi(m)\}^{-2}\{a_m\}^{-1} e^{[s\lambda_m]}$. Clearly g $\in \mathbb{M}$. Moreover $(f \cdot g)(s) = e(s)$. □

THEOREM 4.3. $(\mathbb{M}, \|.\|^*)$ *is a commutative Banach algebra over* $\mathbb{E}$.

Proof. Let $\{f_r\}$ be a Cauchy sequence in $\mathbb{M}$ where $f_r(s) = \sum_{m=1}^{\infty} {a_m}^{(r)} e^{[\lambda_m s]}$ Then for $\epsilon > 0$, there exists some k such that whenever $r, p \geq k$, $\|f_r - f_p\|^* < \epsilon$, which implies

$$\sup_{[m]\geq n} \Psi(m)\|{a_m}^{(r)} - {a_m}^{(p)}\| < \epsilon \quad \text{whenever} \quad r, p \geq k. \tag{5}$$

Here $\{a_m{}^{(r)}\}$, being a Cauchy sequence in $\mathbb{E}$, converges to some $a_m \in \mathbb{E}$.

Letting $p \to \infty$ in (5), we get $\sup_{[m]\geq n} \Psi(m)\|a_m{}^{(r)} - a_m\| < \epsilon$ whenever $r \geq k$. Hence $f_r \to f$ for $f(s) = \sum_{m=1}^{\infty} a_m e^{[\lambda_m s]}$. Also

$$\begin{aligned}\sup_{[m]\geq n} \Psi(m)\|a_m\| &= \sup_{[m]\geq n} \Psi(m)\|a_m - a_m{}^{(r)} + a_m{}^{(r)}\| \\ &\leq \sup_{[m]\geq n} \Psi(m)\|a_m{}^{(r)} - a_m\| + \sup_{[m]\geq n} \Psi(m)\|a_m{}^{(r)}\|.\end{aligned}$$

Therefore, $f \in \mathbb{M}$.

If $f, g \in \mathbb{M}$ then

$$\|f \cdot g\|^* = \sup_{[m]\geq n} \Psi(m)\|a_m b_m \Psi(m)\| \leq \sup_{[m]\geq n} \Psi(m)\|a_m\| \sup_{[m]\geq n} \Psi(m)\|b_m\| = \|f\|^* \cdot \|g\|^*,$$

which proves the theorem. □

Theorem 4.4. *A necessary and sufficient condition for $f(s) = \sum_{m=1}^{\infty} a_m e^{[\lambda_m s]}$ in $\mathbb{M}$ to be a topological divisor of zero is $\lim_{[m]\to\infty} \Psi(m)\|a_m\| = 0$.*

Proof. Let $f(s)$ be a topological divisor of zero. Suppose $\lim_{[m]\to\infty} \Psi(m)\|a_m\| = \alpha(\alpha > 0)$. Then for a given ϵ, $0 < \epsilon < \alpha$, there exists a natural number N such that

$$\Psi(m)\|a_m\| > \alpha - \epsilon \quad \text{whenever} \quad [m] \geq N. \tag{6}$$

As $f \in \mathbb{M}$ is a topological divisor of zero, therefore there exists an arbitrary sequence $\{g_r\}$ of elements in $\mathbb{M}$ having unit norm such that $\sup_{[m]\geq n} \Psi(m)\|b_m{}^{(r)}\| = 1$ for $g_r(s) = \sum_{m=1}^{\infty} b_m{}^{(r)} e^{[\lambda_m s]}$.

For some δ , $0 < \delta < 1$ we can find an integer M_r and a subsequence $\{m_t\}$ of $\{m\}$ such that

$$\Psi(m)\|b_m{}^{(r)}\| > 1 - \delta \quad \text{for all} \quad [m] = [m_t] \geq M_r. \tag{7}$$

If $\Psi(m)\{\Psi(m)\|a_m b_m{}^{(r)}\|\} = 0$ for some $[m] = [m_t] \geq \max\{M_r, N\}$ then $\|a_m b_m{}^{(r)}\| = 0$. Since $\mathbb{E}$ is a field, therefore either $a_m = 0$ or $b_m{}^{(r)} = 0$, which contradicts either (6) or (7). Hence,$\Psi(m)\{\Psi(m)\|a_m b_m{}^{(r)}\|\} > 0$ for all $[m] = [m_t] \geq \max\{M_r, N\}$. Thus $\|f \cdot g_r\|^* \nrightarrow 0$, which is a contradiction to the fact that $f(s)$ is a topological divisor of zero. Hence $\lim_{[m]\to\infty} \Psi(m)\|a_m\| = 0$.

Conversely, let $\lim_{[m]\to\infty} \Psi(m)\|a_m\| = 0$. Construct a sequence $\{g_m\}$ such that $g_m(s) = w\{\Psi(m)\}^{-1} e^{[\lambda_m s]}$. Clearly, $g_m \in \mathbb{M}$ and $\|g_m\|^* = 1$ for all $m \geq 1$. Here $(g_m \cdot f)(s) = (f \cdot g_m)(s) = \{\Psi(m) a_m \{\Psi(m)\}^{-1}\} e^{[\lambda_m s]} = a_m e^{[\lambda_m s]}$. Therefore $\|g_m \cdot f\|^* = \|f \cdot g_m\|^* = \Psi(m)\|a_m\|$. Here $\|g_m \cdot f\|^* = \|f \cdot g_m\|^* \to 0$ as $[m] \to \infty$, therefore, $f(s)$ is a topological divisor of zero. □

Theorem 4.5. *Every continuous linear functional $\theta : \mathbb{M} \to \mathbb{E}$ is of the form $\theta(f) = \sum_{m=1}^{\infty} a_m \alpha_m \Psi(m)$ where $f(s) = \sum_{m=1}^{\infty} a_m e^{[\lambda_m s]}$ and $\{\alpha_m\}$ is a bounded sequence in $\mathbb{E}$.*

Proof. Let $\theta : \mathbb{M} \to \mathbb{E}$ be a continuous linear functional. So,

$$\theta(f) = \theta\big(\sum_{m=1}^{\infty} a_m e^{[\lambda_m s]}\big) = \sum_{m=1}^{\infty} a_m \theta\big(e^{[\lambda_m s]}\big). \tag{8}$$

We define a sequence $\{f_m\}$ in $\mathbb{M}$ as $f_m(s) = w\{\Psi(m)\}^{-1} e^{[\lambda_m s]}$ From (8), $\theta(f) = \sum_{m=1}^{\infty} a_m \Psi(m)\theta(f_m(s))$. Since θ is a continous linear functional, therefore $\|\theta(f_m)\| \leq P\|f_m\|^*$ for some P. As $\|f_m\|^* = 1$, thus $\|\theta(f_m)\| \leq P$. Let $\alpha_m = \theta(f_m)$. Thus α_m is a bounded sequence in $\mathbb{E}$. □

Theorem 4.6. *Let $f(s) = \sum_{m=1}^{\infty} a_m e^{[\lambda_m s]} \in \mathbb{M}$ where $a_m \neq 0$ for every $[m] \geq n$. Let $L \in \mathbb{C}^n$ be a set having at least one finite limit point. Define*

$$f_\tau(s) = \sum_{m=1}^{\infty} a_m \{\Psi(m)\}^{-1} e^{[\lambda_m (s+\tau)]}.$$

Then the set $A_f = \{f_\tau : \tau \in L\}$ is a total set with respect to the family of continuous linear transformations $\theta : \mathbb{M} \to \mathbb{E}$.

Proof. We have

$$f_\tau(s) = \sum_{m=1}^{\infty} a_m \{\Psi(m)\}^{-1} e^{[\lambda_m (s+\tau)]} = \sum_{m=1}^{\infty} a_m \{\Psi(m)\}^{-1} e^{[\lambda_m s]} e^{[\lambda_m \tau]}.$$

Note that for all $\tau \in \mathbb{C}^n$,

$$\Psi(m)\|a_m \{\Psi(m)\}^{-1} e^{[\lambda_m \tau]}\| = \|a_m e^{[\lambda_m \tau]}\| \leq \sum_{m=1}^{\infty} \|a_m e^{[\lambda_m \tau]}\|.$$

Since $f(s)$ is an entire function in $\mathbb{M}$ converging absolutely in the whole $\mathbb{C}^n$, thus $f_\tau(s) \in \mathbb{M}$ for every $\tau \in L$.

Let $\theta : \mathbb{M} \to \mathbb{E}$ be a continuous linear transformation such that $\theta(A_f) \equiv 0$, that is $\theta(f_\tau) = 0$ for all $\tau \in L$. Then by Theorem 4.5, $\sum_{m=1}^{\infty} \Psi(m)(a_m \{\Psi(m)\}^{-1} e^{[\lambda_m \tau]})\alpha_m = 0$ which implies $\sum_{m=1}^{\infty} a_m \alpha_m e^{[\lambda_m \tau]} = 0$, for all $\tau \in L$.

Define $h(s) = \sum_{m=1}^{\infty} a_m \alpha_m e^{[\lambda_m s]}$. Since $\{\alpha_m\}$ is a bounded sequence in $\mathbb{E}$ and $f(s) = \sum_{m=1}^{\infty} a_m e^{[\lambda_m s]}$ is an element of $\mathbb{M}$, therefore $h(s)$ also belongs to $\mathbb{M}$. However, $h(\tau) = \sum_{m=1}^{\infty} a_m \alpha_m e^{[\lambda_m \tau]} = 0$ for all $\tau \in L$. As L has a finite limit point, therefore $h \equiv 0$. This however implies that $a_m \alpha_m = 0$ for all $[m] \geq n$ and as $a_m \neq 0$ for every $[m] \geq n$ implies that $\alpha_m = 0$ for all $[m] \geq n$. □

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(received 29.12.2018; in revised form 26.06.2019; available online 06.01.2020)

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