MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
72, 3 (2020), 215–225
September 2020

research paper
оригинални научни рад

CONFORMAL CURVATURE TENSOR ON PARACONTACT METRIC MANIFOLDS

Venkatesha, Devaraja Mallesha Naik and H. Aruna Kumara

Abstract. In this paper, we consider paracontact metric manifolds satisfying certain flatness conditions on the conformal curvature tensor. Specifically, we study ξ-conformally flat K-paracontact manifolds and φ-conformally flat K-paracontact and paraSasakian manifolds. Also we discuss φ-conformally flat compact regular K-paracontact manifolds. Finally, we study conformally flat paracontact metric manifolds.

1. Introduction

If a pseudo-Riemannian metric g on a manifold M is conformally related with a flat pseudo-Euclidean metric, then g is called conformally flat. A pseudo-Riemannian manifold with a conformally flat pseudo-Riemannian metric is called a conformally flat manifold. Using the tools of conformal transformation, Weyl (see [12, 13]) introduced a generalized curvature tensor which vanishes whenever the metric is conformally flat (for this reason named the confomal curvature tensor). It is well-known that pseudo-Riemannian manifold M of dimension m is conformally flat if and only if the Weyl conformal curvature tensor field C, which is a tensor field of type $(1,3)$ defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{m-2}\{S(Y,Z)X + g(Y,Z)QX - S(X,Z)Y \\ - g(X,Z)QY\} + \frac{r}{(m-1)(m-2)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned}$$

vanishes for $m > 3$, and for $m = 3$,

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4}\{(Xr)Y - (Yr)X\},$$

where Q is the Ricci operator determined by $S(X,Y) = g(QX,Y)$, and r is the scalar curvature of the metric g. The Weyl conformal curvature tensor field C vanishes identically for $m = 3$ and it is invariant under conformal changes of the metric.

2010 Mathematics Subject Classification: 53C15, 53C25, 53D10
Keywords and phrases: Conformal curvature tensor; paracontact metric manifold; K-paracontact manifold; paraSasakian manifold; regular K-paracontact manifold.

Now suppose that $(M, \varphi, \xi, \eta, g)$ is a paracontact metric manifold (see Section 2). Then the module of a differentiable vector field on M can be decomposed into the direct sum $TM = \varphi(TM) \oplus L$, where L is the distribution generated by the characteristic vector field ξ. Thus the conformal curvature tensor C is defined as a map

$$C : TM \times TM \times TM \to \varphi(TM) \oplus L.$$

The three following cases can arise:
(i) The projection of the image of C in $\varphi(TM)$ is zero, i.e., $C(X, Y, Z, \varphi W) = 0$, for all $X, Y, Z, W \in TM$.

(ii) Projection of $C(X, Y)\varphi TM$ onto L is zero for every $X, Y \in TM$, i.e., $C(X, Y)\xi = 0$, for all $X, Y \in TM$.

(iii) Projection to the contact subbundle of the restriction of C to the contact subbundle vanishes, i.e., $\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0$, for all $X, Y, Z \in TM$.
In contact metric manifolds, the above three cases are studied respectively in [15, 16], and [3]. In the present paper, we study various geometric properties of conformal curvature tensor on paracontact metric manifolds. Though some of the present results are analogous to those obtained in [1, 3, 15, 16], our approach to get corresponding results in paracontact metric manifold is quite different.

Our work is structured as follows. In Section 2, we recall some basic information about paracontact metric manifolds. In Section 3, we focus on ξ-conformally flat K-paracontact manifold and we show that a K-paracontact metric manifold is ξ-conformally flat if and only if it is η-Einstein paraSasakian. As a consequence, some other results about ξ-conformally flat K-paracontact manifolds are obtained. Section 4 is devoted to the study of φ-conformally flat K-paracontact and paraSasakian manifolds, and some curvature identities are obtained on K-paracontact manifold. Further we show that a paraSasakian manifold of dimension > 3 is φ-conformally flat if and only if it is of constant curvature -1. Next we prove that φ-conformally flat K-contact manifold is a principal S^1-bundle over an almost paraKaehler space of paraholomorphic sectional curvature $\frac{4n+r}{2n(2n-1)} - 3$. Conformally flat paracontact metric manifolds are considered in Section 5 and we prove that a $(2n+1)$-dimensional conformally flat paracontact metric manifold with $Q\varphi = \varphi Q$ is of constant curvature -1 if $n > 1$ and 0 or -1 if $n = 1$.

2. Preliminaries

In this section, we collect some definitions and properties on almost paracontact metric manifolds which we will use later. For more information and details, we recommend the references [6, 14].

A $(2n + 1)$-dimensional smooth connected manifold M is said to be an *almost paracontact manifold* if there exist a $(1, 1)$ tensor field φ, a vector field ξ and a 1-form η defined on M such that $\varphi^2 = I - \eta \otimes \xi$, $\eta(\xi) = 1$, $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and the eigendistributions D^+ and D^- of φ corresponding to the respective eigenvalues 1 and

-1 have equal dimension n. If an almost paracontact manifold is endowed with a semi-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{1}$$

where signature of g is necessarily $(n+1, n)$ for all $X, Y \in TM$, then $(M, \varphi, \xi, \eta, g)$ is called an *almost paracontact metric manifold*. The curvature tensor R is taken with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ (this convention is opposite in [4, 5, 10]).

The *fundamental 2-form* of an almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by $\Phi(X, Y) = g(X, \varphi Y)$. If $d\eta = \Phi$, then the manifold $(M, \varphi, \xi, \eta, g)$ is said to be *paracontact metric manifold* and g the *associated metric*. In such case η is a contact form (that is, $\eta \wedge (d\eta)^n \neq 0$), ξ is its Reeb vector field and M is a contact manifold (see [7, 10]). The tensor $h = \frac{1}{2}\pounds_\xi \varphi$, where $\pounds$ is the usual Lie derivative, is symmetric and satisfies $h\varphi = -\varphi h, \quad h\xi = 0$. Furthermore, we also have

$$\nabla_X \xi = -\varphi X + \varphi h X, \tag{2}$$

$$(\nabla_\xi h)X = -\varphi X + h^2 \varphi X + \varphi R(\xi, X)\xi. \tag{3}$$

A paracontact metric manifold is said to be a *K-paracontact manifold* if ξ is a Killing vector field, equivalently, $h = 0$. In a K-paracontact manifold, we have the following formulas (see [11]).

$$\nabla_X \xi = -\varphi X, \tag{4}$$

$$R(X, \xi)\xi = -X + \eta(X)\xi, \tag{5}$$

$$S(X, \xi) = -2n\eta(X) \quad (\text{or } Q\xi = -2n\xi), \tag{6}$$

for all $X \in TM$, where S is the Ricci tensor.

From (1) and (5), we also have

$$R(\xi, Y, Z, \xi) = g(\varphi Y, \varphi Z), \tag{7}$$

for all $Y, Z \in TM$.

A paracontact metric manifold is said to be *paraSasakian* if the almost paracomplex structure J on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X + f\xi, \eta(X)\frac{d}{dt}\right),$$

is integrable, where $X \in TM$, t is the coordinate on R and f is a C^∞ function on $M \times R$. It is well known that a paracontact metric manifold M is paraSasakian if and only if

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{8}$$

for all $X, Y \in TM$. A paraSasakian manifold is always a K-paracontact manifold. A 3-dimensional K-paracontact manifold is a paraSasakian manifold [4], which may not be true in higher dimensions [6]. On any paraSasakian manifold, the following relations are well known (see [8, 11])

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{9}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{10}$$

$$S(X,\xi) = -2n\eta(X),$$

for all $X, Y \in TM$. Unlike the contact metric case, the condition (9) does not imply that the manifold is paraSasakian [6]. Further, on a paraSasakian manifold $Q\varphi = \varphi Q$ holds (see [8, Lemma 4]).

A paracontact metric manifold M is said to be *η-Einstein* if there exist smooth functions a and b such that $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, for all $X, Y \in TM$ and if M is paraSasakian then a and b are constants [14, Proposition 4.7]. If $b = 0$, M becomes an *Einstein manifold.* On K-paracontact manifold, from (6) we have

$$a + b = -2n. \tag{11}$$

On the other hand, the scalar curvature satisfies $r = a(2n+1) + b$. So K-paracontact metric manifold is η-Einstein if and only if

$$S(X,Y) = \left(\frac{r}{2n} + 1\right) g(X,Y) - \left(\frac{r}{2n} + 2n + 1\right) \eta(X)\eta(Y). \tag{12}$$

Hence K-paracontact manifold is Einstein if and only if

$$S(X,Y) = -2ng(X,Y), \tag{13}$$

for all $X, Y \in TM$.

Weyl conformal curvature tensor field C on a paracontact manifold is given by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\{S(Y,Z)X + g(Y,Z)QX - S(X,Z)Y \\ - g(X,Z)QY\} + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{14}$$

As discussed in the introduction, there arise the cases (i), (ii) and (iii). Analoguesly to the contact metric case, an almost paracontact metric manifold satisfying these three cases are said to be conformally symmetric [15], ξ-conformally flat [16] and φ-conformally flat [3], respectively.

3. ξ-conformally flat K-paracontact manifold

First we prove the following theorem.

THEOREM 3.1. *Let M be a ξ-conformally flat K-paracontact manifold of dimension > 3. Then the scalar curvature r of M satisfies*

$$\nabla r = \xi(r)\xi, \tag{15}$$

where ∇r is the gradient of scalar curvature r.

Proof. If M is ξ-conformally flat K-paracontact manifold, then the equation (14) becomes

$$\begin{aligned} R(X,Y)\xi =& \frac{1}{2n-1}\{S(Y,\xi)X + \eta(Y)QX - S(X,\xi)Y - \eta(X)QY\} \\ &- \frac{r}{2n(2n-1)}\{\eta(Y)X - \eta(X)Y\}, \end{aligned} \tag{16}$$

which gives

$$R(X,\xi)Y = \frac{1}{2n-1}\{g(Y,Q\xi)X + \eta(Y)QX - g(QX,Y)\xi - g(X,Y)Q\xi\} - \frac{r}{2n(2n-1)}\{\eta(Y)X - g(X,Y)\xi\}. \tag{17}$$

Putting $Y = \xi$ in (16), differentiating this covariantly along an arbitrary vector field W and using (17), we get:

$$(\nabla_W R)(X,\xi)\xi = \frac{1}{2n-1}\{g((\nabla_W Q)\xi,\xi)X + (\nabla_W Q)X - g((\nabla_W Q)X,\xi)\xi - \eta(X)(\nabla_W Q)\xi\} - \frac{Wr}{2n(2n-1)}\{X - \eta(X)\xi\}.$$

Taking inner product of above equation with Y and contracting with respect to X and W yields

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(e_i,\xi)\xi, Y) = \frac{1}{2n-1}\{g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y,\xi)\} + \frac{2n-2}{4n(2n-1)}\{Yr - \eta(Y)\xi(r)\}, \tag{18}$$

where $\{e_i\}$ is an orthonormal basis of vector fields in M and $\varepsilon_i = g(e_i,e_i)$. From the second Bianchi identity we easily obtain

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(Y,\xi)\xi, e_i) = g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y,\xi). \tag{19}$$

Then from (18) and (19), noting that $n > 1$ we get

$$g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y,\xi) = \frac{1}{4n}\{Yr - \eta(Y)\xi(r)\}. \tag{20}$$

On the other hand, from (16), (5) and (6), it follows that $QX = aX + b\eta(X)\xi$, where $a = \frac{r}{2n} + 1$ and $b = \frac{-r}{2n} - (2n+1)$, and so

$$(\nabla_\xi Q)Y - (\nabla_Y Q)\xi = (\xi a + \xi b)(Y + \eta(Y)\xi) - (Ya + Yb)\xi + b(\varphi Y).$$

Using (11), the above equation becomes $(\nabla_Y Q)\xi - (\nabla_\xi Q)Y = b(\varphi Y)$. Then (20) leads to $Yr = \eta(Y)\xi(r)$ which gives (15). □

Now, we present a necessary and sufficient condition for K-paracontact manifold to be ξ-conformally flat.

THEOREM 3.2. *A K-paracontact metric manifold is ξ-conformally flat if and only if it is η-Einstein paraSasakian.*

Proof. Let M be ξ-conformally flat. Then we have (16). Thus from (5) and (6), it follows that

$$QX = aX + b\eta(X)\xi, \tag{21}$$

where $a = \frac{r}{2n} + 1$ and $b = \frac{-r}{2n} - (2n+1)$, which is equivalent to (12). Making use of (21) in (16) yields

$$R(X,Y)\xi = -\{\eta(Y)X - \eta(X)Y\}. \tag{22}$$

We know from [9, p. 259] that $R(\xi, X)Y = -\nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y}\xi$, and it follows from (4) that $R(\xi, X)Y = (\nabla_X \varphi)Y$ which in view of (22) implies

$$g((\nabla_X \varphi)Y, Z) = g(R(\xi, X)Y, Z) = g(R(Y, Z)\xi, X) = g(-g(X, Y)\xi + \eta(Y)X, Z),$$

and thus (8) holds and M is paraSasakian.

Conversely, if M is η-Einstein paraSasakian, then using (9), (10) and (12) in (14) we have

$$\begin{aligned} C(X, Y)\xi &= R(X, Y)\xi - \left(\frac{2a + b}{2n - 1} - \frac{r}{2n(2n - 1)}\right)(\eta(Y)X - \eta(X)Y) \\ &= R(X, Y)\xi + (\eta(Y)X - \eta(X)Y = 0. \end{aligned}$$

So M is ξ-conformally flat. This completes the proof. □

Corollary 3.3. *Let M be a ξ-conformally flat K-paracontact manifold. If there exist functions λ and μ on M such that $(\nabla_X Q)Y - (\nabla_Y Q)X = \lambda X + \mu Y$, then M is Einstein.*

Proof. From Theorem 3.2, M is paraSasakian and so $Q\varphi = \varphi Q$ holds on M. Now the result follows from Lemma 5.1. □

Corollary 3.4 ([11]). *A conformally flat K-paracontact manifold is paraSasakian and of constant curvature -1.*

Proof. For $n > 1$, conformally flatness implies $C = 0$ which gives $\operatorname{div} C = 0$ or equivalently $(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4n}\{(Xr)Y - (Yr)X\}$. For $n = 1$, conformally flatness means $(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4}\{(Xr)Y - (Yr)X\}$. Hence from Corollary 3.3, M is paraSasakian and Einstein, that is, equation (13) holds. Thus $C(X, Y)Z = 0$ and (13) implies $R(X, Y)Z = -\{g(Y, Z)X - g(X, Z)Y\}$, which means that M is of constant curvature -1. □

4. φ-conformally flat K-paracontact and paraSasakian manifold

Theorem 4.1. *A φ-conformally flat K-paracontact manifold M is η-Einstein.*

Proof. One can easily observe that (iii) holds if and only if

$$g(C(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0, \tag{23}$$

for all vector fields $X, Y, Z, W \in TM$. Using (14) in (23), we get

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = {} & \frac{1}{2n - 1}[S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) + g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \\ & - S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)] \\ & - \frac{r}{2n(2n - 1)}[g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \tag{24}$$

Note that if $\{e_i, \xi\}_{i=1}^{2n}$ is a local pseudo-orthonormal φ-basis, then $\{\varphi e_i, \xi\}_{i=1}^{2n}$ is also a local pseudo-orthonormal φ-basis. Putting $X = W = e_i$ and summing over i

gives

$$\sum_{i=1}^{2n}\varepsilon_i g(R(\varphi e_i,\varphi Y)\varphi Z,\varphi e_i)=\sum_{i=1}^{2n}\varepsilon_i\Big\{\frac{1}{2n-1}[S(\varphi Y,\varphi Z)g(\varphi e_i,\varphi e_i)$$
$$+g(\varphi Y,\varphi Z)S(\varphi e_i,\varphi e_i)-S(\varphi e_i,\varphi Z)g(\varphi Y,\varphi e_i)-g(\varphi e_i,\varphi Z)S(\varphi Y,\varphi e_i)]$$
$$-\frac{r}{2n(2n-1)}[g(\varphi Y,\varphi Z)g(\varphi e_i,\varphi e_i)-g(\varphi e_i,\varphi Z)g(\varphi Y,\varphi e_i)]\Big\},$$

which further, using (7), yields

$$S(\varphi Y,\varphi Z)=ag(\varphi Y,\varphi Z), \tag{25}$$

where $a=1+\frac{r}{2n}$. Replacing Y by φY gives

$$S(Y,\varphi Z)=ag(Y,\varphi Z), \tag{26}$$

and so $\varphi QY=a\varphi Y$. Further operating by φ, it gives $QY=aY+(2n+a)\eta(X)\xi$, and so M is η-Einstein. □

Making use of Theorem 3.2 and Theorem 4.1, we obtain the following:

Corollary 4.2. *A φ-conformally flat paraSasakian manifold is always ξ-conformally flat.*

Theorem 4.3. *Let M be a φ-conformally flat K-paracontact manifold of dimension $2n+1$. Then the Ricci operator Q commutes with φ, the curvature tensor R and the scalar curvature r of M satisfy*

$$g(R(\varphi X,\varphi Y)\varphi Z,\varphi W)=\frac{r+4n}{2n(2n-1)}[g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)$$
$$-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)], \tag{27}$$
$$(\varphi X)r=0,\quad \textit{if } n>1. \tag{28}$$

Proof. The commutativity of Q and φ follows from (26) and using (25) in (24), we get (27). Now differentiating (25) covariantly along an arbitrary vector field X and using (26) we get $g((\nabla_X Q)\varphi Y,\varphi Z)=(Xa)g(\varphi Y,\varphi Z)$, from which we obtain $\sum_{i=1}^{2n}\varepsilon_i g((\nabla_{\varphi e_i}Q)\varphi e_i,\varphi Z)=\sum_{i=1}^{2n}\varepsilon_i(\varphi e_i a)g(\varphi e_i,\varphi Z)$. Therefore, we have

$$\frac{1}{2}(\varphi Z)r-g((\nabla_\xi Q)\xi,\varphi Z)=\frac{(\varphi Z)r}{2n}. \tag{29}$$

On the other hand, from (6) we have $(\nabla_\xi Q)\xi=0$. Thus (29) leads to (28), if $n>1$. This completes the proof. □

Theorem 4.4. *Let M be a paraSasakian manifold of dimension >3. Then M is φ-conformally flat if and only if it is of constant curvature -1.*

Proof. Replacing X,Y,Z,W by $\varphi X,\varphi Y,\varphi Z,\varphi W$ respectively in (27), using (9), (10) and (1), followed by a simple computation, we obtain

$$R(X,Y,Z,W)=\frac{4n+r}{2n(2n-1)}\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}$$
$$+\frac{r+2n(2n+1)}{2n(2n-1)}\{g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)$$

$$+ g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)\}. \tag{30}$$

Applying the above equation to the equation [14, (3.36)], we get

$$\begin{aligned}
&\frac{r+4n}{2n(2n-1)}\{g(Y,\varphi Z)g(X,W) - g(X,\varphi Z)g(Y,W) + g(Y,Z)g(X,\varphi W)\\
&- g(X,Z)g(Y,\varphi W)\} + \frac{r+4n^2+2n}{2n(2n-1)}\{g(X,\varphi Z)\eta(Y)\eta(W)\\
&- g(Y,\varphi Z)\eta(X)\eta(W) + g(Y,\varphi W)\eta(X)\eta(Z) - g(X,\varphi W)\eta(Y)\eta(Z)\}\\
=& - g(X,\varphi W)g(Y,Z) + g(X,\varphi Z)g(Y,W) - g(Y,\varphi Z)g(X,W) + g(Y,\varphi W)g(X,Z).
\end{aligned}$$

Contracting the above equation with respect to X and W gives $(r+2n(2n+1))(n-1)g(Y,\varphi Z) = 0$. Since $n > 1$, the above equation shows that $r = -2n(2n+1)$. Substituting this in (30) gives $R(X,Y)Z = -\{g(Y,Z)X - g(X,Z)Y\}$, which means M is of constant curvature -1. □

In [2], the authors introduced the notion of *regular contact manifold*, in which every point p of a contact manifold has a cubical coordinate neighborhood U of p such that the integral curves of ξ in U pass through U only once. Further, the orbits of ξ are closed curves if the manifold is compact. We denote by B the space of orbits of ξ. Then there is a natural projection $\pi : M \to B$ and B is a $2n$-dimensional differentiable manifold such that π is a differentiable map. If M^{2n+1} is a compact regular contact manifold, then M is a principal S^1-bundle over B, where S^1 is the 1-dimensional compact Lie group which is isomorphic to the 1-parameter group of global transformations generated by ξ (see [2]).

Let M be a compact regular K-paracontact manifold. Note that M is a contact manifold with the contact form η (see Section 2). Since ξ is Killing, the metric g is invariant under the action of the group S^1. Hence we can define a metric $\tilde{g}$ and a $(1,1)$ tensor field J by $\widetilde{g}(X,Y) = g(\widehat{X},\widehat{Y})$, and $JX = \pi_*\varphi\widehat{X}$ for all $X,Y \in TB$, where $\widehat{\ }$ denotes the horizontal lift with respect to η. The induced structure $(J,\widetilde{g})$ is an almost paraKaehler structure on B (see [11]).

Let us denote by $\widetilde{\nabla}$ the Riemannian connection associated with $\widetilde{g}$ and by $\widetilde{R}$ the corresponding Riemannian curvature tensor. Then we have (see [11])

$$\begin{aligned}
\widetilde{R}(X,Y,Z,W) &= R(\widehat{X},\widehat{Y},\widehat{Z},\widehat{W}) + 2g(\widehat{X},\varphi\widehat{Y})g(\varphi\widehat{Z},\widehat{W})\\
&\quad - g(\widehat{Z},\varphi\widehat{X})g(\varphi\widehat{Y},\widehat{W}) + g(\widehat{Z},\varphi\widehat{Y})g(\varphi\widehat{X},\widehat{W}),
\end{aligned} \tag{31}$$

for all $X,Y,Z,W \in TB$. Approaching in a similar manner as in [11], from (31) and (27), we find the paraholomorphic sectional curvature of the base manifold B:

$$-\frac{\widetilde{R}(X,JX,JX,X)}{\widetilde{g}(X,X)^2} = \frac{4n+r}{2n(2n-1)} - 3.$$

Hence we have the following theorem.

THEOREM 4.5. *A φ-conformally flat compact regular K-paracontact manifold is a principal S^1-bundle over an almost paraKaehler space of paraholomorphic sectional curvature* $\frac{4n+r}{2n(2n-1)} - 3$.

5. Conformally flat paracontact metric manifold

We begin with the following lemma.

LEMMA 5.1. *Let M be a paracontact metric manifold with $Q\varphi = \varphi Q$. If there exist functions λ and μ on M such that*

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \lambda X + \mu Y, \tag{32}$$

then M is Einstein.

Proof. As $Q\varphi = \varphi Q$, we have $\varphi^2 Q\xi = \varphi Q\varphi\xi = 0$. Thus $\varphi^2 Q\xi = Q\xi - g(Q\xi, \xi)\xi$ gives $Q\xi = g(Q\xi, \xi)\xi$ which means $Q\xi = (\operatorname{tr}\ell)\xi$. Differentiating this along an arbitrary vector field X and using (2), we get

$$(\nabla_X Q)\xi = Q(\varphi X - \varphi hX) + X(\operatorname{tr}\ell)\xi - (\operatorname{tr}\ell)(\varphi X - \varphi hX). \tag{33}$$

Taking inner product of (32) with ξ, we get

$$g((\nabla_X Q)Y - (\nabla_Y Q)X, \xi) = \lambda\eta(X) + \mu\eta(Y).$$

Since $g((\nabla_X Q)Y, Z) = g((\nabla_X Q)Z, Y)$, the above equation becomes

$$g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X) = \lambda\eta(X) + \mu\eta(Y).$$

Making use of (33) in above, one can get

$$\begin{aligned} g(Q(\varphi X - \varphi hX), Y) + X(\operatorname{tr}\ell)\eta(Y) - (\operatorname{tr}\ell)g(\varphi X - \varphi hX, Y) - g(Q(\varphi Y - \varphi hY), X) \\ - Y(\operatorname{tr}\ell)\eta(X) + (\operatorname{tr}\ell)g(\varphi Y - \varphi hY, X) = \lambda\eta(X) + \mu\eta(Y). \end{aligned} \tag{34}$$

Replacing X by φX and Y by φY in (34), using $Q\varphi = \varphi Q$ and $h\varphi = -\varphi h$ gives

$$-2Q\varphi X - Q\varphi hX - hQ\varphi X + 2(\operatorname{tr}\ell)\varphi X = 0. \tag{35}$$

Substituting φX for X in (35) yields $QhX - hQX = 2QX - 2(\operatorname{tr}\ell)X$. Applying φ on both sides of the above equation, we get

$$\varphi QhX - \varphi hQX = 2(\varphi QX - (\operatorname{tr}\ell)\varphi X). \tag{36}$$

Since $Q\varphi = \varphi Q$ and $h\varphi = -\varphi h$, the equation (35) becomes

$$-\varphi QhX + \varphi hQX = 2(\varphi QX - (\operatorname{tr}\ell)\varphi X). \tag{37}$$

Adding (36) and (37) gives $\varphi QX = (\operatorname{tr}\ell)\varphi X$. Applying φ on both sides of above equation gives

$$QX = (\operatorname{tr}\ell)X. \tag{38}$$

Hence M is an Einstein manifold. □

Note that $\operatorname{div} C = 0$ is equivalent to

$$g((\nabla_X Q)Y - (\nabla_Y Q)X, Z) = \frac{1}{4n}\{(Xr)g(Y, Z) - (Yr)g(X, Z)\}, \tag{39}$$

which is (32) with $\lambda = \frac{1}{4n}Xr$ and $\lambda = \frac{-1}{4n}Yr$. This gives the following

COROLLARY 5.2. *A paracontact metric manifold M of dimension > 3 with $Q\varphi = \varphi Q$ which has harmonic conformal curvature tensor is Einstein.*

At this time we recall the following results obtained by Zamkovoy [14] and Calvaruso [4].

Lemma 5.3 ([14, Theorem 3.12]). *If a paracontact manifold M is of constant sectional curvature c and dimension $2n+1>3$, then $c=-1$ and $\|h\|^2=0$.*

Lemma 5.4 ([4, Theorem 3.5]). *A three-dimensional locally symmetric paracontact metric manifold is either flat or of constant sectional curvature -1.*

These lemmas are used to prove the following paracontact analogue result of [1], and is one of the main result of present paper.

Theorem 5.5. *Let M be a $(2n+1)$-dimensional conformally flat paracontact metric manifold with $Q\varphi=\varphi Q$. Then M is of constant curvature -1 if $n>1$ and 0 or -1 if $n=1$.*

Proof. For $n>1$, conformally flatness means $C=0$ which gives $\operatorname{div} C=0$ or equivalently (39). Then Lemma 5.1 implies that M is Einstein. Thus $C(X,Y)Z=0$ and (38) implies that M is of constant curvature. Then Lemma 5.3 implies that M is of constant curvature -1

For $n=1$, conformally flatness means $(\nabla_X Q)Y-(\nabla_Y Q)X=\frac{1}{4}\{(Xr)Y-(Yr)X\}$, and it follows from Lemma 5.1 that M is Einstein. As 3-dimensional Einstein manifold is of constant curvature, it is locally symmetric. Now Lemma 5.4 completes the proof. □

Theorem 5.6. *Let M be a $(2n+1)$-dimensional conformally flat paracontact metric manifold such that ξ is the eigenvector of the Ricci operator everywhere on M and $K(\xi,X)=K(\xi,\varphi X)$ for every unit vector field $X\perp\xi$. Then M is of constant curvature -1 if $n>1$ and 0 or -1 if $n=1$.*

Proof. The conformally flatness implies

$$R(X,\xi)\xi=\frac{1}{2n-1}\{(\operatorname{tr}\ell)X+QX-S(X,\xi)\xi-\eta(X)Q\xi\}+\frac{r}{2n(2n-1)}\{X-\eta(X)\xi\}.$$

Using (3) and the hypothesis $Q\xi=(\operatorname{tr}\ell)\xi$, we get

$$QX-2(\operatorname{tr}\ell)\eta(X)\xi+(\operatorname{tr}\ell)X-\frac{r}{2n}(X-\eta(X)\xi)=(2n-1)(h^2X-\varphi^2X-\varphi(\nabla_\xi h)X). \tag{40}$$

Applying φ on both sides gives

$$\varphi QX+(\operatorname{tr}\ell)\varphi X-\frac{r}{2n}(\varphi X)=(2n-1)(\varphi h^2X-\varphi X-(\nabla_\xi h)X). \tag{41}$$

Replacing X by φX in (40) gives

$$Q\varphi X+(\operatorname{tr}\ell)\varphi X-\frac{r}{2n}(\varphi X)=(2n-1)(h^2\varphi X-\varphi X-\varphi(\nabla_\xi h)X). \tag{42}$$

Now subtracting (42) from (41) yields

$$\varphi QX-Q\varphi X=-2(2n-1)(\nabla_\xi h)X. \tag{43}$$

The following formula is obtained by Perrone in [10]:

$$K(\xi,X)-K(\xi,\varphi X)=2\varepsilon_X g((\nabla_\xi h)X,\varphi X),$$

for any unit vector field $X\perp\xi$, where $\varepsilon_X=g(X,X)=\pm1$. Thus the hypothesis $K(\xi,X)=K(\xi,\varphi X)$ for all unit vector field $X\perp\xi$ shows that $\nabla_\xi h=0$. Then (43) yields $Q\varphi=\varphi Q$. Now the conclusion follows from Theorem 5.5. □

Acknowledgement. The second author (D.M.N.) is grateful to University Grants Commission, New Delhi (Ref. No.:20/12/2015(ii)EU-V) for financial support in the form of Junior Research Fellowship.

References

[1] D.E. Blair, T. Koufogiorgos, *When is the tangent sphere bundle conformally flat?*, J Geom, **49** (1994), 55–66.

[2] W.M. Boothby, H.C. Wang, *On contact manifolds*, Ann. Math. **68** (1958), 721–734.

[3] J.L. Cabrerizo, L.M. Fernandez, M. Fernandez, G. Zhen, *The strucure of a class of K-contact manifolds*, Acta Math. Hungar., **82(4)** (1999), 331–340.

[4] G. Calvaruso, *Homogeneous paracontact metric three-manifolds*, Illinois J. Math., **55** (2011), 697–718.

[5] G. Calvaruso, A. Perrone, *Ricci solitons in three-dimensional paracontact geometry*, J. Geom. Phys. **98** (2015), 1–12.

[6] V. Martin-Molina, *Paracontact metric manifolds without a contact metric counterpart*, Taiwanese J. of Math. **19(1)** (2015), 175–191.

[7] B.C. Montano, A. Carriazo, V. Martin-Molina, *Sasaki-Einstein and paraSasaki- Einstein metrics from (k,μ)-structures*, J. Geom. Physics. **73** (2013), 20–36.

[8] D.M. Naik, V. Venkatesha, *η-Ricci solitons and almost η-Ricci solitons on paraSasakian manifolds*, Int. J. Geom. Methods Mod. Phys. **16(9)** (2019), 1950134 (18 pages).

[9] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[10] A. Perrone, *Some results on almost paracontact metric manifolds*, Mediterr. J. Math. **13(5)** (2016), 3311–3326.

[11] Venkatesha, D.M. Naik, *Certain results on K-paracontact and paraSasakian manifolds.* J. Geom. **108(3)** (2017), 939–952.

[12] H. Weyl, *Reine Infinitesimalgeometrie*, Math. Z. **2** (1918), 384–411.

[13] H. Weyl, *Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung*, Göttingen Nachrichten, (1921), 99–112.

[14] S. Zamkovoy, *Canonical connections on paracontact manifolds.* Ann. Global Anal. Geom., **36** (2009), 37–60.

[15] G. Zhen, *Conformally symmetric K-contact manifolds.* Bull. Belg. Chinese Quart. J. Math., **7(1)** (1992), 5–10.

[16] G. Zhen, J.L. Cabrerizo, L.M. Fernandez, M. Fernandez, *On ξ-conformally flat contact metric manifolds.* Indian J. Pure Appl. Math., **28** (1997), 725–734.

(received 15.12.2018; in revised form 01.06.2019; available online 17.12.2019)

Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Karnataka, India

E-mail: vensmath@gmail.com

Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Karnataka, India

E-mail: devarajamaths@gmail.com

Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Karnataka, India

E-mail: arunmathsku@gmail.com